CAMBRIDGE AS & A LEVEL

Further Mathematics (9231) 2023

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TooMates Coolección

Toomates Coolección

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Trigonometría , Desigualdades , Funciones

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PAU España: Cataluña TEC Cataluña CCSS Valencia Galicia País Vasco Baleares Aragón
PAU Internacional: Portugal A Portugal B Italia A Level 9709 2023 9231 2023 IB Francia
Canguro: ESP CAT FR USA UK AUS
USA: Mathcounts AMC 8 10 12 AIME USAJMO USAMO TSTST TST ELMO Putnam
España: OME OMEFL OMEC OMEA OMEM CDP
Internacional: IMO OMI IGO SMT INMO CMO REOIM Arquimede HMMT BMO
Pruebas acceso: ACM4 , CFGS , PAP
Pizzazz!: Book A Book B Book C Book D Book E Pre-Algebra Algebra
AHSME: Book 1 Book 2 Book 3 Book 4 Book 5 Book 6 Book 7 Book 8 Book 9

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Visita mi **Canal de Youtube**: **https://www.youtube.com/c/GerardRomo**

Versión de este documento: **18/06/2024**

Presentación.

Cambridge International AS & A Level Further Mathematics (9231) ofrece al estudiante un nivel más profundo de las matemáticas en comparación con Pure Mathematics (9709). Esta titulación se tiene en cuenta para entrar en algunas carreras de ciertas universidades de prestigio. Se supone que el estudiante ha cursado o está cursando Pure Mathematics (9709).

Obtención del título.

Para la obtención de un título "**AS Level Further Mathematics**":

Es necesario superar 2 Papers, siguiendo una de las siguientes opciones:

Ruta 1: Paper 1 & Paper 3

Ruta 2: Paper 1 & Paper 4

Para la obtención de un título "**A Level Further Mathematics**":

Es necesario superar los 4 Papers.

Para su distribución en dos años se aconseja seguir una de las siguientes rutas:

Ruta 1: Primer año: Paper 1 y Paper 3
Segundo año: Paper 2 y Paper 4

Ruta 2: Primer año: Paper 1 y Paper 4
Segundo año: Paper 2 y Paper 3

Nivel previo recomendable.

It is expected that learners will have studied the majority of the Cambridge International AS & A Level Mathematics (9709) subject content before studying Cambridge International AS & A Level Further Mathematics (9231).

Further Mathematics (9231) → Prior knowledge required from Mathematics (9709)

Paper 1 → Papers 1 and 3
Paper 2 → Papers 1 and 3
Paper 3 → Papers 1, 3 and 4
Paper 4 → Papers 1, 3, 5 and 6

Papers (Pruebas).

Paper 1 Further Pure Mathematics 1 **(**2 hours)

1.1 Roots of polynomial equations
1.2 Rational functions and graphs
1.3 Summation of series
1.4 Matrices
1.5 Polar coordinates
1.6 Vectors
1.7 Proof by induction

Paper 2 Further Pure Mathematics 2 **(**2 hours)

2.1 Hyperbolic functions
2.2 Matrices
2.3 Differentiation
2.4 Integration
2.5 Complex numbers
2.6 Differential equations

Paper 3 Further Mechanics **(**1 hour 30 minutes)

3.1 Motion of a projectile
3.2 Equilibrium of a rigid body
3.3 Circular motion
3.4 Hooke's law
3.5 Linear motion under a variable force
3.6 Momentum

Paper 4 Further Probability & Statistics **(**1 hour 30 minutes)

4.1 Continuous random variables
4.2 Inference using normal and t-distributions
4.3 2 χ^2 -tests
4.4 Non-parametric tests
4.5 Probability generating functions

Índice.

Cambridge Assessment International Education

Grade thresholds – June 2023

Cambridge International A Level Further Mathematics (9231)

Grade thresholds taken for Syllabus 9231 (Further Mathematics) in the June 2023 examination.

		Minimum raw mark required for grade:				
	Maximum raw mark available	A	B	C	D	E
Component 11	75	56	47	39	31	24
Component 12	75	56	47	39	31	24
Component 13	75	52	43	37	29	20
Component 21	75	66	58	50	42	33
Component 22	75	66	58	50	42	33
Component 23	75	66	58	50	42	33
Component 31	50	35	27	22	17	11
Component 32	50	35	27	22	17	11
Component 33	50	35	27	22	17	11
Component 41	50	42	37	30	23	16
Component 42	50	42	37	30	23	16
Component 43	50	42	37	30	23	16
Component 51	75	59	51	42	34	26
Component 60	75	59	48	40	31	23

Grade A* does not exist at the level of an individual component.

The overall thresholds for the different grades were set as follows.

Option	Maximum mark after weighting	Combination of components	A*	A	B	C	D	E
AX	250	11, 21, 31, 41	224	199	169	140	112	84
AY	250	12, 22, 32, 42	224	199	169	140	112	84
AZ	250	13, 23, 33, 43	222	195	165	136	108	80
BY	250	22, 42, 85	221	193	164	135	106	77
BZ	250	23, 43, 86	222	195	165	136	107	79
CY	250	22, 32, 88	221	192	163	134	105	76
CZ	250	23, 33, 89	222	195	164	135	106	78

Grade thresholds continued

Cambridge A Level Further Mathematics (9231)

Option	Maximum mark after weighting	Combination of components	A*	A	B	C	D	E
DT	250	22, 42, 94	207	180	153	128	103	78
DU	250	23, 43, 95	207	180	153	128	103	78
DY	250	22, 42, 95	207	180	153	128	103	78
DZ	250	23, 43, 96	207	180	153	128	103	78
ET	250	22, 32, 97	215	185	155	128	101	74
EU	250	23, 33, 98	215	185	155	128	101	74
EY	250	22, 32, 98	215	185	155	128	101	74
P3	150	51, 60	134	118	99	82	65	49
S1	125	11, 31	–	91	74	61	48	35
S3	125	12, 32	–	91	74	61	48	35
S4	125	12, 42	–	98	84	69	54	40
S5	125	13, 33	–	87	70	57	44	31
S6	125	13, 43	–	94	80	65	50	36

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

1842585985

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

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1 Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$ [5]

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(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

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2 The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

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(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2 + (\beta+r)^2 + (\gamma+r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

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3 **(a)** Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k, where k is a positive constant. [4]

(b) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(kr+1)(kr-k+1)}$. [1]

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(c) Find also $\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k. [2]

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4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}$, where a, b, c are real constants and $b \neq 0$.

(a) Show that $\mathbf{M}$ does not represent a rotation about the origin. [2]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by $\mathbf{M}$. [5]

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It is given that **M** represents the sequence of two transformations in the $x-y$ plane given by an enlargement, centre the origin, scale factor 5 followed by a shear, x-axis fixed, with $(0, 1)$ mapped to $(5, 1)$.

(c) Find **M**. [3]

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(d) The triangle DEF in the $x-y$ plane is transformed by **M** onto triangle PQR.

Given that the area of triangle DEF is $12\,\text{cm}^2$, find the area of triangle PQR. [2]

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5 The curve C has polar equation $r^2 = \dfrac{1}{\theta^2 + 1}$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Find the area of the region enclosed by C, the initial line, and the half-line $\theta = \pi$. [4]

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(c) Show that, at the point of C furthest from the initial line,

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

and verify that this equation has a root between 1.1 and 1.2. [5]

6 The curve C has equation $y = \dfrac{x^2 + 2x - 15}{x - 2}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that C has no stationary points. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - 2x - 15}{x - 2}\right|$. [2]

(e) Find the set of values of x for which $\left|\dfrac{2x^2+4x-30}{x-2}\right| < 15$. [4]

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7 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

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(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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The line l passes through the point A with position vector $a\mathbf{i}+a\mathbf{j}+(a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i}+b\mathbf{j}+b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 11 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:
- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:
- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$2\mathbf{A}=\begin{pmatrix}6&0\\2&2\end{pmatrix}=\begin{pmatrix}2\times 3&0\\3-1&2\end{pmatrix}$ so true when $n=1$.	**B1**	States base case.
	Assume that it is true for $n=k$, so $2\mathbf{A}^k=\begin{pmatrix}2\times 3^k&0\\3^k-1&2\end{pmatrix}$.	**B1**	States inductive hypothesis.
	Then $2\mathbf{A}^{k+1}=\begin{pmatrix}2\times 3^k&0\\3^k-1&2\end{pmatrix}\begin{pmatrix}3&0\\1&1\end{pmatrix}=\begin{pmatrix}2\times 3^{k+1}&0\\3^{k+1}-3+2&2\end{pmatrix}$	**M1A1**	Multiplies $2\mathbf{A}^k$ with $\mathbf{A}$.
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	
1(b)	$\det\mathbf{A}^n=\det\begin{pmatrix}3^n&0\\\frac{1}{2}\left(3^n-1\right)&1\end{pmatrix}=3^n$ Or a multiple of $\mathbf{A}^{-n}=2^{-1}3^{-n}\begin{pmatrix}2&0\\1-3^n&2\times 3^n\end{pmatrix}$ seen.	**B1**	
	$\mathbf{A}^{-n}=3^{-n}\begin{pmatrix}1&0\\\frac{1}{2}\left(1-3^n\right)&3^n\end{pmatrix}$	**B1**	OE $\mathbf{A}^{-n}=2^{-1}3^{-n}\begin{pmatrix}2&0\\1-3^n&2\times 3^n\end{pmatrix}$
		2	

Question	Answer	Marks	Guidance
2(a)	$(-4)^2 - 2(6)$	**M1**	Uses formula for sum of squares.
	4	**A1**	
		2	
2(b)	$(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = \sum_{r=1}^{n}\left(4 + 2(-4)r + 3r^2\right)$	**M1 A1**	Collects like terms and uses $\alpha + \beta + \lambda = -4$ and *their* result from part **(a)**.
	$4n - 4n(n+1) + \frac{1}{2}n(n+1)(2n+1)$	**M1**	Applies formulae from MF19.
	$-4n^2 + \frac{1}{2}n(n+1)(2n+1)$ $= n\left(-4n + \frac{1}{2}(2n^2 + 3n + 1)\right)$ $= n\left(n^2 - \frac{5}{2}n + \frac{1}{2}\right)$	**M1 A1**	Simplifies.
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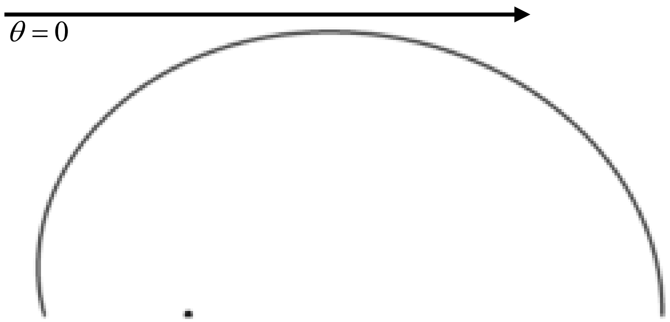
Question	Answer	Marks	Guidance
3(a)	$\dfrac{1}{(kr+1)(kr-k+1)}=\dfrac{1}{k}\left(\dfrac{1}{k(r-1)+1}-\dfrac{1}{kr+1}\right)$	**M1 A1**	Finds partial fractions.
	$\displaystyle\sum_{r=1}^{n}\frac{1}{(kr+1)(kr-k+1)}=\frac{1}{k}\left(1-\frac{1}{k+1}+\frac{1}{k+1}-\frac{1}{2k+1}+\ldots+\frac{1}{k(n-1)+1}-\frac{1}{kn+1}\right)$	**M1**	Writes at least three complete terms, including last.
	$\dfrac{1}{k}\left(1-\dfrac{1}{kn+1}\right)$	**A1**	OE e.g. $\dfrac{n}{kn+1}$
		4	
3(b)	$\dfrac{1}{k}$	**B1**	
		1	
3(c)	$\displaystyle\sum_{r=n}^{n^2}\frac{1}{(kr+1)(kr-k+1)}=\sum_{r=1}^{n^2}\frac{1}{(kr+1)(kr-k+1)}-\sum_{r=1}^{n-1}\frac{1}{(kr+1)(kr-k+1)}$	**M1**	Or applies the method of differences again.
	$\dfrac{1}{k}\left(1-\dfrac{1}{kn^2+1}-\left(1-\dfrac{1}{k(n-1)+1}\right)\right)=\dfrac{1}{k}\left(\dfrac{1}{k(n-1)+1}-\dfrac{1}{kn^2+1}\right)$	**A1**	OE e.g. $\dfrac{n^2}{kn^2+1}-\dfrac{(n-1)}{k(n-1)+1}$
		2	

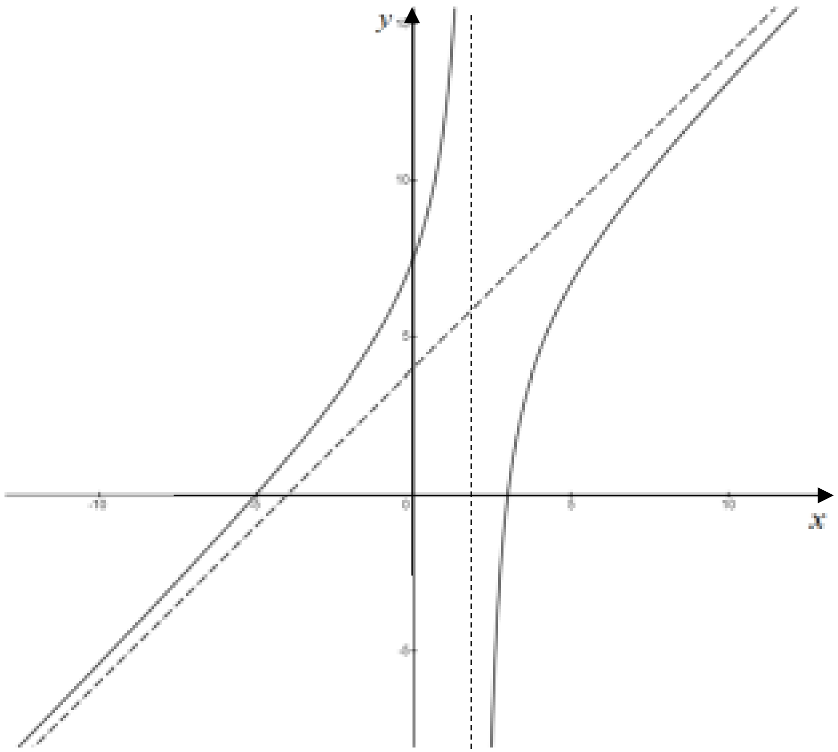
Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	**M1**	Uses correct matrix for rotation.
	$b^2 = -c^2$ which is impossible since b and c are real and $b \neq 0$.	**A1**	AG
		2	
4(b)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b^2 y \\ c^2 x + ay \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$c^2 x + amx = m\left(ax + b^2 mx\right)$	**M1 A1**	Uses $y = mx$ and $Y = mX$.
	$c^2 + am = ma + b^2 m^2 \Rightarrow c^2 = b^2 m^2$	**A1**	
	$y = \frac{c}{b}x$ and $y = -\frac{c}{b}x$	**A1**	
		5	
4(c)	$\mathbf{M} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$	**M1 A1***	Award M1 if matrices correct but order is wrong.
	$\begin{pmatrix} 5 & 5^2 \\ 0 & 5 \end{pmatrix}$	**DA1**	Dep: previous A1
		3	
4(d)	$12 \times \det \mathbf{M}$	**M1**	Using *their* **M**.
	$300\ \text{cm}^2$	**A1FT**	
		2	

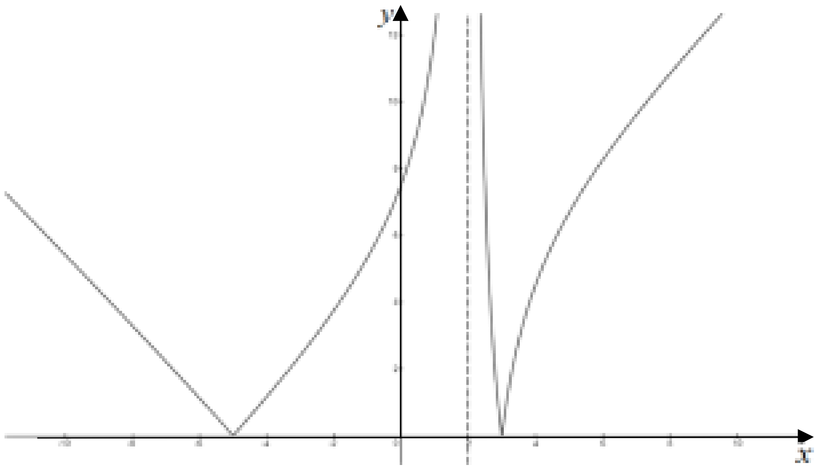
Question	Answer	Marks	Guidance
5(a)		B1*	Correct shape and domain, polar graph with r strictly decreasing but condone if not strictly decreasing close to $\theta = \pi$.
		DB1	Fully correct including shape at $\theta = 0$ correct shape at $\theta = \pi$ correct
	$(1, 0)$	B1	Identified as point furthest from the pole and given as coordinates.
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \frac{1}{\theta^2+1}\,\mathrm{d}\theta$	M1	Uses correct formula with correct limits.
	$\frac{1}{2}\left[\tan^{-1}\theta\right]_0^{\pi}$	M1 A1	Integrates $\frac{1}{\theta^2+1}$.
	$\frac{1}{2}\tan^{-1}\pi = 0.631$	A1	
		4	

Question	Answer	Marks	Guidance
5(c)	$y = \dfrac{\sin\theta}{\sqrt{\theta^2+1}}$	B1	Uses $y = r\sin\theta$
	$\dfrac{dy}{d\theta} = \dfrac{\left(\theta^2+1\right)^{\frac{1}{2}}\cos\theta - \theta\left(\theta^2+1\right)^{-\frac{1}{2}}\sin\theta}{\theta^2+1} = 0$	M1 A1	Sets derivative equal to zero.
	$\theta \neq 0 \Rightarrow \left(\theta + \dfrac{1}{\theta}\right)\cot\theta - 1 = 0$	A1	AG.
	$\left(1.1 + \dfrac{1}{1.1}\right)\cot 1.1 - 1 = 0.02\ldots$ and $\left(1.2 + \dfrac{1}{1.2}\right)\cot 1.2 - 1 = -0.209\ldots$	B1	Shows sign change (1sf or better).
		5	

Question	Answer	Marks	Guidance
6(a)	$x = 2$	B1	States vertical asymptote.
	$x^2 + 2x - 15 = (x-2)(x+4) - 7 \Rightarrow y = x + 4$	M1 A1	Finds oblique asymptote.
		3	

Question	Answer	Marks	Guidance
6(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)(2x+2)-(x^2+2x-15)}{(x-2)^2}$	**M1**	Differentiates.
	$x^2 - 4x + 11 = 0$ $\left(\text{or } \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{7}{(x-2)^2}\right)$	**A1**	Forms quadratic equation or simplifies $\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$4^2 - 4(1)(11) = -28 < 0$ (or $y' > 0$) There are no turning points.	**A1**	Correct conclusion.
		3	

Question	Answer	Marks	Guidance
6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
	$(0, 7.5), (-5, 0), (3, 0)$	**B1**	States coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
6(d)		B1FT	FT from sketch in **(c)**.
		B1	Correct shape at infinity and on x axis.
		2	
6(e)	$\frac{x^2+2x-15}{x-2}=\frac{15}{2}$ or $\frac{x^2+2x-15}{x-2}=-\frac{15}{2}$ $x^2-\frac{11}{2}x=0$ or $x^2+\frac{19}{2}x-30=0$	M2	Finds critical points, award M1 for each case.
	$x=0,\frac{11}{2}$ or $x=-12,\frac{5}{2}$	A1	
	$-12<x<0$ or $\frac{5}{2}<x<\frac{11}{2}$	A1FT	
		4	

Question	Answer	Marks	Guidance
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$(-4)+(-3)=-7 \Rightarrow y+z=-7$	**M1 A1**	Substitutes point.
		4	
7(b)	States point common to both planes e.g. $\begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$.	**B1 FT**	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	**M1 A1FT**	Finds direction of line.
	$\mathbf{r} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	**A1**	OE.
		4	
7(c)	$\left\lvert \dfrac{a+a-7+7}{\sqrt{2}} \right\rvert = \sqrt{2}$	**M1**	Uses correct formula for distance from A to Π_1.
	$a=1$	**A1**	
		2	

Question	Answer	Marks	Guidance
7(d)	$\left\lvert \dfrac{b+b}{\sqrt{2}\sqrt{(1-b)^2+2b^2}} \right\rvert = \frac{1}{2}\sqrt{2}$	**M1 A1**	Uses correct formula.
	$2b=\sqrt{(1-b)^2+2b^2} \Rightarrow b^2+2b-1=0$	**M1**	Solves for *b*.
	$b=-1+\sqrt{2}$	**A1**	CAO
		4	

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5572680205

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Any blank pages are indicated.

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1 Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$ [5]

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(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

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2 The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

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(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

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3 **(a)** Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k, where k is a positive constant. [4]

(b) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(kr+1)(kr-k+1)}$. [1]

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(c) Find also $\displaystyle\sum_{r=n}^{n^2} \frac{1}{(kr+1)(kr-k+1)}$ in terms of n and k. [2]

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4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}$, where a, b, c are real constants and $b \neq 0$.

(a) Show that $\mathbf{M}$ does not represent a rotation about the origin. [2]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{M}$. [5]

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It is given that **M** represents the sequence of two transformations in the $x-y$ plane given by an enlargement, centre the origin, scale factor 5 followed by a shear, x-axis fixed, with $(0, 1)$ mapped to $(5, 1)$.

(c) Find **M**. [3]

(d) The triangle DEF in the $x-y$ plane is transformed by **M** onto triangle PQR.

Given that the area of triangle DEF is $12\ \text{cm}^2$, find the area of triangle PQR. [2]

5 The curve C has polar equation $r^2 = \dfrac{1}{\theta^2 + 1}$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Find the area of the region enclosed by C, the initial line, and the half-line $\theta = \pi$. [4]

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(c) Show that, at the point of C furthest from the initial line,

$$\left(\theta + \frac{1}{\theta}\right)\cot\theta - 1 = 0$$

and verify that this equation has a root between 1.1 and 1.2. [5]

6 The curve C has equation $y = \dfrac{x^2 + 2x - 15}{x - 2}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that C has no stationary points. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - 2x - 15}{x - 2}\right|$. [2]

(e) Find the set of values of x for which $\left|\dfrac{2x^2+4x-30}{x-2}\right| < 15$. [4]

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7 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

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(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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The line l passes through the point A with position vector $a\mathbf{i}+a\mathbf{j}+(a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i}+b\mathbf{j}+b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

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(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

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Additional page

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 12 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

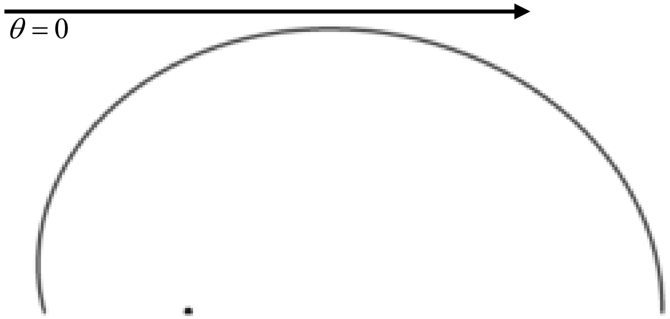
AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$2\mathbf{A}=\begin{pmatrix}6&0\\2&2\end{pmatrix}=\begin{pmatrix}2\times3&0\\3-1&2\end{pmatrix}$ so true when $n=1$.	**B1**	States base case.
	Assume that it is true for $n=k$, so $2\mathbf{A}^k=\begin{pmatrix}2\times3^k&0\\3^k-1&2\end{pmatrix}$.	**B1**	States inductive hypothesis.
	Then $2\mathbf{A}^{k+1}=\begin{pmatrix}2\times3^k&0\\3^k-1&2\end{pmatrix}\begin{pmatrix}3&0\\1&1\end{pmatrix}=\begin{pmatrix}2\times3^{k+1}&0\\3^{k+1}-3+2&2\end{pmatrix}$	**M1A1**	Multiplies $2\mathbf{A}^k$ with $\mathbf{A}$.
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	
1(b)	$\det\mathbf{A}^n=\det\begin{pmatrix}3^n&0\\\frac{1}{2}\left(3^n-1\right)&1\end{pmatrix}=3^n$ Or a multiple of $\mathbf{A}^{-n}=2^{-1}3^{-n}\begin{pmatrix}2&0\\1-3^n&2\times3^n\end{pmatrix}$ seen.	**B1**	
	$\mathbf{A}^{-n}=3^{-n}\begin{pmatrix}1&0\\\frac{1}{2}\left(1-3^n\right)&3^n\end{pmatrix}$	**B1**	OE $\mathbf{A}^{-n}=2^{-1}3^{-n}\begin{pmatrix}2&0\\1-3^n&2\times3^n\end{pmatrix}$
		2	

Question	Answer	Marks	Guidance
2(a)	$(-4)^2 - 2(6)$	**M1**	Uses formula for sum of squares.
	4	**A1**	
		2	
2(b)	$(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha+r)^2 + (\beta+r)^2 + (\gamma+r)^2\right) = \sum_{r=1}^{n}\left(4 + 2(-4)r + 3r^2\right)$	**M1 A1**	Collects like terms and uses $\alpha + \beta + \lambda = -4$ and *their* result from part **(a)**.
	$4n - 4n(n+1) + \frac{1}{2}n(n+1)(2n+1)$	**M1**	Applies formulae from MF19.
	$-4n^2 + \frac{1}{2}n(n+1)(2n+1)$ $= n\left(-4n + \frac{1}{2}(2n^2 + 3n + 1)\right)$ $= n\left(n^2 - \frac{5}{2}n + \frac{1}{2}\right)$	**M1 A1**	Simplifies.
		6	

Question	Answer	Marks	Guidance
3(a)	$\dfrac{1}{(kr+1)(kr-k+1)}=\dfrac{1}{k}\left(\dfrac{1}{k(r-1)+1}-\dfrac{1}{kr+1}\right)$	M1 A1	Finds partial fractions.
	$\displaystyle\sum_{r=1}^{n}\frac{1}{(kr+1)(kr-k+1)}=\frac{1}{k}\left(1-\frac{1}{k+1}+\frac{1}{k+1}-\frac{1}{2k+1}+\ldots+\frac{1}{k(n-1)+1}-\frac{1}{kn+1}\right)$	M1	Writes at least three complete terms, including last.
	$\dfrac{1}{k}\left(1-\dfrac{1}{kn+1}\right)$	A1	OE e.g. $\dfrac{n}{kn+1}$
		4	
3(b)	$\dfrac{1}{k}$	B1	
		1	
3(c)	$\displaystyle\sum_{r=n}^{n^2}\frac{1}{(kr+1)(kr-k+1)}=\sum_{r=1}^{n^2}\frac{1}{(kr+1)(kr-k+1)}-\sum_{r=1}^{n-1}\frac{1}{(kr+1)(kr-k+1)}$	M1	Or applies the method of differences again.
	$\dfrac{1}{k}\left(1-\dfrac{1}{kn^2+1}-\left(1-\dfrac{1}{k(n-1)+1}\right)\right)=\dfrac{1}{k}\left(\dfrac{1}{k(n-1)+1}-\dfrac{1}{kn^2+1}\right)$	A1	OE e.g. $\dfrac{n^2}{kn^2+1}-\dfrac{(n-1)}{k(n-1)+1}$
		2	

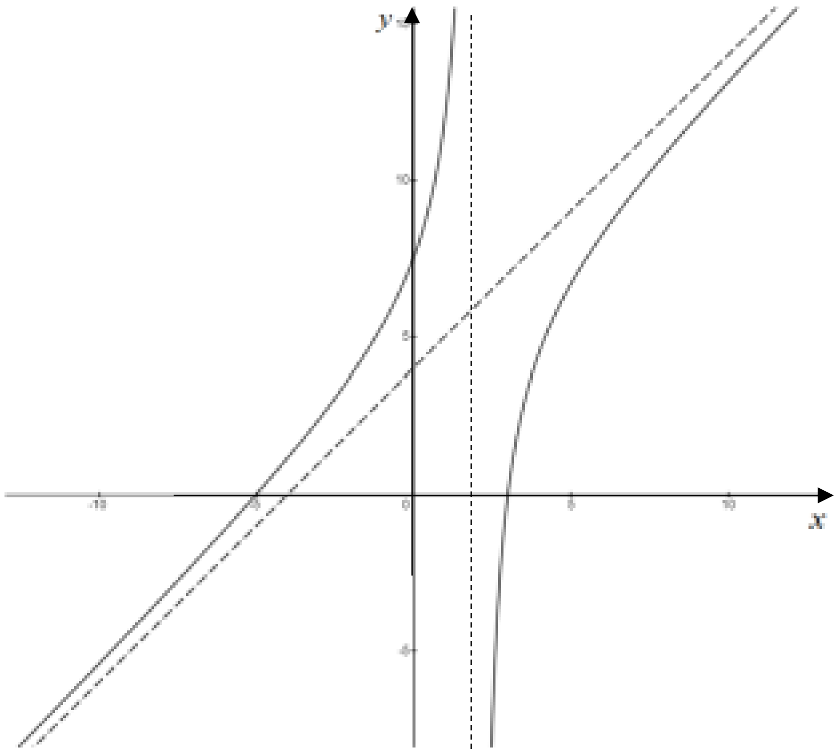
Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	**M1**	Uses correct matrix for rotation.
	$b^2 = -c^2$ which is impossible since b and c are real and $b \neq 0$.	**A1**	AG
		2	
4(b)	$\begin{pmatrix} a & b^2 \\ c^2 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b^2 y \\ c^2 x + ay \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$c^2 x + amx = m\left(ax + b^2 mx\right)$	**M1 A1**	Uses $y = mx$ and $Y = mX$.
	$c^2 + am = ma + b^2 m^2 \Rightarrow c^2 = b^2 m^2$	**A1**	
	$y = \frac{c}{b}x$ and $y = -\frac{c}{b}x$	**A1**	
		5	
4(c)	$\mathbf{M} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$	**M1 A1***	Award M1 if matrices correct but order is wrong.
	$\begin{pmatrix} 5 & 5^2 \\ 0 & 5 \end{pmatrix}$	**DA1**	Dep: previous A1
		3	
4(d)	$12 \times \det \mathbf{M}$	**M1**	Using *their* **M**.
	$300\ \text{cm}^2$	**A1FT**	
		2	

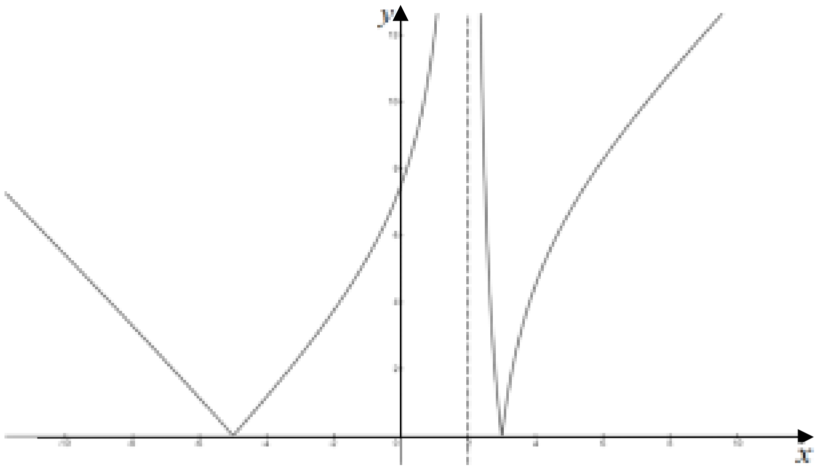
Question	Answer	Marks	Guidance
5(a)	$\theta = 0$ 	**B1***	Correct shape and domain, polar graph with r strictly decreasing but condone if not strictly decreasing close to $\theta = \pi$.
		DB1	Fully correct including shape at $\theta = 0$ correct shape at $\theta = \pi$ correct
	$(1, 0)$	**B1**	Identified as point furthest from the pole and given as coordinates.
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \frac{1}{\theta^2 + 1} \mathrm{d}\theta$	**M1**	Uses correct formula with correct limits.
	$\frac{1}{2}\left[\tan^{-1}\theta\right]_0^{\pi}$	**M1 A1**	Integrates $\frac{1}{\theta^2+1}$.
	$\frac{1}{2}\tan^{-1}\pi = 0.631$	**A1**	
		4	

Question	Answer	Marks	Guidance
5(c)	$y = \dfrac{\sin\theta}{\sqrt{\theta^2+1}}$	**B1**	Uses $y = r\sin\theta$
	$\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \dfrac{\left(\theta^2+1\right)^{\frac{1}{2}}\cos\theta - \theta\left(\theta^2+1\right)^{-\frac{1}{2}}\sin\theta}{\theta^2+1} = 0$	**M1 A1**	Sets derivative equal to zero.
	$\theta \neq 0 \Rightarrow \left(\theta + \dfrac{1}{\theta}\right)\cot\theta - 1 = 0$	**A1**	AG.
	$\left(1.1 + \dfrac{1}{1.1}\right)\cot 1.1 - 1 = 0.02\ldots$ and $\left(1.2 + \dfrac{1}{1.2}\right)\cot 1.2 - 1 = -0.209\ldots$	**B1**	Shows sign change (1sf or better).
		5	

Question	Answer	Marks	Guidance
6(a)	$x = 2$	**B1**	States vertical asymptote.
	$x^2 + 2x - 15 = (x-2)(x+4) - 7 \Rightarrow y = x + 4$	**M1 A1**	Finds oblique asymptote.
		3	

Question	Answer	Marks	Guidance
6(b)	$\dfrac{dy}{dx}=\dfrac{(x-2)(2x+2)-(x^2+2x-15)}{(x-2)^2}$	M1	Differentiates.
	$x^2-4x+11=0$ $\left(\text{or } \dfrac{dy}{dx}=1+\dfrac{7}{(x-2)^2}\right)$	A1	Forms quadratic equation or simplifies $\dfrac{dy}{dx}$.
	$4^2-4(1)(11)=-28<0$ (or $y'>0$) There are no turning points.	A1	Correct conclusion.
		3	

Question	Answer	Marks	Guidance
6(c)		B1	Axes and asymptotes.
		B1	Branches correct.
	$(0,7.5),(-5,0),(3,0)$	B1	States coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
6(d)		B1FT	FT from sketch in **(c)**.
		B1	Correct shape at infinity and on x axis.
		2	
6(e)	$\frac{x^2+2x-15}{x-2}=\frac{15}{2}$ or $\frac{x^2+2x-15}{x-2}=-\frac{15}{2}$ $x^2-\frac{11}{2}x=0$ or $x^2+\frac{19}{2}x-30=0$	M2	Finds critical points, award M1 for each case.
	$x=0,\frac{11}{2}$ or $x=-12,\frac{5}{2}$	A1	
	$-12<x<0$ or $\frac{5}{2}<x<\frac{11}{2}$	A1FT	
		4	

Question	Answer	Marks	Guidance
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$(-4)+(-3)=-7 \Rightarrow y+z=-7$	**M1 A1**	Substitutes point.
		4	
7(b)	States point common to both planes e.g. $\begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$.	**B1 FT**	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	**M1 A1FT**	Finds direction of line.
	$\mathbf{r} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}$	**A1**	OE.
		4	
7(c)	$\left\lvert \dfrac{a+a-7+7}{\sqrt{2}} \right\rvert = \sqrt{2}$	**M1**	Uses correct formula for distance from A to Π_1.
	$a=1$	**A1**	
		2	

Question	Answer	Marks	Guidance
7(d)	$\left\lvert \dfrac{b+b}{\sqrt{2}\sqrt{(1-b)^2+2b^2}} \right\rvert = \frac{1}{2}\sqrt{2}$	M1 A1	Uses correct formula.
	$2b = \sqrt{(1-b)^2 + 2b^2} \Rightarrow b^2 + 2b - 1 = 0$	M1	Solves for b.
	$b = -1 + \sqrt{2}$	A1	CAO
		4	

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0834557711

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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1 Prove by mathematical induction that, for all positive integers n, $5^{3n} + 32^n - 33$ is divisible by 31. [6]

2 **(a)** Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(6r^2+6r-5) = an^3+bn^2+cn,$$

where a, b and c are integers to be determined. [2]

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(b) Use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}$ in terms of n. [4]

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(c) Find also $\displaystyle\sum_{r=n+1}^{2n} \frac{6r^2+6r-5}{r^2+r}$ in terms of n. [2]

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3 The equation $x^4 - x^2 + 2x + 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [3]

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(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

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4 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where $0 < \theta < \pi$ and k is a non-zero constant.

The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations, one of which is a shear.

(a) Describe fully the other transformation and state the order in which the transformations are applied. [3]

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(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

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(c) Find, in terms of k, the value of $\tan\theta$ for which $\mathbf{M}-\mathbf{I}$ is singular. [5]

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(d) Given that $k = 2\sqrt{3}$ and $\theta = \frac{1}{3}\pi$, show that the invariant points of the transformation represented by **M** lie on the line $3y + \sqrt{3}x = 0$. [4]

5 **(a)** Show that the curve with Cartesian equation

$$x^2 - y^2 = a,$$

where a is a positive constant, has polar equation $r^2 = a\sec 2\theta$. [3]

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The curve C has polar equation $r^2 = a\sec 2\theta$, where a is a positive constant, for $0 \leqslant \theta < \frac{1}{4}\pi$.

(b) Sketch C and state the minimum distance of C from the pole. [3]

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(c) Find, in terms of a, the exact value of the area of the region enclosed by C, the initial line, and the half-line $\theta = \frac{1}{12}\pi$. [You may use any result from the list of formulae (MF19) without proof.] [4]

6 The points A, B, C have position vectors

$$\mathbf{i}+\mathbf{j}, \quad -\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

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(b) Find the perpendicular distance from O to the plane ABC. [2]

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(c) Find a vector equation of the common perpendicular to the lines OC and AB. [8]

7 The curve C has equation $y = \dfrac{x^2 + 2x + 1}{x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of the turning points on C. [3]

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(c) Sketch C. [3]

(d) Sketch the curves with equations $y = \left|\frac{x^2+2x+1}{x-3}\right|$ and $y^2 = \frac{x^2+2x+1}{x-3}$ on a single diagram, clearly identifying each curve. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 13 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$5^3+32-33=124$ is divisible by 31.	B1	Checks base case.
	Assume that $5^{3k}+32^k-33$ is divisible by 31 for some positive integer k.	B1	States inductive hypothesis.
	$5^{3k+3}+32^{k+1}-33=(124+1)5^{3k}+(31+1)32^k-33$	M1 A1	Separates $5^{3k}+32^k-33$ or considers difference.
	is divisible by 31 because $124\times5^{3k}+31\times32^k$ is divisible by 31.	A1	
	Hence, by induction, true for every positive integer n.	A1	
		6	

Question	Answer	Marks	Guidance
2(a)	$6\left(\frac{1}{6}n(n+1)(2n+1)\right)+6\left(\frac{1}{2}n(n+1)\right)[-5n]$	M1	Substitutes formulae for $\sum r^2$ and $\sum r$.
	$2n^3+6n^2-n$	A1	
		2	

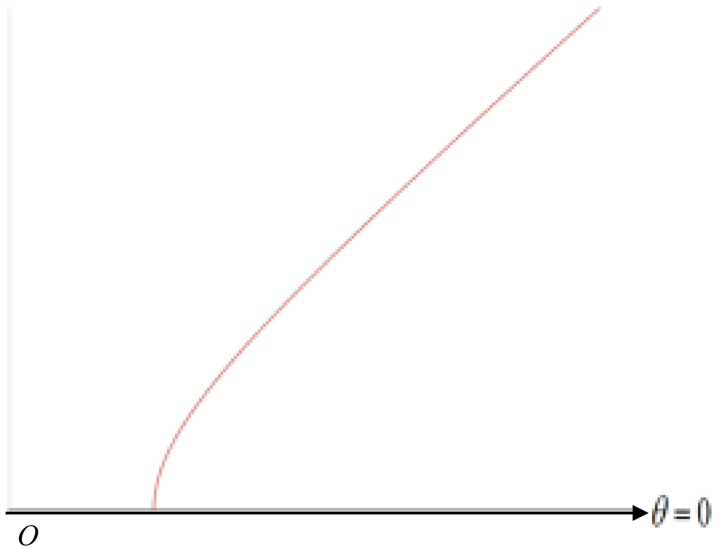
Question	Answer	Marks	Guidance
2(b)	$\frac{6r^2+6r-5}{r^2+r}=6-\frac{5}{r(r+1)}$	**B1**	Divides by denominator.
	$\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$	**B1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{r(r+1)}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}$	**M1**	Writes at least three complete terms, including the last term, to show cancelation.
	$\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}=6n-5+\frac{5}{n+1}$	**A1**	OE e.g. $\frac{6n^2+n}{n+1}$
		4	
2(c)	$\sum_{r=1}^{2n}\frac{6r^2+6r-5}{r^2+r}-\sum_{r=1}^{n}\frac{6r^2+6r-5}{r^2+r}$ $=12n-5+\frac{5}{2n+1}-6n+5-\frac{5}{n+1}$	**M1**	Or uses method of differences again.
	$12n-5+\frac{5}{2n+1}-6n+5-\frac{5}{n+1}=6n+\frac{5}{2n+1}-\frac{5}{n+1}$	**A1**	Or $6n-\frac{5n}{(n+1)(2n+1)}$. OE, like terms collected.
		2	

Question	Answer	Marks	Guidance
3(a)	$y = x^2$ so, $x = y^{\frac{1}{2}}$	B1	Correct substitution.
	$y^2 - y + 2y^{\frac{1}{2}} + 5 = 0$ so, $\left(2y^{\frac{1}{2}}\right)^2 = \left(-y^2 + y - 5\right)^2$	M1	Obtains an equation which eliminates radicals.
	$y^4 - 2y^3 + 11y^2 - 14y + 25 = 0$	A1	
	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 2$	B1FT	
		4	
3(b)	$\alpha^2\beta^2\gamma^2\delta^2 = 25$	B1FT	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	M1	Relates to coefficients.
	$\frac{14}{25}$	A1	CAO
		3	
3(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = \left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right)^2 - 2\left(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2\right) = 2^2 - 2(11)$	M1	Uses formula for sum of squares or uses original equation.
	-18	A1	
		2	

Question	Answer	Marks	Guidance
4(a)	Rotation [anticlockwise]	**B1**	
	about the origin through angle 2θ .	**B1**	
	Shear [in the *x*-direction] followed by a rotation [anticlockwise about the origin through angle 2θ].	**B1**	
		3	
4(b)	$\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & -k\\0 & 1\end{pmatrix}$, $\begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}^{-1}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$	**B1**	
	$\mathbf{M}^{-1}=\begin{pmatrix}1 & -k\\0 & 1\end{pmatrix}\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$	**B1**	Correct order
		2	
4(c)	$\mathbf{M}-\mathbf{I}=\begin{pmatrix}\cos 2\theta-1 & k\cos 2\theta-\sin 2\theta\\ \sin 2\theta & k\sin 2\theta+\cos 2\theta-1\end{pmatrix}$	**B1**	
	$(\cos 2\theta-1)(k\sin 2\theta+\cos 2\theta-1)-k\sin 2\theta\cos 2\theta+\sin^2 2\theta[=0]$	**M1**	Evaluates $\det(\mathbf{M}-\mathbf{I})$
	$2-2\cos 2\theta-k\sin 2\theta=0$	**A1**	Brackets removed correctly and = 0
	$4\sin^2\theta=2k\sin\theta\cos\theta$	**M1**	Uses $1-\cos 2\theta=2\sin^2\theta$ and $\sin 2\theta=2\sin\theta\cos\theta$ or all necessary double angle formulae.
	$\tan\theta=\frac{1}{2}k$	**A1**	
		5	

Question	Answer	Marks	Guidance
4(d)	$\mathbf{M}=\begin{pmatrix}-\frac{1}{2} & -\frac{3}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{5}{2}\end{pmatrix}$.	B1	
	$\begin{pmatrix}-\frac{1}{2} & -\frac{3}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{5}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}x-\frac{3}{2}\sqrt{3}y\\ \frac{1}{2}\sqrt{3}x+\frac{5}{2}y\end{pmatrix}$	B1FT	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$
	$-\frac{1}{2}x-\frac{3}{2}\sqrt{3}y=x\left[\Rightarrow-\frac{3}{2}x-\frac{3}{2}\sqrt{3}y=0\Rightarrow x+\sqrt{3}y=0\right]$ and $\frac{1}{2}\sqrt{3}x+\frac{5}{2}y=y\left[\Rightarrow\frac{1}{2}\sqrt{3}x+\frac{3}{2}y=0\Rightarrow\sqrt{3}x+3y=0\right]$	M1	Sets $\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$
	$\sqrt{3}x+3y=0$	A1	AG.
		4	

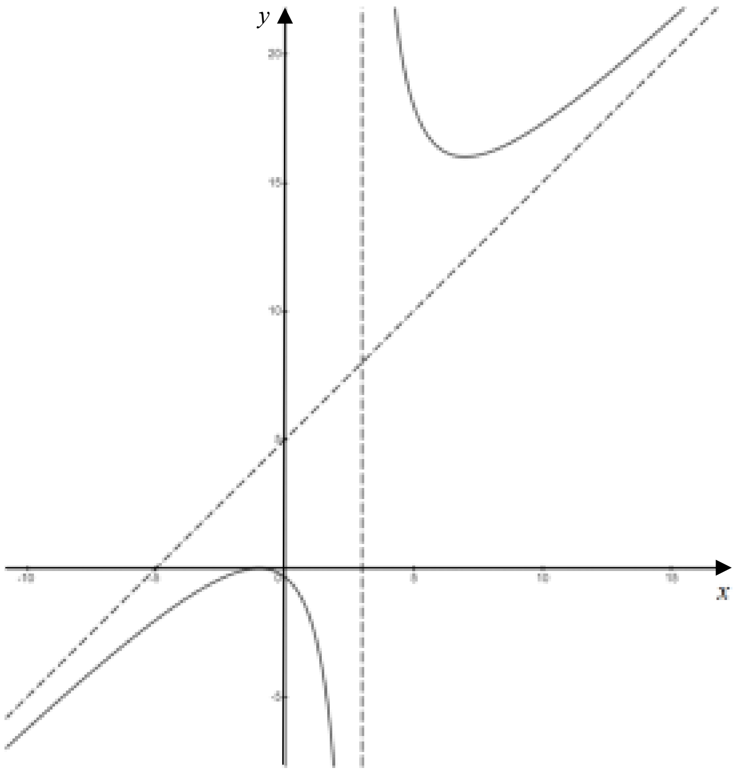
Question	Answer	Marks	Guidance
5(a)	$r^2\left(\cos^2\theta-\sin^2\theta\right)=a$	B1	Uses $x=r\cos\theta$ and/or $y=r\sin\theta$.
	$r^2\cos 2\theta=a$	M1	Applies relevant double angle formulae.
	$r^2=a\sec 2\theta$	A1	AG.
		3	

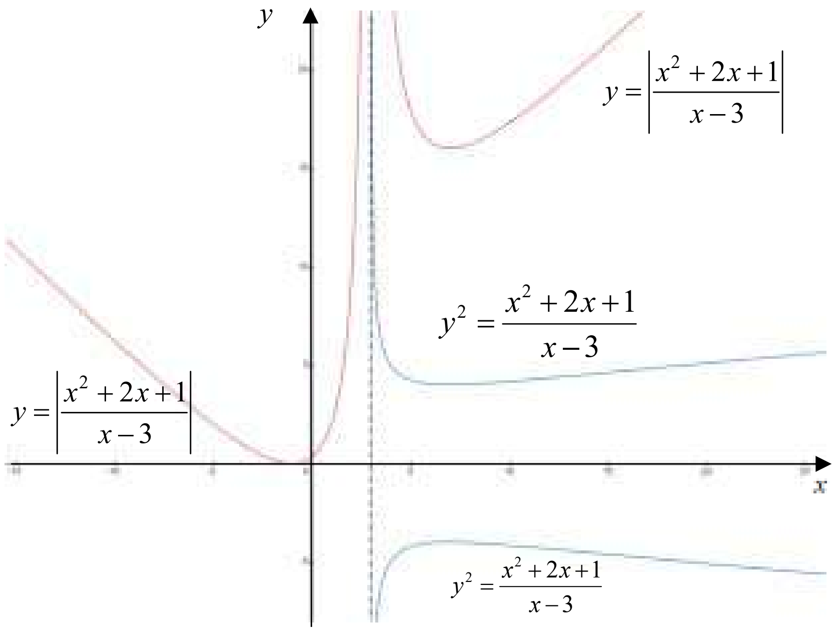
Question	Answer	Marks	Guidance
5(b)		B1*	Initial line drawn. Correct domain and position, r strictly increasing.
		dB1	Also sloping to right, concave on opposite side to pole, correct gradient when $\theta = 0$ and $\theta \to \pi/4$.
	$\sqrt{a}$	B1	
		3	
5(c)	$\frac{1}{2}a\int_0^{\frac{1}{12}\pi} \sec 2\theta \,\mathrm{d}\theta$	M1	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{4}a\left[\ln\tan\left(\theta + \frac{1}{4}\pi\right)\right]_0^{\frac{1}{12}\pi}$ or $\frac{1}{4}a\left[\ln\left(\tan 2\theta + \sec 2\theta\right)\right]_0^{\frac{1}{12}\pi}$	M1 A1	Integrates.
	$\frac{1}{4}a\ln\sqrt{3} = \frac{1}{8}a\ln 3$	A1	
		4	

Question	Answer	Marks	Guidance
6(a)	$\overrightarrow{AB} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ $\overrightarrow{AC} = -3\mathbf{i} + 3\mathbf{k}$ $\overrightarrow{BC} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$	**B1**	Finds direction vectors of **two** lines in the plane.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -3 & 0 & 3 \end{vmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$	**M1 A1FT**	Finds normal to the plane *ABC*.
	$1(1) - 2(1) + 1(0) = -1 \Rightarrow x - 2y + z = -1$	**M1 A1**	Substitutes point. CAO
		5	
6(b)	$\dfrac{1}{\sqrt{1^2 + 2^2 + 1^2}} = \dfrac{1}{\sqrt{6}} = 0.408$	**M1 A1FT**	Divides by magnitude of normal vector. FT *their* **(a)**.
		2	

Question	Answer	Marks	Guidance
6(c)	$\overrightarrow{OP}=\begin{pmatrix}-2\lambda\\ \lambda\\ 3\lambda\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}1-2\mu\\ 1+\mu\\ 4\mu\end{pmatrix} \Rightarrow \overrightarrow{PQ}=\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$, where *P* is a point on *OC* and *Q* is a point on *AB*.
	$\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}\bullet\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}=0$	**M1***	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero.
	$17\mu-14\lambda=1$	**dM1**	Deduces one equation.
	$\begin{pmatrix}1-2\mu+2\lambda\\ 1+\mu-\lambda\\ 4\mu-3\lambda\end{pmatrix}\bullet\begin{pmatrix}-2\\ 1\\ 4\end{pmatrix}=0 \Rightarrow 21\mu-17\lambda=1$	**dM1**	Deduces second equation.
	$\lambda=-\frac{4}{5} \Rightarrow \overrightarrow{OP}=-\frac{4}{5}\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}$	**dM1 A1**	Solves for λ or μ and substitutes into $\overrightarrow{OP}$.
	$\mathbf{r}=-\frac{4}{5}\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}+k\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$	**A1**	OE
		8	

Question	Answer	Marks	Guidance
7(a)	$x = 3$	**B1**	States vertical asymptote.
	$y = \dfrac{(x-3)(x+5)+16}{x-3} = x+5+\dfrac{16}{x-3}$	**M1**	Finds oblique asymptote.
	$y = x+5$	**A1**	
		3	
7(b)	$\dfrac{dy}{dx} = 1 - \dfrac{16}{(x-3)^2} = 0 \Rightarrow (x-3)^2 = 16$	**M1**	Differentiates and sets equal to zero.
	$x = -1, 7$	**A1**	Finds x-coordinates
	$(-1,0), (7,16)$	**A1**	States coordinates of turning points.
		3	

Question	Answer	Marks	Guidance
7(c)		B1FT	Axes and labelled asymptotes.
		B1	Upper branch correct.
		B1	Lower branch correct and no additional branches.
		3	

Question	Answer	Marks	Guidance
7(d)		B1FT	Clear labels, axes and their vertical asymptote.
		B1FT	$y=\left\lvert\frac{x^2+2x+1}{x-3}\right\rvert$ correct, FT from their sketch in **(c)**.
		B1	Upper branch of $y^2=\frac{x^2+2x+1}{x-3}$ (positive square root).
		B1FT	Lower branch of $y^2=\frac{x^2+2x+1}{x-3}$ (negative square root). FT from previous mark.
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

2004404347

FURTHER MATHEMATICS **9231/21**

Paper 2 Further Pure Mathematics 2 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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1 **(a)** Show that the system of equations

$$\begin{aligned} x+2y+3z &= 1, \\ 4x+5y+6z &= 1, \\ 7x+8y+9z &= 1, \end{aligned}$$

does not have a unique solution. [2]

(b) Show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

2 Use the substitution $z = x + y$ to find the solution of the differential equation

$$\frac{dy}{dx} = \frac{1 + 3x + 3y}{3x + 3y - 1}$$

for which $y = 0$ when $x = 1$. Give your answer in the form $a\ln(x+y) + b(x-y) + c = 0$, where a, b and c are constants to be determined. [7]

3 **(a)** By considering the binomial expansion of $(z+z^{-1})^4$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [5]

(b) Use the substitution $x = \sin\theta$ to find the exact value of $\int_0^{\frac{1}{2}} (1-x^2)^{\frac{3}{2}}\,\mathrm{d}x$. [3]

4 The integral I_n is defined by $I_n = \int_0^1 (1+x^5)^n \, dx$.

(a) By considering $\frac{d}{dx}\left(x(1+x^5)^n\right)$, or otherwise, show that

$$(5n+1)I_n = 2^n + 5nI_{n-1}.$$ [5]

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(b) Find the exact value of I_3. [4]

5 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 18 & 5 & -11 \\ 8 & 6 & -4 \\ 32 & 10 & -20 \end{pmatrix}.$$

(a) Show that the characteristic equation of **A** is $\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$ and hence find the eigenvalues of **A**. [4]

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(b) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [6]

6 Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 36x = 37\sin t,$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [11]

7 **(a)** Use the substitution $u = x^2 - 1$ to find $\int \frac{x}{\sqrt{x^2-1}}\,dx$. [3]

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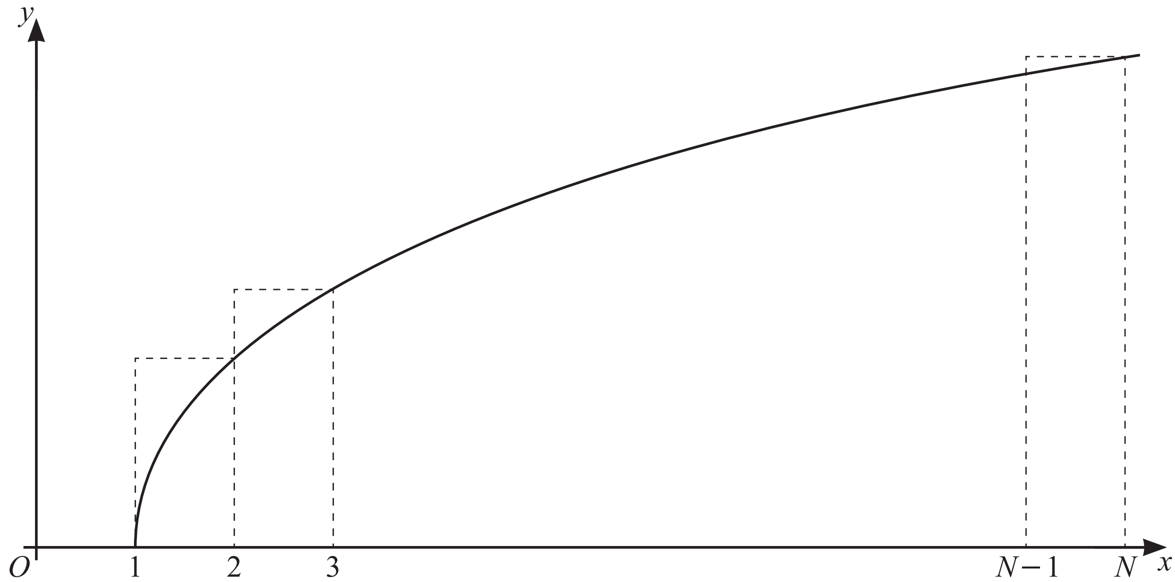

The diagram shows the curve with equation $y = \cosh^{-1} x$ together with a set of $(N-1)$ rectangles of unit width.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2-1}\right) > N\ln\left(N + \sqrt{N^2-1}\right) - \sqrt{N^2-1}.$$ [5]

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(c) Use a similar method to find, in terms of N, an upper bound for $\sum_{r=2}^{N} \ln\left(r+\sqrt{r^2-1}\right)$. [3]

8 **(a)** Starting from the definitions of sech and tanh in terms of exponentials, prove that

$$1-\operatorname{sech}^2 t = \tanh^2 t.$$ [3]

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The curve C has parametric equations

$$x = \tfrac{1}{2}\tanh^2 t + \ln \operatorname{sech} t, \qquad y = 1 + \tanh^4 t, \qquad \text{for } t > 0.$$

(b) Show that $\dfrac{dy}{dx} = -4\operatorname{sech}^2 t$. [5]

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(c) Find the coordinates of the point on C with $\frac{d^2y}{dx^2} = -\frac{9}{2}$, giving your answer in the form $(a + \ln b, c)$ where a, b and c are rational numbers. [6]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/21**

Paper 2 Further Pure Mathematics 21 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (5\times9-8\times6)-2(4\times9-7\times6)+3(4\times8-7\times5)$ $=-3+12-9=0$	**M1 A1**	Shows that determinant is zero or row operations to obtain e.g. $x + 2y + 3z = 1,$ $y + 2z = 1,$
		2	
1(b)	$x + 2y + 3z = 1,$ $4x + 5y + 6z = 1, \Rightarrow z = \frac{1}{3}(1-x-2y) \Rightarrow y = -2x-1$ $7x + 8y + 9z = 1,$	**M1 A1**	Uses *all three* equations to reduce to one equation with two unknowns. Reducing to two equations scores M1 A0.
	The three planes form a sheaf.	**B1**	Accept clear sketch or the three planes intersect along a common line.
		3	

Question	Answer	Marks	Guidance
2	$\frac{dy}{dx} = \frac{dz}{dx} - 1$	**B1**	
	$\frac{dz}{dx} - 1 = \frac{1+3z}{3z-1}$	**M1**	Substitutes and derives first order separable equation.
	$\frac{dz}{dx} = \frac{1+3z}{3z-1} + 1 = \frac{6z}{3z-1}$	**A1**	
	$\int \frac{1}{2} - \frac{1}{6}z^{-1}dz = \int 1\,dx$	**M1**	Separates variables and integrates both sides.
	$\frac{1}{2}z - \frac{1}{6}\ln z = x + C \Rightarrow -\frac{1}{2}x + \frac{1}{2}y - \frac{1}{6}\ln(x+y) = C$	**A1**	
	$C = -\frac{1}{2}$	**M1**	Substitutes initial conditions into their expression.
	$\frac{1}{6}\ln(x+y) + \frac{1}{2}(x-y) - \frac{1}{2} = 0$	**A1**	OE.
		7	

Question	Answer	Marks	Guidance
3(a)	$z+z^{-1}=2\cos\theta$	B1	Use of $z+z^{-1}=2\cos\theta$.
	$\left(z+z^{-1}\right)^4=\left(z^4+z^{-4}\right)+4\left(z^2+z^{-2}\right)+6$	M1 A1	Expands and groups. M1 A0 for no clear grouping. Correct substitution of $z^n=\cos n\theta+\mathrm{i}\sin n\theta$ for each term scores M1 A1.
	$(2\cos\theta)^4=2\cos 4\theta+4(2\cos 2\theta)+6$	M1	Substitutes $z^n+z^{-n}=2\cos n\theta$. If $z^n=\cos n\theta+\mathrm{i}\sin n\theta$ used then must cancel sin.
	$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$	A1	AG. SC B1 expands $(\cos\theta+\mathrm{i}\sin\theta)^4$ and uses trigonometric identities.
		5	
3(b)	$\int_0^{\frac{1}{2}}\left(1-x^2\right)^{\frac{3}{2}}\mathrm{d}x=\int_0^{\frac{1}{6}\pi}\cos^4\theta\,\mathrm{d}\theta=\frac{1}{8}\left[\frac{1}{4}\sin 4\theta+2\sin 2\theta+3\theta\right]_0^{\frac{1}{6}\pi}$	M1 A1	Applies substitution (M1) gets to $\int\cos^4\theta\,d\theta$, changes limits, integration correct (A1).
	$\frac{1}{16}\left(\frac{9}{4}\sqrt{3}+\pi\right)$	A1	
		3	

Question	Answer	Marks	Guidance
4(a)	$\frac{d}{dx}\left(x\left(1+x^5\right)^n\right)=5nx^5\left(1+x^5\right)^{n-1}+\left(1+x^5\right)^n$	M1 A1	Uses the product rule to differentiate.
	$5n\left(1+x^5-1\right)\left(1+x^5\right)^{n-1}+\left(1+x^5\right)^n$	M1*	Uses $x^5=x^5+1-1$.
	$\left[x\left(1+x^5\right)^n\right]_0^1=5nI_n-5nI_{n-1}+I_n$	DM1	Integrates both sides using the limits given. Requires previous method mark.
	$2^n=(5n+1)I_n-5nI_{n-1}\Rightarrow(5n+1)I_n=2^n+5nI_{n-1}$	A1	Substitutes limits and rearranges. AG.
	Alternative method for question 4(a)		
	$I_n=\int_0^1\left(1+x^5\right)^n dx=\left[x\left(1+x^5\right)^n\right]_0^1-5n\int_0^1 x^5\left(1+x^5\right)^{n-1} dx$	M1 A1	Integrates by parts.
	$I_n=\left[x\left(1+x^5\right)^n\right]_0^1-5n\int_0^1\left(1+x^5\right)^n dx+5n\int_0^1\left(1+x^5\right)^{n-1} dx$	M1*	Uses $x^5=x^5+1-1$.
	$I_n=\left[x\left(1+x^5\right)^n\right]_0^1-5nI_n+5nI_{n-1}$	DM1	Forms recursive formula. Requires previous method mark.
	$(5n+1)I_n=2^n+5nI_{n-1}$	A1	Substitutes limits and rearranges. AG.
		5	

Question	Answer	Marks	Guidance
4(b)	$I_1 = \left[x + \frac{1}{6}x^6\right]_0^1 = \frac{7}{6}$ or $I_0 = 1$	**B1**	
	$11I_2 = 2^2 + 10I_1 \Rightarrow I_2 = \frac{47}{33}$	**M1 A1**	Applies reduction formula.
	$16I_3 = 2^3 + 15I_2 \Rightarrow I_3 = \frac{323}{176}$	**A1**	
		4	

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} 18-\lambda & 5 & -11 \\ 8 & 6-\lambda & -4 \\ 32 & 10 & -20-\lambda \end{vmatrix} = 0$	**M1**	Sets determinant equal to zero.
	$(18-\lambda)\begin{vmatrix} 6-\lambda & -4 \\ 10 & -20-\lambda \end{vmatrix} - 5\begin{vmatrix} 8 & -4 \\ 32 & -20-\lambda \end{vmatrix} - 11\begin{vmatrix} 8 & 6-\lambda \\ 32 & 10 \end{vmatrix}$	**A1**	Expands determinant. Can use other rows/columns.
	$\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$	**A1**	AG
	$(\lambda-2)(\lambda+4)(\lambda-6) = 0$ leading to $\lambda = 2, -4, 6$	**B1**	
		4	

Question	Answer	Marks	Guidance
5(b)	$\lambda = 2: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & 5 & -11 \\ 8 & 4 & -4 \end{vmatrix} = \begin{pmatrix} 24 \\ -24 \\ 24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors. Must attempt to solve equations for M1.
	$\lambda = -4: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 10 & -4 \\ 32 & 10 & -16 \end{vmatrix} = \begin{pmatrix} -120 \\ 0 \\ -240 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\lambda = 6: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 5 & -11 \\ 8 & 0 & -4 \end{vmatrix} = \begin{pmatrix} -20 \\ -40 \\ -40 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	**A1 A1**	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 32 & 0 & 0 \\ 0 & -1024 & 0 \\ 0 & 0 & 7776 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns. M0 if a column of zeros appears in **P**.
		6	

Question	Answer	Marks	Guidance
6	$m^2-12m+36=0$	**M1**	Auxiliary equation.
	$x=e^{6t}\left(At+B\right)$	**A1**	Complementary function. Allow "$x=$" missing.
	$x=p\sin t+q\cos t\Rightarrow x'=p\cos t-q\sin t\Rightarrow x''=-p\sin t-q\cos t$	**M1 A1**	Particular integral and its derivatives.
	$-p+12q+36p=37\quad -q-12p+36q=0$	**M1**	Substitutes and equates coefficients. Must have two unknowns.
	$p=\frac{35}{37}\quad q=\frac{12}{37}$	**A1**	
	$x=e^{6t}\left(At+B\right)+\frac{35}{37}\sin t+\frac{12}{37}\cos t$	**A1 FT**	Must have "$x=$". FT on CF.
	$x'=Ae^{6t}+6e^{6t}\left(At+B\right)+\frac{35}{37}\cos t-\frac{12}{37}\sin t$	**M1***	Differentiates using product rule. Must have their PI differentiated.
	$B=-\frac{12}{37}\quad A+6B+\frac{35}{37}=0\Rightarrow A=1$	**DM1 A1**	Forms simultaneous equations using initial conditions.
	$x=e^{6t}\left(t-\frac{12}{37}\right)+\frac{35}{37}\sin t+\frac{12}{37}\cos t$	**A1**	Must have "$x=$".
		11	

Question	Answer	Marks	Guidance
7(a)	$\int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$	**M1 A1**	Applies given substitution.
	$\sqrt{x^2-1}+C$	**A1**	Allow with "$+C$" missing. Answer must be in terms of x.
		3	
7(b)	$\cosh^{-1} r = \ln\left(r+\sqrt{r^2-1}\right)$	**B1**	
	$\cosh^{-1} 2 + \cosh^{-1} 3 + \ldots + \cosh^{-1} N$	**M1**	Forms sum of the areas of the rectangles.
	$> \int_1^N \cosh^{-1} x\,dx$	**M1**	Compares with integral with correct limits.
	$\int_1^N \cosh^{-1} x\,dx = \left[x\cosh^{-1} x\right]_1^N - \int_1^N \frac{x}{\sqrt{x^2-1}}\,dx$	**A1**	Evaluates integral.
	$\sum_{r=2}^{N} \ln\left(r+\sqrt{r^2-1}\right) > N\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$	**A1**	AG.
		5	

Question	Answer	Marks	Guidance
7(c)	$(\cosh^{-1}1)+\cosh^{-1}2+\ldots+\cosh^{-1}(N-1)<\int_1^N \cosh^{-1}x\,\mathrm{d}x$ (or $<\int_2^N \cosh^{-1}x\,\mathrm{d}x$)	M1 A1	Compares with integral with correct limits.
	$\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$ or $\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	Adds $\ln\left(N+\sqrt{N^2-1}\right)$ to both sides.
	Alternative method for question 7(c)		
	$(\cosh^{-1}1)+\cosh^{-1}2+\ldots+\cosh^{-1}(N-1)+\cosh^{-1}N$ $<\int_1^{N+1}\cosh^{-1}x\,\mathrm{d}x=\left[x\ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}\right]_1^{N+1}$ (or $<\int_2^{N+1}\cosh^{-1}x\,\mathrm{d}x$)	M1 A1	Compares with integral with correct limits.
	$\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}$ or $<(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	
	Second alternative method for question 7(c)		
	$\cosh^{-1}(1+1)+\cosh^{-1}(2+1)+\ldots+\cosh^{-1}(N-1+1)$ $<\int_1^{N}\cosh^{-1}(x+1)\,\mathrm{d}x=\left[(x+1)\ln\left(x+1+\sqrt{x^2+2x}\right)-\sqrt{x^2+2x}\right]_1^{N}$	M1 A1	Compares with integral with correct limits.
	$=(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	
		3	

Question	Answer	Marks	Guidance
8(a)	$\operatorname{sech} t = \dfrac{2}{e^t + e^{-t}}$ $\tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$	**B1**	
	$1 - \left(\dfrac{2}{e^t + e^{-t}}\right)^2 = \dfrac{\left(e^t + e^{-t}\right)^2 - 4}{\left(e^t + e^{-t}\right)^2} = \dfrac{e^{2t} + 2 + e^{-2t} - 4}{\left(e^t + e^{-t}\right)^2} = \dfrac{\left(e^t - e^{-t}\right)^2}{\left(e^t + e^{-t}\right)^2}$	**M1 A1**	Expands, gets to $\dfrac{e^{2t} + 2 + e^{-2t} - 4}{\left(e^t + e^{-t}\right)^2}$ for M1, AG.
		3	
8(b)	$\dfrac{dy}{dt} = 4\tanh^3 t \operatorname{sech}^2 t$	**B1**	
	$\dfrac{dx}{dt} = \tanh t \operatorname{sech}^2 t - \tanh t \quad \left(= \tanh t\left(\operatorname{sech}^2 t - 1\right) = -\tanh^3 t\right)$	**M1 A1**	M1 sensible attempt at derivative of x.
	$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{4\tanh^3 t \operatorname{sech}^2 t}{-\tanh^3 t} = -4\operatorname{sech}^2 t$	**M1 A1**	Applies chain rule, must substitute their $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ for M1, AG.
		5	
8(c)	$\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx} = \dfrac{8\operatorname{sech}^2 t \tanh t}{-\tanh^3 t} = -8\dfrac{\operatorname{sech}^2 t}{\tanh^2 t}$	**M1 A1**	Finds $\dfrac{d^2y}{dx^2}$. For M1 $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = c\operatorname{sech}^2 t \tanh t$. AEF. Accept $-8\operatorname{cosech}^2 t$.
	$-8\dfrac{\operatorname{sech}^2 t}{\tanh^2 t} = -\tfrac{9}{2} \Rightarrow 8\left(1 - \tanh^2 t\right) = \tfrac{9}{2}\tanh^2 t \Rightarrow \tanh^2 t = \tfrac{16}{25}$	**M1 A1**	Sets equal to $-\tfrac{9}{2}$, uses identity from **(a)** or equivalent. Accept $\sinh^2 t = \tfrac{16}{9}$ or $\cosh^2 t = \tfrac{25}{9}$.
	$(x, y) = \left(\tfrac{8}{25} + \ln\tfrac{3}{5}, \tfrac{881}{625}\right)$	**A1 A1**	A1 for each correct coordinate. $t = \ln 3$
		6	

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3417140718

FURTHER MATHEMATICS **9231/22**

Paper 2 Further Pure Mathematics 2 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (CJ/SG) 311417/1

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1 **(a)** Show that the system of equations

$$\begin{aligned} x + 2y + 3z &= 1, \\ 4x + 5y + 6z &= 1, \\ 7x + 8y + 9z &= 1, \end{aligned}$$

does not have a unique solution. [2]

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(b) Show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

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2 Use the substitution $z = x + y$ to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + 3x + 3y}{3x + 3y - 1}$$

for which $y = 0$ when $x = 1$. Give your answer in the form $a\ln(x+y) + b(x-y) + c = 0$, where a, b and c are constants to be determined. [7]

3 **(a)** By considering the binomial expansion of $(z+z^{-1})^4$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [5]

(b) Use the substitution $x = \sin\theta$ to find the exact value of $\int_0^{\frac{1}{2}} (1-x^2)^{\frac{3}{2}}\,\mathrm{d}x$. [3]

4 The integral I_n is defined by $I_n = \int_0^1 (1+x^5)^n \, dx$.

(a) By considering $\frac{d}{dx}\left(x(1+x^5)^n\right)$, or otherwise, show that

$$(5n+1)I_n = 2^n + 5nI_{n-1}.$$ [5]

(b) Find the exact value of I_3. [4]

5 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 18 & 5 & -11 \\ 8 & 6 & -4 \\ 32 & 10 & -20 \end{pmatrix}.$$

(a) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$ and hence find the eigenvalues of $\mathbf{A}$. [4]

(b) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [6]

6 Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 36x = 37\sin t,$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [11]

7 **(a)** Use the substitution $u = x^2 - 1$ to find $\int \frac{x}{\sqrt{x^2 - 1}}\,dx$. [3]

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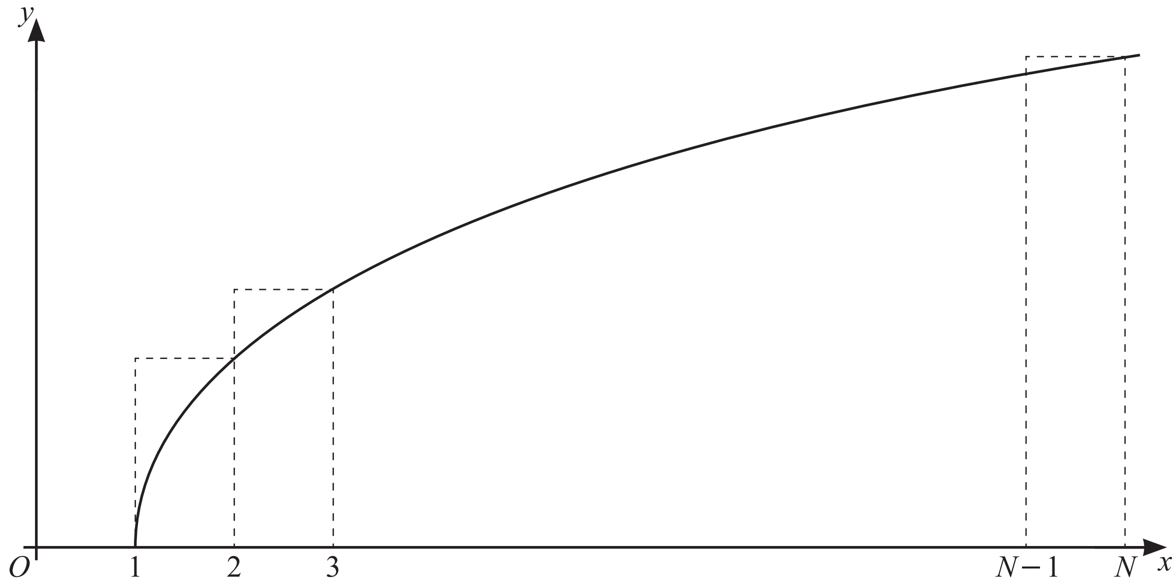

The diagram shows the curve with equation $y = \cosh^{-1}x$ together with a set of $(N-1)$ rectangles of unit width.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2 - 1}\right) > N\ln\left(N + \sqrt{N^2 - 1}\right) - \sqrt{N^2 - 1}.$$ [5]

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(c) Use a similar method to find, in terms of N, an upper bound for $\sum_{r=2}^{N} \ln\left(r+\sqrt{r^2-1}\right)$. [3]

8 **(a)** Starting from the definitions of sech and tanh in terms of exponentials, prove that

$$1-\operatorname{sech}^2 t = \tanh^2 t.$$ [3]

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The curve C has parametric equations

$$x = \tfrac{1}{2}\tanh^2 t + \ln \operatorname{sech} t, \qquad y = 1 + \tanh^4 t, \qquad \text{for } t > 0.$$

(b) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -4\operatorname{sech}^2 t$. [5]

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(c) Find the coordinates of the point on C with $\frac{d^2y}{dx^2} = -\frac{9}{2}$, giving your answer in the form $(a + \ln b, c)$ where a, b and c are rational numbers. [6]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/22**

Paper 2 Further Pure Mathematics 22 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (5\times9-8\times6)-2(4\times9-7\times6)+3(4\times8-7\times5)$ $=-3+12-9=0$	**M1 A1**	Shows that determinant is zero or row operations to obtain e.g. $x+\ 2y+\ 3z=\ 1,$ $y+\ 2z=\ 1,$
		2	
1(b)	$x+\ 2y+\ 3z=\ 1,$ $4x+\ 5y+\ 6z=\ 1,\ \Rightarrow z=\frac{1}{3}(1-x-2y)\Rightarrow y=-2x-1$ $7x+\ 8y+\ 9z=\ 1,$	**M1 A1**	Uses *all three* equations to reduce to one equation with two unknowns. Reducing to two equations scores M1 A0.
	The three planes form a sheaf.	**B1**	Accept clear sketch or the three planes intersect along a common line.
		3	

Question	Answer	Marks	Guidance
2	$\frac{dy}{dx} = \frac{dz}{dx} - 1$	**B1**	
	$\frac{dz}{dx} - 1 = \frac{1+3z}{3z-1}$	**M1**	Substitutes and derives first order separable equation.
	$\frac{dz}{dx} = \frac{1+3z}{3z-1} + 1 = \frac{6z}{3z-1}$	**A1**	
	$\int \frac{1}{2} - \frac{1}{6}z^{-1}dz = \int 1\,dx$	**M1**	Separates variables and integrates both sides.
	$\frac{1}{2}z - \frac{1}{6}\ln z = x + C \Rightarrow -\frac{1}{2}x + \frac{1}{2}y - \frac{1}{6}\ln(x+y) = C$	**A1**	
	$C = -\frac{1}{2}$	**M1**	Substitutes initial conditions into their expression.
	$\frac{1}{6}\ln(x+y) + \frac{1}{2}(x-y) - \frac{1}{2} = 0$	**A1**	OE.
		7	

Question	Answer	Marks	Guidance
3(a)	$z+z^{-1}=2\cos\theta$	**B1**	Use of $z+z^{-1}=2\cos\theta$.
	$\left(z+z^{-1}\right)^4=\left(z^4+z^{-4}\right)+4\left(z^2+z^{-2}\right)+6$	**M1 A1**	Expands and groups. M1 A0 for no clear grouping. Correct substitution of $z^n=\cos n\theta+\mathrm{i}\sin n\theta$ for each term scores M1 A1.
	$(2\cos\theta)^4=2\cos 4\theta+4(2\cos 2\theta)+6$	**M1**	Substitutes $z^n+z^{-n}=2\cos n\theta$. If $z^n=\cos n\theta+\mathrm{i}\sin n\theta$ used then must cancel sin.
	$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$	**A1**	AG. SC B1 expands $(\cos\theta+\mathrm{i}\sin\theta)^4$ and uses trigonometric identities.
		5	
3(b)	$\int_0^{\frac{1}{2}}\left(1-x^2\right)^{\frac{3}{2}}\,\mathrm{d}x=\int_0^{\frac{1}{6}\pi}\cos^4\theta\,\mathrm{d}\theta=\frac{1}{8}\left[\frac{1}{4}\sin 4\theta+2\sin 2\theta+3\theta\right]_0^{\frac{1}{6}\pi}$	**M1 A1**	Applies substitution (M1) gets to $\int\cos^4\theta\,d\theta$, changes limits, integration correct (A1).
	$\frac{1}{16}\left(\frac{9}{4}\sqrt{3}+\pi\right)$	**A1**	
		3	

Question	Answer	Marks	Guidance
4(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1+x^5\right)^n\right) = 5nx^5\left(1+x^5\right)^{n-1} + \left(1+x^5\right)^n$	**M1 A1**	Uses the product rule to differentiate.
	$5n\left(1+x^5-1\right)\left(1+x^5\right)^{n-1} + \left(1+x^5\right)^n$	**M1***	Uses $x^5 = x^5 + 1 - 1$.
	$\left[x\left(1+x^5\right)^n\right]_0^1 = 5nI_n - 5nI_{n-1} + I_n$	**DM1**	Integrates both sides using the limits given. Requires previous method mark.
	$2^n = (5n+1)I_n - 5nI_{n-1} \Rightarrow (5n+1)I_n = 2^n + 5nI_{n-1}$	**A1**	Substitutes limits and rearranges. AG.
	Alternative method for question 4(a)		
	$I_n = \int_0^1 \left(1+x^5\right)^n \mathrm{d}x = \left[x\left(1+x^5\right)^n\right]_0^1 - 5n\int_0^1 x^5\left(1+x^5\right)^{n-1}\mathrm{d}x$	**M1 A1**	Integrates by parts.
	$I_n = \left[x\left(1+x^5\right)^n\right]_0^1 - 5n\int_0^1\left(1+x^5\right)^n \mathrm{d}x + 5n\int_0^1\left(1+x^5\right)^{n-1}\mathrm{d}x$	**M1***	Uses $x^5 = x^5 + 1 - 1$.
	$I_n = \left[x\left(1+x^5\right)^n\right]_0^1 - 5nI_n + 5nI_{n-1}$	**DM1**	Forms recursive formula. Requires previous method mark.
	$(5n+1)I_n = 2^n + 5nI_{n-1}$	**A1**	Substitutes limits and rearranges. AG.
		5	

Question	Answer	Marks	Guidance
4(b)	$I_1 = \left[x + \frac{1}{6}x^6\right]_0^1 = \frac{7}{6}$ or $I_0 = 1$	B1	
	$11I_2 = 2^2 + 10I_1 \Rightarrow I_2 = \frac{47}{33}$	M1 A1	Applies reduction formula.
	$16I_3 = 2^3 + 15I_2 \Rightarrow I_3 = \frac{323}{176}$	A1	
		4	

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} 18-\lambda & 5 & -11 \\ 8 & 6-\lambda & -4 \\ 32 & 10 & -20-\lambda \end{vmatrix} = 0$	M1	Sets determinant equal to zero.
	$(18-\lambda)\begin{vmatrix} 6-\lambda & -4 \\ 10 & -20-\lambda \end{vmatrix} - 5\begin{vmatrix} 8 & -4 \\ 32 & -20-\lambda \end{vmatrix} - 11\begin{vmatrix} 8 & 6-\lambda \\ 32 & 10 \end{vmatrix}$	A1	Expands determinant. Can use other rows/columns.
	$\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$	A1	AG
	$(\lambda-2)(\lambda+4)(\lambda-6) = 0$ leading to $\lambda = 2, -4, 6$	B1	
		4	

Question	Answer	Marks	Guidance
5(b)	$\lambda = 2: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & 5 & -11 \\ 8 & 4 & -4 \end{vmatrix} = \begin{pmatrix} 24 \\ -24 \\ 24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors. Must attempt to solve equations for M1.
	$\lambda = -4: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 10 & -4 \\ 32 & 10 & -16 \end{vmatrix} = \begin{pmatrix} -120 \\ 0 \\ -240 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\lambda = 6: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 5 & -11 \\ 8 & 0 & -4 \end{vmatrix} = \begin{pmatrix} -20 \\ -40 \\ -40 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	**A1 A1**	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 32 & 0 & 0 \\ 0 & -1024 & 0 \\ 0 & 0 & 7776 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns. M0 if a column of zeros appears in **P**.
		6	

Question	Answer	Marks	Guidance
6	$m^2 - 12m + 36 = 0$	**M1**	Auxiliary equation.
	$x = e^{6t}(At + B)$	**A1**	Complementary function. Allow " $x =$ " missing.
	$x = p\sin t + q\cos t \Rightarrow x' = p\cos t - q\sin t \Rightarrow x'' = -p\sin t - q\cos t$	**M1 A1**	Particular integral and its derivatives.
	$-p + 12q + 36p = 37 \quad -q - 12p + 36q = 0$	**M1**	Substitutes and equates coefficients. Must have two unknowns.
	$p = \frac{35}{37} \quad q = \frac{12}{37}$	**A1**	
	$x = e^{6t}(At + B) + \frac{35}{37}\sin t + \frac{12}{37}\cos t$	**A1 FT**	Must have " $x =$ ". FT on CF.
	$x' = Ae^{6t} + 6e^{6t}(At + B) + \frac{35}{37}\cos t - \frac{12}{37}\sin t$	**M1***	Differentiates using product rule. Must have their PI differentiated.
	$B = -\frac{12}{37} \quad A + 6B + \frac{35}{37} = 0 \Rightarrow A = 1$	**DM1 A1**	Forms simultaneous equations using initial conditions.
	$x = e^{6t}\left(t - \frac{12}{37}\right) + \frac{35}{37}\sin t + \frac{12}{37}\cos t$	**A1**	Must have " $x =$ ".
		11	

Question	Answer	Marks	Guidance
7(a)	$\int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$	**M1 A1**	Applies given substitution.
	$\sqrt{x^2-1}+C$	**A1**	Allow with " $+C$ " missing. Answer must be in terms of x.
		3	
7(b)	$\cosh^{-1} r = \ln\left(r+\sqrt{r^2-1}\right)$	**B1**	
	$\cosh^{-1}2+\cosh^{-1}3+\ldots+\cosh^{-1}N$	**M1**	Forms sum of the areas of the rectangles.
	$> \int_1^N \cosh^{-1} x\,dx$	**M1**	Compares with integral with correct limits.
	$\int_1^N \cosh^{-1} x\,dx = \left[x\cosh^{-1}x\right]_1^N - \int_1^N \frac{x}{\sqrt{x^2-1}}\,dx$	**A1**	Evaluates integral.
	$\sum_{r=2}^{N} \ln\left(r+\sqrt{r^2-1}\right) > N\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$	**A1**	AG.
		5	

Question	Answer	Marks	Guidance
7(c)	$(\cosh^{-1}1)+\cosh^{-1}2+\ldots+\cosh^{-1}(N-1)<\int_1^N \cosh^{-1}x\,dx$ (or $<\int_2^N \cosh^{-1}x\,dx$)	M1 A1	Compares with integral with correct limits.
	$\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$ or $\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	Adds $\ln\left(N+\sqrt{N^2-1}\right)$ to both sides.
	Alternative method for question 7(c)		
	$(\cosh^{-1}1)+\cosh^{-1}2+\ldots+\cosh^{-1}(N-1)+\cosh^{-1}N$ $<\int_1^{N+1}\cosh^{-1}x\,dx=\left[x\ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}\right]_1^{N+1}$ (or $<\int_2^{N+1}\cosh^{-1}x\,dx$)	M1 A1	Compares with integral with correct limits.
	$\sum_{r=2}^{N}\ln\left(r+\sqrt{r^2-1}\right)<(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}$ or $<(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	
	Second alternative method for question 7(c)		
	$\cosh^{-1}(1+1)+\cosh^{-1}(2+1)+\ldots+\cosh^{-1}(N-1+1)$ $<\int_1^N\cosh^{-1}(x+1)\,dx=\left[(x+1)\ln\left(x+1+\sqrt{x^2+2x}\right)-\sqrt{x^2+2x}\right]_1^N$	M1 A1	Compares with integral with correct limits.
	$=(N+1)\ln\left(N+1+\sqrt{N^2+2N}\right)-\sqrt{N^2+2N}-2\ln\left(2+\sqrt{3}\right)+\sqrt{3}$	A1	
		3	

Question	Answer	Marks	Guidance
8(a)	$\operatorname{sech} t = \dfrac{2}{e^t + e^{-t}} \quad \tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$	**B1**	
	$1 - \left(\dfrac{2}{e^t + e^{-t}}\right)^2 = \dfrac{\left(e^t + e^{-t}\right)^2 - 4}{\left(e^t + e^{-t}\right)^2} = \dfrac{e^{2t} + 2 + e^{-2t} - 4}{\left(e^t + e^{-t}\right)^2} = \dfrac{\left(e^t - e^{-t}\right)^2}{\left(e^t + e^{-t}\right)^2}$	**M1 A1**	Expands, gets to $\dfrac{e^{2t} + 2 + e^{-2t} - 4}{\left(e^t + e^{-t}\right)^2}$ for M1, AG.
		3	
8(b)	$\dfrac{dy}{dt} = 4\tanh^3 t \operatorname{sech}^2 t$	**B1**	
	$\dfrac{dx}{dt} = \tanh t \operatorname{sech}^2 t - \tanh t \quad \left(= \tanh t\left(\operatorname{sech}^2 t - 1\right) = -\tanh^3 t\right)$	**M1 A1**	M1 sensible attempt at derivative of x.
	$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{4\tanh^3 t \operatorname{sech}^2 t}{-\tanh^3 t} = -4\operatorname{sech}^2 t$	**M1 A1**	Applies chain rule, must substitute their $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ for M1, AG.
		5	
8(c)	$\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx} = \dfrac{8\operatorname{sech}^2 t \tanh t}{-\tanh^3 t} = -8\dfrac{\operatorname{sech}^2 t}{\tanh^2 t}$	**M1 A1**	Finds $\dfrac{d^2y}{dx^2}$. For M1 $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = c\operatorname{sech}^2 t \tanh t$. AEF. Accept $-8\operatorname{cosech}^2 t$.
	$-8\dfrac{\operatorname{sech}^2 t}{\tanh^2 t} = -\frac{9}{2} \Rightarrow 8\left(1 - \tanh^2 t\right) = \frac{9}{2}\tanh^2 t \Rightarrow \tanh^2 t = \frac{16}{25}$	**M1 A1**	Sets equal to $-\frac{9}{2}$, uses identity from **(a)** or equivalent. Accept $\sinh^2 t = \frac{16}{9}$ or $\cosh^2 t = \frac{25}{9}$.
	$(x, y) = \left(\frac{8}{25} + \ln\frac{3}{5}, \frac{881}{625}\right)$	**A1 A1**	A1 for each correct coordinate. $t = \ln 3$
		6	

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

1793694541

FURTHER MATHEMATICS **9231/23**

Paper 2 Further Pure Mathematics 2 **May/June 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (RW/CB) 311419/2

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1 **(a)** Find the Maclaurin series for $\sin^{-1}x$ up to and including the term in x^3. [5]

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(b) Deduce an approximation to $\int_0^{\frac{1}{5}} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u$, giving your answer as a fraction. [1]

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2 The variables x and y are related by the differential equation

$$6\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + x = t^2 + 10t + 13.$$

(a) Find the general solution for x in terms of t. [6]

(b) State an approximate solution for large positive values of t. [1]

3 By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^4$ and $\left(z-\frac{1}{z}\right)^4$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cot^4\theta = \frac{\cos 4\theta + a\cos 2\theta + b}{\cos 4\theta - a\cos 2\theta + b},$$

where a and b are integers to be determined. [7]

4 The curve C has equation

$$4y^3 + (x+y)^6 = 109.$$

(a) Show that, at the point $(-4, 3)$ on C, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{17}$. [3]

(b) Find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $(-4, 3)$. [5]

5 **(a)** Starting from the definitions of cosh and sinh in terms of exponentials, prove that

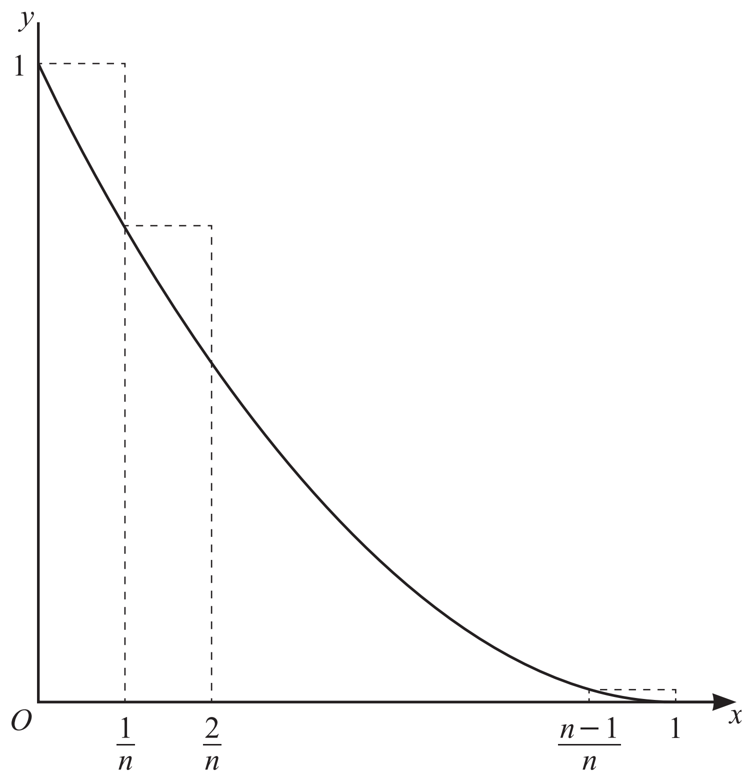
$$2\cosh^2 x = \cosh 2x + 1.$$ [3]

(b) Find the solution of the differential equation

$$\frac{dy}{dx} + 2y\tanh x = 1$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. [8]

6



The diagram shows the curve with equation $y = (1 - x)^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (1-x)^2 \, dx < U_n$, where

$$U_n = \frac{2n^2 + 3n + 1}{6n^2}.$$

[5]

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(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 (1-x)^2 \mathrm{d}x$. [4]

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(c) Show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

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7 The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_{-1} giving your answer in the form $\ln a$, where a is an integer to be determined. [2]

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(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^{\frac{1}{2}n}\right)$, or otherwise, show that

$$(n+1)I_n = nI_{n-2} + \tfrac{4}{3}\left(\tfrac{5}{3}\right)^n.$$ [5]

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(c) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The arc length of the curve is denoted by s.

Use the substitution $u = 2x$ to show that $s = \frac{1}{2}I_1$ and find the exact value of s. [4]

8 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & -6a & 2a+2 \\ 0 & 1-a & 0 \\ 0 & 2-a & -1 \end{pmatrix}$$

where a is a constant with $a \neq 0$ and $a \neq 1$.

(a) Show that the equation $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution and interpret this situation geometrically. [3]

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(b) Show that the eigenvalues of $\mathbf{A}$ are a, $1-a$ and -1. [2]

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(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^4 = \mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of **A** to find $\mathbf{A}^4$ in terms of **A** and a. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/23**

Paper 2 Further Pure Mathematics 23 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Partial Marks	Guidance
1(a)	$f'(x) = \left(1-x^2\right)^{-\frac{1}{2}}$	B1	Correct first derivative.
	$f''(x) = x\left(1-x^2\right)^{-\frac{3}{2}}$	B1	Correct second derivative.
	$f'''(x) = 3x^2\left(1-x^2\right)^{-\frac{5}{2}} + \left(1-x^2\right)^{-\frac{3}{2}}$	M1	Differentiates their $f''(x)$ using product rule.
	$f(0)=0 \quad f'(0)=1 \quad f''(0)=0 \quad f'''(0)=1$	M1	Evaluates their derivatives at $x=0$. Must have attempted all three derivatives.
	$\sin^{-1} x = x + \frac{1}{6}x^3$	A1	CWO.
	Alternative method for question 1(a)		
	$\sin y = x \Rightarrow f'(x) = \sec y$	(B1)	Finds first derivative.
	$f''(x) = \tan y\left(\sec y\right)^2$	(B1)	Finds second derivative.
	$f'''(x) = \left(2\tan^2 y\sec^2 y + \sec^4 y\right)\sec y = 3\sec^5 y - 2\sec^3 y$	(M1)	Differentiates $\tan y\left(\sec y\right)^2$ using product and chain rule.
	$f(0)=0 \quad f'(0)=1 \quad f''(0)=0 \quad f'''(0)=1$	(M1)	Evaluates their derivatives at $x=0$. Must have attempted all three derivatives.
	$\sin^{-1} x = x + \frac{1}{6}x^3$	(A1)	CWO.
		5	

Question	Answer	Partial Marks	Guidance
1(b)	$\frac{151}{750}$	**B1**	
		1	

Question	Answer	Partial Marks	Guidance
2(a)	$6m^2+5m+1=0 \Rightarrow m=-\frac{1}{2},-\frac{1}{3}$	**M1**	Auxiliary equation.
	$x=Ae^{-\frac{1}{2}t}+Be^{-\frac{1}{3}t}$	**A1**	Complementary function. Allow with '$x=$' missing.
	$x=p+qt+rt^2 \Rightarrow x'=q+2rt \Rightarrow x''=2r$	**B1**	Particular integral and its derivatives.
	$12r+5q+10rt+p+qt+rt^2=t^2+10t+13$	**M1**	Substitutes and equates coefficients.
	$r=1,\quad q=0,\quad p=1$	**A1**	
	$x=Ae^{-\frac{1}{2}t}+Be^{-\frac{1}{3}t}+t^2+1$	**A1**	Must have $x=$.
		6	
2(b)	$x=t^2+1$	**B1FT**	Must have '$x=$'. Accept 'x is approximately' but do not accept '$x\to$'. FT on their PI.
		1	

Question	Answer	Partial Marks	Guidance
3	$z - z^{-1} = 2\mathrm{i}\sin\theta$ and $z + z^{-1} = 2\cos\theta$	**B1**	Use of $z - z^{-1} = 2\mathrm{i}\sin\theta$ and $z + z^{-1} = 2\cos\theta$
	$\left(z + z^{-1}\right)^4 = \left(z^4 + z^{-4}\right) + 4\left(z^2 + z^{-2}\right) + 6$	**M1 A1**	Expands and groups.
	$\left(z - z^{-1}\right)^4 = \left(z^4 + z^{-4}\right) - 4\left(z^2 + z^{-2}\right) + 6$	**M1 A1**	Expands and groups. Only withhold one A1 mark (this or the previous) for no clear grouping.
	$\dfrac{2^4\cos^4\theta}{2^4\sin^4\theta} = \dfrac{2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6}{2\cos 4\theta - 4\left(2\cos 2\theta\right) + 6}$	**M1**	Substitutes $z^n + z^{-n} = 2\cos n\theta$ once in LHS.
	$\cot^4\theta = \dfrac{\cos 4\theta + 4\cos 2\theta + 3}{\cos 4\theta - 4\cos 2\theta + 3}$	**A1**	
		7	

Question	Answer	Partial Marks	Guidance
4(a)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(4y^3\right) = 12y^2y'$	**B1**	Differentiates y^3 correctly.
	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left((x+y)^6\right) = 6(x+y)^5(1+y')$	**B1**	Differentiates $(x+y)^6$ correctly.
	$12\left(3^2\right)y' + 6(-1)(1+y') = 0 \Rightarrow y' = \frac{1}{17}$	**B1**	Substitutes $(-4,3)$, AG.
		3	

Question	Answer	Partial Marks	Guidance
4(b)	$2y^2y''+4y\left(y''\right)^2$	**B1**	Differentiates y^2y'.
	$+\left(x+y\right)^5y''+5\left(x+y\right)^4\left(1+y'\right)^2=0$	**B1 B1**	Differentiates $(x+y)^5\left(1+y'\right)$.
	$17y''+12\left(\frac{1}{17}\right)^2+5\times(-1)^4\times\left(\frac{18}{17}\right)^2=0\Rightarrow 17y''+\frac{96}{17}=0$	**M1**	Substitutes $(-4,3)$.
	$y''=-\frac{96}{289}$	**A1**	
		5	

Question	Answer	Partial Marks	Guidance
5(a)	$\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$	**B1**	
	$\frac{2}{4}\left(e^x+e^{-x}\right)^2=\frac{1}{2}\left(e^{2x}+e^{-2x}+2\right)=\cosh 2x+1$	**M1 A1**	Expands, AG.
		3	

Question	Answer	Partial Marks	Guidance
5(b)	$e^{2\int \tanh x \mathrm{d}x} = e^{2\ln\cosh x} = \cosh^2 x$	**M1 A1**	Finds integrating factor.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\cosh^2 x\right) = \cosh^2 x$	**M1**	Correct form on LHS, $\frac{\mathrm{d}}{\mathrm{d}x}(yI)$ for *their* integrating factor I, and attempt to integrate *their* RHS.
	$y\cosh^2 x = \int \cosh^2 x \,\mathrm{d}x = \frac{1}{2}\int \cosh 2x + 1 \,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{2}\sinh 2x + x\right) + C$	**M1 A1**	Uses $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$. M1 A1 is for RHS.
	$1 = C$	**M1**	Finds C.
	$y = \mathrm{sech}^2 x\left(\frac{1}{4}\sinh 2x + \frac{1}{2}x + 1\right)$	**M1 A1**	Divides through by *their* coefficient of y.
		8	

Question	Answer	Partial Marks	Guidance
6(a)	$\int_0^1 (1-x)^2\,dx < \left(\frac{1}{n}\right)\left(1-\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$	M1 A1	Forms the sum of the areas of the rectangles. Must have first two terms and last term for A1. Accept $\left(\frac{1}{n}\right)\left(1-\frac{n}{n}\right)^2$ for the last term. Accept written in summation form with correct limits.
	$\frac{1}{n}+\frac{1}{n^3}\sum_{r=1}^{n-1}(n-r)^2 = \frac{1}{n}+\frac{1}{n^3}\sum_{r=1}^{n-1}\left(n^2-2nr+r^2\right)$ $\left(=1-\frac{2}{n^2}\sum_{r=1}^{n-1}r+\frac{1}{n^3}\sum_{r=1}^{n-1}r^2\right)$	M1 A1	Setting up correct series so that formulae from MF19 can be applied. Can also recognise sum as $\frac{1}{n^3}\sum_{r=1}^{n}r^2$
	$\frac{1}{n}+\frac{n^2(n-1)}{n^3}-\frac{n^2(n-1)}{n^3}+\frac{n(n-1)(2n-1)}{6n^3}=\frac{1}{n}+\frac{2n^2-3n+1}{6n^2}=\frac{2n^2+3n+1}{6n^2}$	A1	AG. Must have gained previous accuracy mark.
		5	

Question	Answer	Partial Marks	Guidance
6(b)	$\int_0^1 (1-x)^2\,dx > \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(1-\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$	M1 A1	Forms the sum of the areas of appropriate rectangles. Must have first two and last terms for A1. Accept $\left(\frac{1}{n}\right)\left(1-\frac{n}{n}\right)^2$ for the last term. Accept written in summation form with correct limits.
	$\frac{1}{n^3}\sum_{r=1}^{n-1}\left(n^2-2nr+r^2\right) = \frac{n^2(n-1)}{n^3} - \frac{n^2(n-1)}{n^3} + \frac{n(n-1)(2n-1)}{6n^3}$ $\left(=1-\frac{2}{n^2}\sum_{r=1}^{n} r + \frac{1}{n^3}\sum_{r=1}^{n} r^2\right)$	M1	Setting up correct series so that formulae from MF19 can be applied. Recognising sum as $\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{1}{n^3}\sum_{r=1}^{n} r^2 - \frac{1}{n}$ without wrong working scores M1 A1 M1.
	$\frac{2n^2-3n+1}{6n^2}$	A1	
		4	
6(c)	$U_n - L_n = \frac{1}{n}$	M1	Simplifies $U_n - L_n$ to $\frac{c}{n}$.
	$\frac{1}{n} \to 0$ as $n \to \infty$	A1	AG.
		2	

Question	Answer	Partial Marks	Guidance
7(a)	$I_{-1} = \int_0^{\frac{4}{3}} \left(1+x^2\right)^{-\frac{1}{2}} \mathrm{d}x = \left[\sinh^{-1} x\right]_0^{\frac{4}{3}} = \ln 3$	**M1 A1**	Recognises integral.
		2	
7(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1+x^2\right)^{\frac{1}{2}n}\right) = nx^2\left(1+x^2\right)^{\frac{1}{2}n-1} + \left(1+x^2\right)^{\frac{1}{2}n}$	**M1 A1**	Uses the product rule to differentiate.
	$n\left(1+x^2-1\right)\left(1+x^2\right)^{\frac{1}{2}n-1} + \left(1+x^2\right)^{\frac{1}{2}n}$	**M1***	Uses $x^2 = 1+x^2-1$.
	$\left[x\left(1+x^2\right)^{\frac{1}{2}n}\right]_0^{\frac{4}{3}} = nI_n - nI_{n-2} + I_n$	**DM1**	Integrates both sides using the limits given. Requires previous method mark.
	$\frac{4}{3}\left(\frac{5}{3}\right)^n = (n+1)I_n - nI_{n-2} \Rightarrow (n+1)I_n = nI_{n-2} + \frac{4}{3}\left(\frac{5}{3}\right)^n$	**A1**	Substitutes limits and rearranges. AG.
		5	
7(c)	$s = \int_0^{\frac{2}{3}} \sqrt{1+4x^2}\,\mathrm{d}x$	**M1**	Forms correct integral with correct limits.
	$\frac{1}{2}\int_0^{\frac{4}{3}} \sqrt{1+u^2}\,\mathrm{d}u = \frac{1}{2}I_1$	**A1**	AG.
	$2I_1 = I_{-1} + \frac{4}{3}\left(\frac{5}{3}\right)$	**M1**	Applies reduction formula with $n=1$.
	$s = \frac{1}{4}\ln 3 + \frac{5}{9}$	**A1**	
		4	

Question	Answer	Partial Marks	Guidance
8(a)	$\begin{vmatrix} a & -6a & 2a+2 \\ 0 & 1-a & 0 \\ 0 & 2-a & -1 \end{vmatrix} = a(a-1) \neq 0$ or $ax = 1 + \frac{12a}{1-a} - 2\frac{(a+1)^2}{1-a} = \frac{-2a^2+7a-1}{1-a}$, $y = \frac{2}{1-a}$, $z = \frac{2(2-a)}{1-a} - 3 = \frac{a+1}{1-a}$	M1 A1	Shows that determinant is non-zero or solves equations.
	Three planes intersect at a *single* point.	B1	Must be clear that there is just one point where all 3 planes intersect.
		3	
8(b)	$\begin{vmatrix} a-\lambda & -6a & 2a+2 \\ 0 & 1-a-\lambda & 0 \\ 0 & 2-a & -1-\lambda \end{vmatrix} = 0$	M1	Equates determinant to zero or working to find characteristic equation.
	$(a-\lambda)(1-a-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda = a, 1-a, -1$	A1	AG. Factorisation must be clear.
		2	

Question	Answer	Partial Marks	Guidance
8(c)	$\lambda = a: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1-2a & 0 \\ 0 & 2-a & -1-a \end{vmatrix} = \begin{pmatrix} 2a^2+a-1 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = -1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a+1 & -6a & 2a+2 \\ 0 & 2-a & 0 \end{vmatrix} = \begin{pmatrix} -(2-a)(2a+2) \\ 0 \\ (a+1)(2-a) \end{pmatrix} \sim \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$	A1	
	$\lambda = 1-a: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2a-1 & -6a & 2a+2 \\ 0 & 2-a & -2+a \end{vmatrix} = \begin{pmatrix} -4a^2+10a-4 \\ -2a^2+5a-2 \\ -2a^2+5a-2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$	A1	
	$\mathbf{P} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} a^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-a)^4 \end{pmatrix}$	M1 A1	Correctly matched permutations of columns. Their eigenvectors must be non-zero for M1.
		6	
8(d)	$(a-\lambda)(1-a-\lambda)(-1-\lambda) = \lambda^3 - (a^2-a+1)\lambda - (a^2-a) = 0$	B1	Finds characteristic equation multiplied out.
	$\mathbf{A}^4 = (a^2-a+1)\mathbf{A}^2 + (a^2-a)\mathbf{A}$	M1 A1	Multiplies through by $\mathbf{A}$.
		3	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME			
CENTRE NUMBER		CANDIDATE NUMBER	

8225278573

FURTHER MATHEMATICS **9231/31**

Paper 3 Further Mechanics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\mathrm{m\,s^{-2}}$.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 One end of a light elastic string, of natural length a and modulus of elasticity $3mg$, is attached to a fixed point O. The other end of the string is attached to a particle P of mass m. The string hangs with P vertically below O. The particle P is pulled vertically downwards so that the extension of the string is $2a$. The particle P is then released from rest.

(a) Find the speed of P when it is at a distance $\frac{3}{4}a$ below O. [3]

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(b) Find the initial acceleration of P when it is released from rest. [2]

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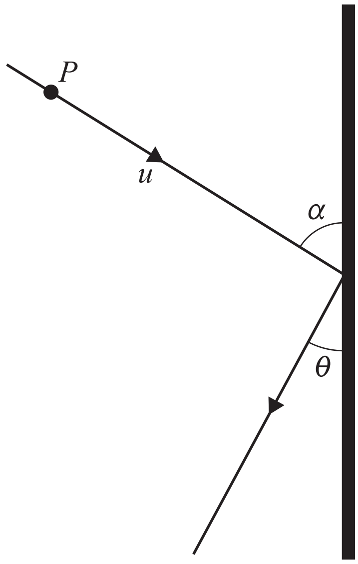

A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. It collides at an angle α with a fixed smooth vertical barrier. After the collision, P moves at an angle θ with the barrier, where $\tan\theta = \frac{1}{2}$ (see diagram). The coefficient of restitution between P and the barrier is e. The particle P loses 20% of its kinetic energy as a result of the collision.

Find the value of e. [5]

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3 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle θ with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed u. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where angle AOB is 90° and the speed of P is $\sqrt{\frac{4}{5}ag}$.

(a) Find the value of $\sin\theta$. [2]

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(b) Find, in terms of m and g, the tension in the string when P is at A. [5]

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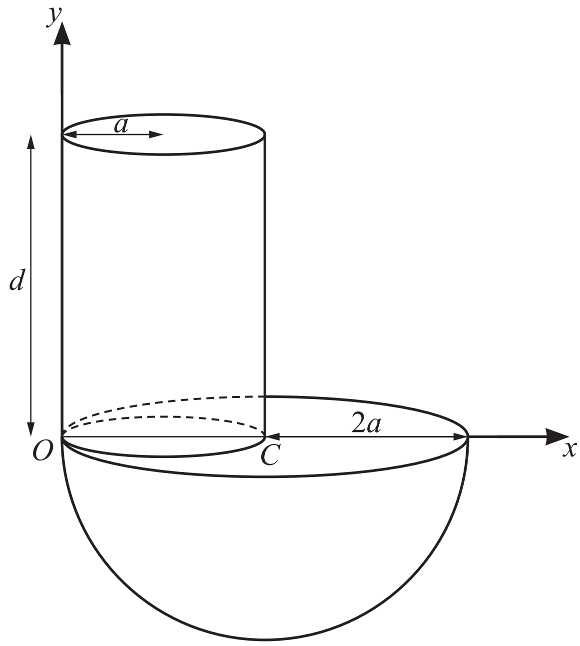

An object is formed from a solid hemisphere, of radius $2a$, and a solid cylinder, of radius a and height d. The hemisphere and the cylinder are made of the same material. The cylinder is attached to the plane face of the hemisphere. The line OC forms a diameter of the base of the cylinder, where C is the centre of the plane face of the hemisphere and O is common to both circumferences (see diagram). Relative to axes through O, parallel and perpendicular to OC as shown, the centre of mass of the object is $(\overline{x}, \overline{y})$.

(a) Show that $\overline{x} = \dfrac{32a^2 + 3ad}{16a + 3d}$ and find an expression, in terms of a and d, for $\overline{y}$. [5]

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The object is placed on a rough plane which is inclined to the horizontal at an angle θ where $\sin\theta = \frac{1}{6}$. The object is in equilibrium with CO horizontal, where CO lies in a vertical plane through a line of greatest slope.

(b) Find d in terms of a. [3]

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5 A light elastic string of natural length a and modulus of elasticity λmg has one end attached to a fixed point O on a smooth horizontal surface. When a particle of mass m is attached to the free end of the string, it moves with speed v in a horizontal circle with centre O and radius x. When, instead, a particle of mass $2m$ is attached to the free end of the string, this particle moves with speed $\frac{1}{2}v$ in a horizontal circle with centre O and radius $\frac{3}{4}x$.

(a) Find x in terms of a. [5]

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(b) Given that $v = \sqrt{12ag}$, find the value of λ. [2]

6 A particle P moving in a straight line has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The acceleration of P, in $\mathrm{m\,s^{-2}}$, is given by $6v\sqrt{v+9}$. When $t = 0$, $x = 2$ and $v = 72$.

(a) Find an expression for v in terms of x. [4]

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(b) Find an expression for x in terms of t. [5]

7 At time t s, a particle P is projected with speed $40\,\text{m}\,\text{s}^{-1}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The greatest height achieved by P during its flight is H m and the corresponding time is T s.

(a) Obtain expressions for H and T in terms of θ. [2]

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During the time between $t = T$ and $t = 3$, P descends a distance $\frac{1}{4}H$.

(b) Find the value of θ. [4]

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(c) Find the speed of P when $t = 3$. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/31**

Paper 3 Further Mechanics 31 **May/June 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\frac{3mg}{2a}(2a)^2$	**B1**	Correct EPE term seen
	$\frac{1}{2}mv^2 + mg \times \left(3a - \frac{3}{4}a\right) = \frac{3mg}{2a}(2a)^2$	**M1**	Dimensionally correct energy equation. Must have one KE, one EPE term and at least one GPE. Allow sign errors.
	$v = \sqrt{\frac{15}{2}ag}$ $\left[2.74\sqrt{ag}\right]$	**A1**	AEF
		3	
1(b)	$T - mg = mA$ and $T = \frac{3mg}{a} \times 2a$	**M1**	N2L and Hooke's law
	Acceleration = $5g$ [upwards]	**A1**	Allow ± 50 or $\pm 5g$
		2	

Question	Answer	Marks	Guidance
2	Parallel to wall $v\cos\theta = u\cos\alpha$ Perpendicular to wall $v\sin\theta = eu\sin\alpha$	**M1**	Both
	Dividing, $e = \dfrac{1}{2\tan\alpha}$	**A1**	AEF
	KE reduced by 20%, so $\frac{1}{2}mu^2\left(\cos^2\alpha + e^2\sin^2\alpha\right) = \frac{4}{5}\times\frac{1}{2}mu^2$	**M1**	Dimensionally correct equation in u or v, but not both. Must have either α or θ, but not both. Must see $\frac{4}{5}$ on the correct side of the equation.
	Eliminate e: $\cos\alpha = \dfrac{4}{5}$	**A1**	
	$e = \dfrac{2}{3}$	**A1**	

Question	Answer	Marks	Guidance
2	**Alternative method for question 2**		
	Parallel to wall $v\cos\theta = u\cos\alpha$ Perpendicular to wall $v\sin\theta = eu\sin\alpha$	**M1**	Both
	$\left[\sin(\theta) = \frac{\sqrt{5}}{5},\ \cos(\theta) = \frac{2\sqrt{5}}{5}\right]$ $u\sin(\alpha) = \frac{\sqrt{5}v}{5e},\ u\cos(\alpha) = \frac{2\sqrt{5}v}{5}$	**A1**	
	$u^2 = \left[u^2 cos^2(\alpha) + u^2 sin^2(\alpha)\right] = \frac{4v^2}{5} + \frac{v^2}{5e^2}$	**A1**	AEF, e.g. $\frac{v^2}{5}\left(4 + \frac{1}{e^2}\right)$
	$\frac{1}{2}mv^2 = \left[\frac{4}{5} \times \frac{1}{2}mu^2 = \right]\frac{2}{5}m\frac{v^2}{5}\left(4 + \frac{1}{e^2}\right)$	**M1**	Dimensionally correct equation in v. Must have either α or θ, but not both. Must see $\frac{4}{5}$ or $\frac{2}{5}$ on the correct side of the equation.
	$e = \frac{2}{3}$	**A1**	
		5	

Question	Answer	Marks	Guidance
3(a)	At B, $mg\sin\theta = \frac{m4ag}{5a}$	**M1**	Allow cos instead of sin for M1. Do not award until tension $=0$ used. Mass must be seen. No sign error.
	$\sin\theta = \frac{4}{5}$	**A1**	
		2	
3(b)	At A, $T - mg\cos\theta = \frac{mu^2}{a}$	**B1**	
	Energy $\frac{1}{2}mu^2 - \frac{1}{2}m \times \frac{4ag}{5} = mga(\cos\theta + \sin\theta)$	**M1 A1**	Energy equation with 4 terms, dimensionally correct. Mass must be present, allow sign errors. Must see $\frac{1}{2}$ in the KE terms.
	Solve to find T	**M1**	Complete method leading to an expression in mg for T.
	$T = \frac{21}{5}mg$	**A1**	CWO
		5	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>4(a)</td><td>[Mass is proportional to volume]

	Volume	Distance of centre of mass from vertical axis	Distance of centre of mass from OC
Hemisphere	$\frac{2}{3}\pi(2a)^3$	$2a$	$-\frac{3}{8}\times 2a$
Cylinder	$\pi a^2 d$	a	$\frac{1}{2}d$
Object	$\frac{2}{3}\pi(2a)^3+\pi a^2 d$	$\bar{x}$	$\bar{y}$

$\left(\frac{2}{3}\pi(2a)^3+\pi a^2 d\right)\bar{x}=\frac{16}{3}\pi a^3\times 2a+\pi a^2 d\times a$</td><td>M1 A1</td><td>Moments equation, dimensionally correct, correct number of terms. Allow sign errors.</td></tr>
<tr><td></td><td>Simplify to $\bar{x}=\frac{32a^2+3ad}{16a+3d}$</td><td>A1</td><td>AG. At least one line of intermediate working.</td></tr>
<tr><td></td><td>$\left(\frac{2}{3}\pi(2a)^3+\pi a^2 d\right)\bar{y}=\frac{16}{3}\pi a^3\times\left(-\frac{3}{8}\times 2a\right)+\pi a^2 d\times\frac{1}{2}d$</td><td>M1</td><td>Moments equation, dimensionally correct, correct number of terms. Allow sign errors.</td></tr>
<tr><td></td><td>$\bar{y}=\frac{3\left(d^2-8a^2\right)}{2(16a+3d)}$</td><td>A1</td><td>AEF</td></tr>
<tr><td></td><td></td><td>5</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
4(b)	$\sin\theta = \dfrac{2a-\bar{x}}{2a}$	**B1**	
	$2a\times\dfrac{1}{6} = 2a - \dfrac{32a^2+3ad}{16a+3d}$ $\dfrac{5}{3}(16a+3d) = (32a+3d)$	**M1**	Remove fractions
	$d = \dfrac{8}{3}a$	**A1**	
		3	

Question	Answer	Marks	Guidance
5(a)	Hooke's law, $T_1 = \dfrac{\lambda mg}{a}(x-a)$ or $T_2 = \dfrac{\lambda mg}{a}\left(\dfrac{3}{4}x - a\right)$	B1	
	Also, $T_1 = \dfrac{mv^2}{x}$ and equate $\dfrac{mv^2}{x} = \dfrac{\lambda mg}{a}(x-a)$	M1	$v^2 = \dfrac{\lambda gx(x-a)}{a}$ Dimensionally correct terms.
	Similarly: $\dfrac{\lambda mg\left(\dfrac{3x}{4} - a\right)}{a} = \dfrac{2m\left(\dfrac{1}{2}v\right)^2}{\dfrac{3}{4}x}$	M1	$v^2 = \dfrac{3\lambda gx}{2a}\left(\dfrac{3}{4}x - a\right)$ Must have $\dfrac{1}{2}v$ and $\dfrac{3x}{4}$ on the RHS. Their dimensionally correct T_2.
	Equate expressions for v^2 and solve for x in terms of a.	M1	
	$x = 4a$	A1	WWW
		5	**SC B3** for answer of $4a$ using λ instead of λmg.
5(b)	$\lambda = \dfrac{a}{xg(x-a)}v^2$ or $\lambda = \dfrac{2a}{3xg\left(\dfrac{3}{4}x - a\right)}v^2$ and substitute $x = 4a$, $v = \sqrt{12ag}$	M1	FT their expression for x.
	1	A1	CAO
		2	

Question	Answer	Marks	Guidance
6(a)	$v\frac{dv}{dx} = 6v\sqrt{v+9}$ and attempt to separate variables and integrate	M1	
	$2\sqrt{v+9} = 6x + A$	A1	
	$x = 2, v = 72 \quad A = 6$	M1	Use initial condition to find constant.
	$v = 9(x+1)^2 - 9$	A1	Correct, AEF.
		4	
6(b)	$\left[\frac{dx}{dt} = 9(x^2+2x)\ ,\ \frac{dx}{x(x+2)} = 9dt\right]\ \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = 9dt$	M1	Separate variables and write in the form $\left(\frac{a}{x} - \frac{b}{x+c}\right)dx = dt$
	$\frac{1}{2}\ln\left(\frac{x}{x+2}\right) = 9t + B$	A1	Integrate, any correct form.
	$t = 0, x = 2 \quad B = \frac{1}{2}\ln\frac{1}{2}$	M1	Use initial condition to find constant.
	$18t = \ln\left(\frac{2x}{x+2}\right) \quad e^{18t} = \frac{2x}{x+2}$	M1	Take logarithms
	$x = \frac{2e^{18t}}{2-e^{18t}}$ or $x = \frac{2}{2e^{-18t}-1}$	A1	Any correct form
		5	

Question	Answer	Marks	Guidance
7(a)	$H = 80\sin^2\theta$ or $\dfrac{800\sin^2\theta}{g}$	B1	
	$T = 4\sin\theta$ or $\dfrac{40\sin\theta}{g}$	B1	
		2	
7(b)	Between $t = T$ and $t = 3$ $\uparrow$ $\frac{1}{4}H = \frac{1}{2}\times 10\times(3-T)^2$	M1 A1	No extra terms.
	Use results from part **(a)** $\frac{1}{4}80\sin^2\theta = 5(3-4\sin\theta)^2$ $4\sin^2\theta - 8\sin\theta + 3 = 0$	M1	Substitute their expressions for H and T from part **(a)** and obtain a quadratic equation in $\sin\theta$ with no more than three terms.
	$\sin\theta = \frac{1}{2}$, $\theta = 30°$	A1	Single answer. NFWW.
	Alternative method for question 7 part (b)		
	$\frac{3}{4}H = y(3) = 40\times 3\ \sin\theta - \frac{1}{2}\times 10\times 3^2$	M1 A1	$120\ \sin\theta - 45$
	Use results from (a): $\frac{3}{4}80\sin^2\theta = 120\ \sin\theta - 45$ $4\sin^2\theta - 8\sin\theta + 3 = 0$	M1	Substitute their expressions for H and T from part **(a)** and obtain a quadratic equation in $\sin\theta$ with no more than three terms.
	$\sin\theta = \frac{1}{2}$, $\theta = 30°$	A1	Single answer. NFWW.
		4	

Question	Answer	Marks	Guidance
7(c)	When $t = 3$ speeds $\rightarrow 40\cos\theta$ and $\uparrow 40\sin\theta - 10\times 3$	B1	
	Square and add to find square of speed: $v^2 = \left(20\sqrt{3}\right)^2 + \left(-10\right)^2$	M1	Must be numerical.
	$v^2 = 1300$, $v = 10\sqrt{13}$ [= 36.1]	A1	
		3	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6081966353

FURTHER MATHEMATICS **9231/32**

Paper 3 Further Mechanics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\text{m}\,\text{s}^{-2}$.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 One end of a light elastic string, of natural length a and modulus of elasticity $3mg$, is attached to a fixed point O. The other end of the string is attached to a particle P of mass m. The string hangs with P vertically below O. The particle P is pulled vertically downwards so that the extension of the string is $2a$. The particle P is then released from rest.

(a) Find the speed of P when it is at a distance $\frac{3}{4}a$ below O. [3]

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(b) Find the initial acceleration of P when it is released from rest. [2]

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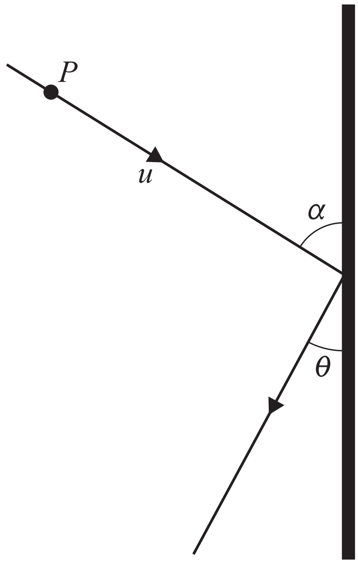

A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. It collides at an angle α with a fixed smooth vertical barrier. After the collision, P moves at an angle θ with the barrier, where $\tan\theta = \frac{1}{2}$ (see diagram). The coefficient of restitution between P and the barrier is e. The particle P loses 20% of its kinetic energy as a result of the collision.

Find the value of e. [5]

3 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle θ with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed u. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where angle AOB is 90° and the speed of P is $\sqrt{\frac{4}{5}ag}$.

(a) Find the value of $\sin\theta$. [2]

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(b) Find, in terms of m and g, the tension in the string when P is at A. [5]

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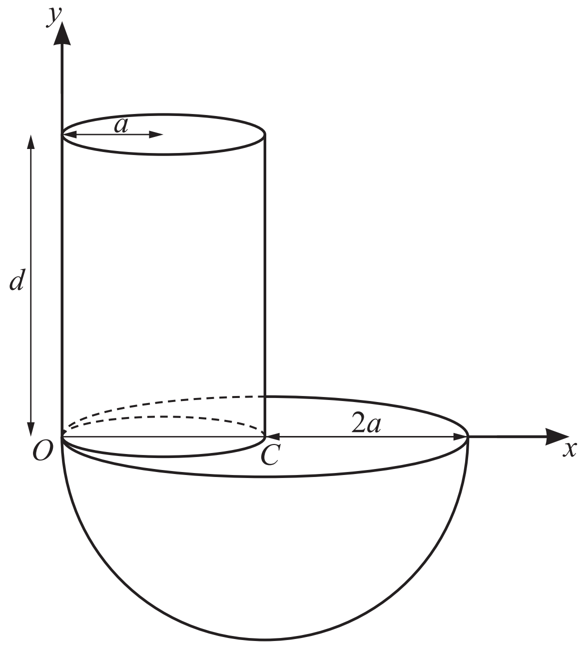

An object is formed from a solid hemisphere, of radius $2a$, and a solid cylinder, of radius a and height d. The hemisphere and the cylinder are made of the same material. The cylinder is attached to the plane face of the hemisphere. The line OC forms a diameter of the base of the cylinder, where C is the centre of the plane face of the hemisphere and O is common to both circumferences (see diagram). Relative to axes through O, parallel and perpendicular to OC as shown, the centre of mass of the object is $(\overline{x}, \overline{y})$.

(a) Show that $\overline{x} = \dfrac{32a^2 + 3ad}{16a + 3d}$ and find an expression, in terms of a and d, for $\overline{y}$. [5]

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The object is placed on a rough plane which is inclined to the horizontal at an angle θ where $\sin\theta = \frac{1}{6}$. The object is in equilibrium with CO horizontal, where CO lies in a vertical plane through a line of greatest slope.

(b) Find d in terms of a. [3]

5 A light elastic string of natural length a and modulus of elasticity λmg has one end attached to a fixed point O on a smooth horizontal surface. When a particle of mass m is attached to the free end of the string, it moves with speed v in a horizontal circle with centre O and radius x. When, instead, a particle of mass $2m$ is attached to the free end of the string, this particle moves with speed $\frac{1}{2}v$ in a horizontal circle with centre O and radius $\frac{3}{4}x$.

(a) Find x in terms of a. [5]

(b) Given that $v = \sqrt{12ag}$, find the value of λ. [2]

6 A particle P moving in a straight line has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The acceleration of P, in $\mathrm{m\,s^{-2}}$, is given by $6v\sqrt{v+9}$. When $t = 0$, $x = 2$ and $v = 72$.

(a) Find an expression for v in terms of x. [4]

(b) Find an expression for x in terms of t. [5]

7 At time t s, a particle P is projected with speed $40\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The greatest height achieved by P during its flight is H m and the corresponding time is T s.

(a) Obtain expressions for H and T in terms of θ. [2]

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During the time between $t = T$ and $t = 3$, P descends a distance $\frac{1}{4}H$.

(b) Find the value of θ. [4]

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(c) Find the speed of P when $t = 3$. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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FURTHER MATHEMATICS **9231/32**

Paper 3 Further Mechanics 32 **May/June 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate’s intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\frac{3mg}{2a}(2a)^2$	**B1**	Correct EPE term seen
	$\frac{1}{2}mv^2 + mg \times \left(3a - \frac{3}{4}a\right) = \frac{3mg}{2a}(2a)^2$	**M1**	Dimensionally correct energy equation. Must have one KE, one EPE term and at least one GPE. Allow sign errors.
	$v = \sqrt{\frac{15}{2}ag} \quad \left[2.74\sqrt{ag}\right]$	**A1**	AEF
		3	
1(b)	$T - mg = mA$ and $T = \frac{3mg}{a} \times 2a$	**M1**	N2L and Hooke's law
	Acceleration = $5g$ [upwards]	**A1**	Allow ± 50 or $\pm 5g$
		2	

Question	Answer	Marks	Guidance
2	Parallel to wall $v\cos\theta = u\cos\alpha$ Perpendicular to wall $v\sin\theta = eu\sin\alpha$	**M1**	Both
	Dividing, $e = \dfrac{1}{2\tan\alpha}$	**A1**	AEF
	KE reduced by 20%, so $\frac{1}{2}mu^2\left(\cos^2\alpha + e^2\sin^2\alpha\right) = \frac{4}{5}\times\frac{1}{2}mu^2$	**M1**	Dimensionally correct equation in u or v, but not both. Must have either α or θ, but not both. Must see $\frac{4}{5}$ on the correct side of the equation.
	Eliminate e: $\cos\alpha = \dfrac{4}{5}$	**A1**	
	$e = \dfrac{2}{3}$	**A1**	

Question	Answer	Marks	Guidance
2	**Alternative method for question 2**		
	Parallel to wall $v\cos\theta = u\cos\alpha$ Perpendicular to wall $v\sin\theta = eu\sin\alpha$	**M1**	Both
	$\left[\sin(\theta)=\frac{\sqrt{5}}{5},\ \cos(\theta)=\frac{2\sqrt{5}}{5}\right]$ $u\sin(\alpha)=\frac{\sqrt{5}v}{5e},\ u\cos(\alpha)=\frac{2\sqrt{5}v}{5}$	**A1**	
	$u^2=\left[u^2cos^2(\alpha)+u^2sin^2(\alpha)\right]=\frac{4v^2}{5}+\frac{v^2}{5e^2}$	**A1**	AEF, e.g. $\frac{v^2}{5}\left(4+\frac{1}{e^2}\right)$
	$\frac{1}{2}mv^2=\left[\frac{4}{5}\times\frac{1}{2}mu^2=\right]\frac{2}{5}m\frac{v^2}{5}\left(4+\frac{1}{e^2}\right)$	**M1**	Dimensionally correct equation in v. Must have either α or θ, but not both. Must see $\frac{4}{5}$ or $\frac{2}{5}$ on the correct side of the equation.
	$e=\frac{2}{3}$	**A1**	
		5	

Question	Answer	Marks	Guidance
3(a)	At B, $mg\sin\theta = \frac{m4ag}{5a}$	**M1**	Allow cos instead of sin for M1. Do not award until tension $= 0$ used. Mass must be seen. No sign error.
	$\sin\theta = \frac{4}{5}$	**A1**	
		2	
3(b)	At A, $T - mg\cos\theta = \frac{mu^2}{a}$	**B1**	
	Energy $\frac{1}{2}mu^2 - \frac{1}{2}m \times \frac{4ag}{5} = mga(\cos\theta + \sin\theta)$	**M1 A1**	Energy equation with 4 terms, dimensionally correct. Mass must be present, allow sign errors. Must see $\frac{1}{2}$ in the KE terms.
	Solve to find T	**M1**	Complete method leading to an expression in mg for T.
	$T = \frac{21}{5}mg$	**A1**	CWO
		5	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="5">4(a)</td><td>[Mass is proportional to volume]
<table><tr><th></th><th>Volume</th><th>Distance of centre of mass from vertical axis</th><th>Distance of centre of mass from OC</th></tr><tr><td>Hemisphere</td><td>$\frac{2}{3}\pi(2a)^3$</td><td>$2a$</td><td>$-\frac{3}{8}\times 2a$</td></tr><tr><td>Cylinder</td><td>$\pi a^2 d$</td><td>a</td><td>$\frac{1}{2}d$</td></tr><tr><td>Object</td><td>$\frac{2}{3}\pi(2a)^3+\pi a^2 d$</td><td>$\bar{x}$</td><td>$\bar{y}$</td></tr></table>
$\left(\frac{2}{3}\pi(2a)^3+\pi a^2 d\right)\bar{x}=\frac{16}{3}\pi a^3\times 2a+\pi a^2 d\times a$</td><td>M1 A1</td><td>Moments equation, dimensionally correct, correct number of terms. Allow sign errors.</td></tr>
<tr><td>Simplify to $\bar{x}=\frac{32a^2+3ad}{16a+3d}$</td><td>A1</td><td>AG. At least one line of intermediate working.</td></tr>
<tr><td>$\left(\frac{2}{3}\pi(2a)^3+\pi a^2 d\right)\bar{y}=\frac{16}{3}\pi a^3\times\left(-\frac{3}{8}\times 2a\right)+\pi a^2 d\times\frac{1}{2}d$</td><td>M1</td><td>Moments equation, dimensionally correct, correct number of terms. Allow sign errors.</td></tr>
<tr><td>$\bar{y}=\frac{3(d^2-8a^2)}{2(16a+3d)}$</td><td>A1</td><td>AEF</td></tr>
<tr><td></td><td>5</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
4(b)	$\sin\theta = \dfrac{2a - \bar{x}}{2a}$	**B1**	
	$2a \times \dfrac{1}{6} = 2a - \dfrac{32a^2 + 3ad}{16a + 3d}$ $\dfrac{5}{3}(16a + 3d) = (32a + 3d)$	**M1**	Remove fractions
	$d = \dfrac{8}{3}a$	**A1**	
		3	

Question	Answer	Marks	Guidance
5(a)	Hooke's law, $T_1 = \frac{\lambda mg}{a}(x-a)$ or $T_2 = \frac{\lambda mg}{a}\left(\frac{3}{4}x-a\right)$	B1	
	Also, $T_1 = \frac{mv^2}{x}$ and equate $\frac{mv^2}{x} = \frac{\lambda mg}{a}(x-a)$	M1	$v^2 = \frac{\lambda gx(x-a)}{a}$ Dimensionally correct terms.
	Similarly: $\frac{\lambda mg\left(\frac{3x}{4}-a\right)}{a} = \frac{2m\left(\frac{1}{2}v\right)^2}{\frac{3}{4}x}$	M1	$v^2 = \frac{3\lambda gx}{2a}\left(\frac{3}{4}x-a\right)$ Must have $\frac{1}{2}v$ and $\frac{3x}{4}$ on the RHS. Their dimensionally correct T_2.
	Equate expressions for v^2 and solve for x in terms of a.	M1	
	$x = 4a$	A1	WWW
		5	**SC B3** for answer of $4a$ using λ instead of λmg.
5(b)	$\lambda = \frac{a}{xg(x-a)}v^2$ or $\lambda = \frac{2a}{3xg\left(\frac{3}{4}x-a\right)}v^2$ and substitute $x = 4a$, $v = \sqrt{12ag}$	M1	FT their expression for x.
	1	A1	CAO
		2	

Question	Answer	Marks	Guidance
6(a)	$v\dfrac{dv}{dx} = 6v\sqrt{v+9}$ and attempt to separate variables and integrate	M1	
	$2\sqrt{v+9} = 6x + A$	A1	
	$x = 2, v = 72 \quad A = 6$	M1	Use initial condition to find constant.
	$v = 9(x+1)^2 - 9$	A1	Correct, AEF.
		4	
6(b)	$\left[\dfrac{dx}{dt} = 9\left(x^2 + 2x\right) \ , \ \dfrac{dx}{x(x+2)} = 9dt \ \right] \dfrac{1}{2}\left(\dfrac{1}{x} - \dfrac{1}{x+2}\right)dx = 9dt$	M1	Separate variables and write in the form $\left(\dfrac{a}{x} - \dfrac{b}{x+c}\right)dx = dt$
	$\dfrac{1}{2}\ln\left(\dfrac{x}{x+2}\right) = 9t + B$	A1	Integrate, any correct form.
	$t = 0, x = 2 \quad B = \dfrac{1}{2}\ln\dfrac{1}{2}$	M1	Use initial condition to find constant.
	$18t = \ln\left(\dfrac{2x}{x+2}\right) \quad e^{18t} = \dfrac{2x}{x+2}$	M1	Take logarithms
	$x = \dfrac{2e^{18t}}{2 - e^{18t}}$ or $x = \dfrac{2}{2e^{-18t} - 1}$	A1	Any correct form
		5	

Question	Answer	Marks	Guidance
7(a)	$H = 80\sin^2\theta$ or $\dfrac{800\sin^2\theta}{g}$	**B1**	
	$T = 4\sin\theta$ or $\dfrac{40\sin\theta}{g}$	**B1**	
		2	
7(b)	Between $t = T$ and $t = 3$ $\uparrow$ $\frac{1}{4}H = \frac{1}{2}\times10\times(3-T)^2$	**M1 A1**	No extra terms.
	Use results from part **(a)** $\frac{1}{4}80\sin^2\theta = 5(3-4\sin\theta)^2$ $4\sin^2\theta - 8\sin\theta + 3 = 0$	**M1**	Substitute their expressions for H and T from part **(a)** and obtain a quadratic equation in $\sin\theta$ with no more than three terms.
	$\sin\theta = \frac{1}{2}$, $\theta = 30°$	**A1**	Single answer. NFWW.
	Alternative method for question 7 part (b)		
	$\frac{3}{4}H = y(3) = 40\times3\ \sin\theta - \frac{1}{2}\times10\times3^2$	**M1 A1**	$120\ \sin\theta - 45$
	Use results from (a): $\frac{3}{4}80\sin^2\theta = 120\ \sin\theta - 45$ $4\sin^2\theta - 8\sin\theta + 3 = 0$	**M1**	Substitute their expressions for H and T from part **(a)** and obtain a quadratic equation in $\sin\theta$ with no more than three terms.
	$\sin\theta = \frac{1}{2}$, $\theta = 30°$	**A1**	Single answer. NFWW.
		4	

Question	Answer	Marks	Guidance
7(c)	When $t = 3$ speeds $\rightarrow 40\cos\theta$ and $\uparrow 40\sin\theta - 10\times 3$	**B1**	
	Square and add to find square of speed: $v^2 = \left(20\sqrt{3}\right)^2 + \left(-10\right)^2$	**M1**	Must be numerical.
	$v^2 = 1300, \quad v = 10\sqrt{13}$ [= 36.1]	**A1**	
		3	

Cambridge International AS & A Level

CANDIDATE NAME	
CENTRE NUMBER	
CANDIDATE NUMBER	

1370453044

FURTHER MATHEMATICS **9231/33**

Paper 3 Further Mechanics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
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- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\mathrm{m\,s^{-2}}$.

INFORMATION
- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (CE/SG) 316276/1

1 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle α with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed $\sqrt{3ag}$. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where OB makes an angle θ with the upward vertical.

Given that $\cos\alpha = \frac{4}{5}$, find the value of $\cos\theta$. [4]

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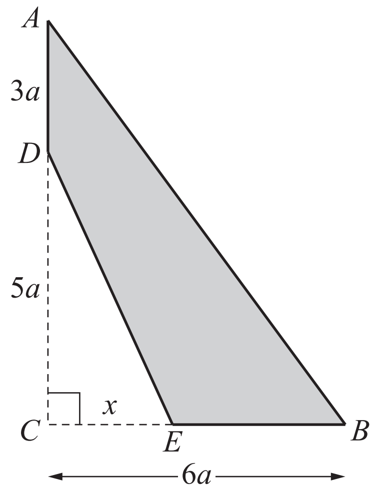
2 One end of a light elastic string, of natural length a and modulus of elasticity λmg, is attached to a fixed point O. The string lies on a smooth horizontal surface. A particle P of mass m is attached to the other end of the string. The particle P is projected in the direction OP. When the length of the string is $\frac{4}{3}a$, the speed of P is $\sqrt{2ag}$. When the length of the string is $\frac{5}{3}a$, the speed of P is $\frac{1}{2}\sqrt{2ag}$.

Find the value of λ. [4]

3



A uniform lamina is in the form of a triangle ABC, with $AC = 8a$, $BC = 6a$ and angle $ACB = 90°$. The point D on AC is such that $AD = 3a$. The point E on CB is such that $CE = x$ (see diagram). The triangle CDE is removed from the lamina.

(a) Find, in terms of a and x, the distance of the centre of mass of the remaining object $ADEB$ from AC. [4]

The object $ADEB$ is on the point of toppling about the point E when the object is in the vertical plane with its edge EB on a smooth horizontal surface.

(b) Find x in terms of a. [3]

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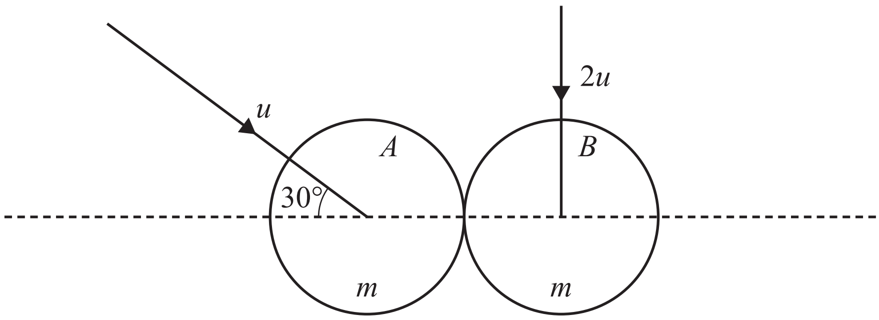

Two identical smooth uniform spheres A and B each have mass m. The two spheres are moving on a smooth horizontal surface when they collide with speeds u and $2u$ respectively. Immediately before the collision, A's direction of motion makes an angle of 30° with the line of centres, and B's direction of motion is perpendicular to the line of centres (see diagram). After the collision, A and B are moving in the same direction. The coefficient of restitution between the spheres is e.

(a) Find the value of e. [5]

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(b) Find the loss in the total kinetic energy of the spheres as a result of the collision. [3]

5 One end of a light elastic string, of natural length $12a$ and modulus of elasticity kmg, is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle moves with constant speed $\frac{3}{2}\sqrt{3ag}$ in a horizontal circle with centre at a distance $12a$ below O. The string is inclined at an angle θ to the downward vertical through O.

(a) Find, in terms of a, the extension of the string. [5]

(b) Find the value of k. [3]

6 A particle of mass m kg falls vertically under gravity, from rest. At time t s, P has fallen x m and has velocity $v\,\mathrm{m\,s^{-1}}$. The only forces acting on P are its weight and a resistance of magnitude $kmgv$ N, where k is a constant.

(a) Find an expression for v in terms of t, g and k. [5]

(b) Given that $k = 0.05$, find, in metres, how far P has fallen when its speed is $12\,\mathrm{m\,s^{-1}}$. [5]

7 The points O and P are on a horizontal plane, a distance 8 m apart. A ball is thrown from O with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. At the same instant, a model aircraft is launched with speed $5\,\mathrm{m\,s^{-1}}$ parallel to the horizontal plane from a point 4 m vertically above P. The model aircraft moves in the same vertical plane as the ball and in the same horizontal direction as the ball. The model aircraft moves horizontally with a constant speed of $5\,\mathrm{m\,s^{-1}}$. After T s, the ball and the model aircraft collide.

(a) Find the value of T. [6]

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(b) Find the direction in which the ball is moving immediately before the collision. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/33**

Paper 3 Further Mechanics 33 **May/June 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Question	Answer	Marks	Guidance
1	$\frac{1}{2}m.3ag - \frac{1}{2}mv^2 = mga(\cos\alpha + \cos\theta)$	**M1**	Energy equation, 4 terms, dimensionally correct, mass must be present, allow sign errors, allow sin in both terms on RHS
	$mg\cos\theta = \frac{mv^2}{a}$	**B1**	N2L, may include tension initially but not awarded until tension = 0 used
	$\frac{3}{2}mag - \frac{1}{2}m.ag\cos\theta = mga\left(\frac{4}{5} + \cos\theta\right)$ $\frac{3}{2}\cos\theta = \frac{7}{10}$	**M1**	Dependent on tension = 0 and on an energy equation, eliminate v^2.
	$\cos\theta = \frac{7}{15}$	**A1**	If no *m* in energy equation and no further errors, award SCB2 for correct final answer
		4	

Question	Answer	Marks	Guidance
2	$\frac{1}{2}m\left(v^2 - \frac{v^2}{4}\right) = \frac{\frac{1}{2}\lambda mg}{a}\left(\left(\frac{2}{3}a\right)^2 - \left(\frac{1}{3}a\right)^2\right)$	**M1**	Kinetic energy = elastic potential energy, 4 terms, dimensionally correct, allow sign errors.
	$\frac{1}{2}m\left(2ag - \frac{1}{2}ag\right) = \frac{\frac{1}{2}\lambda mg}{a}\left(\left(\frac{2}{3}a\right)^2 - \left(\frac{1}{3}a\right)^2\right)$	**A1**	With v substituted.
	Solve [$\frac{3}{4}v^2 = \lambda g\left(\frac{3}{9}a\right)$]	**M1**	Solve to find value for λ dependent on energy equation with 3 or 4 terms
	$\lambda = \frac{9}{2}$	**A1**	SCB2 for $\lambda = \frac{9}{2}mg$ if given λ not used
		4	

Question	Answer	Marks	Guidance
3(a)	[Mass is proportional to area] <table><tr><td></td><td>Area</td><td>Centre of mass from AC</td></tr><tr><td>ABC</td><td>$\frac{1}{2}.6a.8a\ (=24a^2)$</td><td>$2a$</td></tr><tr><td>$DEC$</td><td>$\frac{1}{2}x.5a$</td><td>$\frac{1}{3}x$</td></tr><tr><td>$ADEB$</td><td>$24a^2-\frac{5}{2}xa$</td><td>$\bar{x}$</td></tr></table>	**B1**	All correct for ABC and DEC.
	Moments [about AC] $\bar{x}\left(24a^2-\frac{5}{2}xa\right)=24a^2\times 2a-\frac{1}{3}x\times\frac{5}{2}ax$	**M1**	All moment terms present, dimensionally correct, allow sign error.
		A1	All correct moments about AC.
	$\bar{x}=\dfrac{288a^2-5x^2}{3(48a-5x)}$	**A1**	AEF
		4	

Question	Answer	Marks	Guidance
3(b)	On the point of toppling about E: $\bar{x} = x$, $\dfrac{288a^2 - 5x^2}{3(48a - 5x)} = x$	**B1 FT**	FT *their* expression for $\bar{x}$ from part **(a)**.
	Rearrange to 3-term quadratic: $10x^2 - 144ax + 288a^2 = 0$	**M1**	Allow 3-term inequality.
	$2(5x - 12a)(x - 12a) = 0$, $x = \dfrac{12}{5}a$	**A1**	Single correct answer, no inequality, CWO.
		3	

Question	Answer	Marks	Guidance
4(a)	Let speeds of *A* and *B* along line of centres after collision be V_A and V_B $V_A + V_B = u\cos 30°$ (1)	M1	Allow sign errors, allow missing *m*.
	$-V_A + V_B = eu\cos 30°$ (2)	M1	Signs on LHS must be consistent with (1).
	Speeds perpendicular to line of centres after collision are $u\sin 30°$ and $2u$ Moving in same direction, so $\dfrac{V_A}{u\sin 30°} = \dfrac{V_B}{2u}$ (3)	B1	SOI $V_B = 4V_A$
	Use $V_B = 4V_A$ in (1): $5V_A = u\cos 30°$ From (2): $3V_A = eu\cos 30°$ then Combine to find equation in *e* only.	M1	A complete method to find equation in *e* only
	$e = \frac{3}{5}$	A1	

Question	Answer	Marks	Guidance
4(a)	**Alternative method for question 4(a)**		
	Let speeds of A and B along line of centres after collision be V_A and V_B $V_A + V_B = u\cos 30°$ (1)	M1	Allow sign errors, allow missing m.
	$-V_A + V_B = eu\cos 30°$ (2)	M1	Signs on LHS must be consistent with (1).
	Speeds perpendicular to line of centres after collision are $u\sin 30°$ and $2u$ Moving in same direction, so $\frac{V_A}{u\sin 30°} = \frac{V_B}{2u}$ (3)	B1	SOI $V_B = 4V_A$
	Solve (1) and (2): $V_A = \frac{1}{2}u(1-e)\cos 30°$, $V_B = \frac{1}{2}u(1+e)\cos 30°$ Substitute in (3) to find equation in e only .	M1	Note: $V_A = \frac{u}{10}\sqrt{3}$, $V_B = \frac{4u}{10}\sqrt{3}$
	$e = \frac{3}{5}$	A1	
		5	
4(b)	KE after = $\frac{1}{2}m\left(V_A^2 + \left(\frac{u}{2}\right)^2\right) + \frac{1}{2}m\ ((2u)^2 + V_B^2\ )$	B1	Correct expression for KE for one of the spheres, after collision, with both components.
	KE for A after = $\frac{7}{50}mu^2$ or KE for B after = $\frac{56}{25}mu^2$ or KE loss for A = $\frac{9}{25}mu^2$ or KE **gain** for B = $\frac{6}{25}mu^2$	B1	Implied by total KE after = $\frac{119}{50}mu^2$.
	Total loss in KE = $\frac{3}{25}mu^2$	B1	Term $\frac{1}{2}m(2u)^2$ may be omitted from KE of B before and after.
		3	

Question	Answer	Marks	Guidance
5(a)	$\uparrow T\cos\theta = mg$	**B1**	
	$\rightarrow T\sin\theta = \left[\dfrac{mv^2}{r} =\right] m \times \dfrac{27ag}{4r}$	**B1**	
	$r = 12a\tan\theta$ used	**M1**	
	Divide: $\tan\theta = \dfrac{27}{4\times 12\tan\theta}$, so $\tan\theta = \dfrac{3}{4}$	**M1**	Finds value for $\tan\theta$ OE. Reduces to equation in θ or x, no k.
	$r = 9a$, extension of string $= 3a$	**A1**	
	Alternative method for question 5(a)		
	Let L be stretched length of string. $\uparrow T\cos\theta = mg$	**B1**	Or $T \times \dfrac{12a}{L} = mg$
	$\rightarrow T\sin\theta = \left[\dfrac{mv^2}{r} =\right] m \times \dfrac{27ag}{4r}$	**B1**	
	$r = L\sin\theta$ used	**M1**	
	Use $\cos\theta = \dfrac{12a}{L}$ and $\sin\theta = \dfrac{\left(L^2 - 144a^2\right)^{0.5}}{L}$ and eliminate T.	**M1**	
	$[L^2 - 144a^2 = 81a^2]$ $L = 15a$, extension of string $= 3a$	**A1**	
		5	

Question	Answer	Marks	Guidance
5(b)	Hooke's law: $T = \dfrac{kmg(L-12a)}{12a}$	B1	
	Eliminate T: $\dfrac{kmg(L-12a)}{12a} = \dfrac{mgL}{12a}$	M1	
	$k = \dfrac{L}{L-12a} = 5$	A1	
		3	

Question	Answer	Marks	Guidance
6(a)	$m\dfrac{dv}{dt} = mg(1-kv)$	B1	Mass must be seen at this point or earlier. [SUVAT does **not** apply.]
	$-\dfrac{1}{k}\ln(1-kv) = gt + A$	M1	Separate variables and integrate to logarithm.
		A1	Correct, with constant of integration.
	$t=0, v=0$ $[A=0]$	M1	Use initial condition to evaluate their constant.
	$v = \dfrac{1}{k}\left(1-e^{-kgt}\right)$	A1	Any correct form with v as subject. Final A0 if numerical value of g present.
		5	

Question	Answer	Marks	Guidance
6(b)	$k = 0.05$ and so $\dfrac{dx}{dt} = 20\left(1 - e^{-0.5t}\right)$	*M1	Attempt to integrate if expression contains a term of the form be^{ct}.
	Integrate: $x = 20\left(t + 2e^{-0.5t}\right) + B$	A1	$x = \dfrac{1}{k}\left(t + \dfrac{1}{gk}e^{-kgt}\right) + B$
	$t = 0,\ x = 0\ \left[B = -40\right]$	DM1	Use initial condition to evaluate their constant.
	When $v = 12$, from part **(a)**, $e^{-0.5t} = 1 - 0.05 \times 12 = 0.4$, $t = -2\ln 0.4$	M1	1.83…
	$x = -40\ln 0.4 + 40 \times 0.4 - 40 = 12.7$	A1	$40\ln\dfrac{5}{2} - 24$
	Alternative method for question 6(b)		
	$v\dfrac{dv}{dx} = g(1 - kv)$ leading to $\left(1 - \dfrac{1}{1 - kv}\right)dv = -kgdx$	*M1	Separate variables and write in integrable form
	$v + \dfrac{1}{k}\ln(1 - kv) = -kgx + B$	DM1 A1	Dependent on previous M1. Attempt to integrate.
	$v = 0,\ x = 0\ \left[B = 0\right]$ and $k = 0.05,\ v = 12\ \ 12 + 20\ln 0.4 = -0.5x$	M1	Dependent on both previous M1s. Use initial condition to evaluate their constant and use $v = 12$
	$x = 12.7$	A1	$40\ln\dfrac{5}{2} - 24$
		5	

Question	Answer	Marks	Guidance
7(a)	For aircraft, $d = 5T$	**B1**	
	For ball, $\uparrow 4 = u\sin\theta\, T - \frac{1}{2}\times 10\times T^2$	**B1**	To point of collision.
	For ball, $\rightarrow u\cos\theta\, T = d + 8 = 5T + 8$	**B1**	
	Eliminate u: $4 = \frac{u4}{5}T - \frac{1}{2}\times 10\times T^2$, $u = \frac{5\left(4+5T^2\right)}{4T}$ and $u = \frac{5(5T+8)}{3T}$ $3\left(4+5T^2\right) = 4(5T+8)$	***M1**	Dependent on LHS of second B1 being 4, expression involving only T
	$3T^2 - 4T - 4 = 0$	**DM1**	Dependent on previous M1. Obtain and solve 3-term quadratic.
	$T = 2$	**A1**	Single correct answer.
		6	Note $d - 8$ used leads to $T = \frac{2}{3}$ B1B1B0M1M1A0

Question	Answer	Marks	Guidance
7(b)	$\tan^{-1}\dfrac{(u\sin\theta - 10T)}{u\cos\theta}$	M1	OE Accept 'tan = ……'
	$\tan^{-1}\dfrac{8}{9}$	A1	OE
	Direction is 41.6° below the horizontal	A1	CAO Note: $d-8$ used leads to $20.9°$ above the horizontal.
		3	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

* 5 5 4 8 6 1 2 3 5 1 *

FURTHER MATHEMATICS **9231/41**

Paper 4 Further Probability & Statistics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **12** pages.

DC (LO) 327443

1 The lengths of the leaves of a particular type of tree are normally distributed with mean μ cm. The lengths, x cm, of a random sample of 12 leaves of this type are recorded. The results are summarised as follows.

$$\Sigma x = 91.2 \qquad \Sigma x^2 = 695.8$$

Find a 95% confidence interval for μ. [4]

2 The children at two large schools, P and Q, are all given the same puzzle to solve. A random sample of size 10 is taken from the children at school P. Their individual times to complete the puzzle give a sample mean of 9.12 minutes and an unbiased variance estimate of 2.16 minutes^2. A random sample of size 12 is taken from the children at school Q. Their individual times, x minutes, to complete the puzzle are summarised by

$$\Sigma x = 99.6 \qquad \Sigma(x-\overline{x})^2 = 21.5,$$

where $\overline{x}$ is the sample mean. Times to complete the puzzle are assumed to be normally distributed with the same population variance.

Test at the 5% significance level whether the population mean time taken to complete the puzzle by children at school P is greater than the population mean time taken to complete the puzzle by children at school Q. [8]

3 A random sample of 50 values of the continuous random variable X was taken. These values are summarised in the following table.

Interval	$1 \leqslant x < 1.5$	$1.5 \leqslant x < 2$	$2 \leqslant x < 2.5$	$2.5 \leqslant x < 3$	$3 \leqslant x < 3.5$	$3.5 \leqslant x \leqslant 4$
Observed frequency	3	3	8	11	13	12

It is required to test the goodness of fit of the distribution with probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \dfrac{1}{24}\left(\dfrac{4}{x^2} + x^2\right) & 1 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

The expected frequencies, correct to 4 decimal places, are given in the following table.

Interval	$1 \leqslant x < 1.5$	$1.5 \leqslant x < 2$	$2 \leqslant x < 2.5$	$2.5 \leqslant x < 3$	$3 \leqslant x < 3.5$	$3.5 \leqslant x \leqslant 4$
Expected frequency	4.4271	a	6.1285	8.4549	b	14.9678

(a) Show that $a = 4.6007$ and find the value of b. [3]

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(b) Carry out a goodness of fit test, at the 10% significance level, to test whether f is a satisfactory model for the data. [6]

4 A random sample of 13 technology companies is chosen and the numbers of employees in 2018 and in 2022 are recorded.

Company	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*
Number in 2018	104	19	126	234	970	514	35	149	429	12	86	304	1104
Number in 2022	106	24	127	228	1012	525	32	156	449	24	78	294	1154

A researcher claims that there has been an increase in the median number of employees at technology companies between 2018 and 2022.

(a) Carry out a Wilcoxon matched-pairs signed-rank test, at the 5% significance level, to test whether the data supports this claim. [7]

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The researcher notices that the figures for company G have been recorded incorrectly. In fact, the number of employees in 2018 was 32 and the number of employees in 2022 was 35.

(b) Explain, with numerical justification, whether or not the conclusion of the test in part **(a)** remains the same. [2]

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5 Harry has three coins.

- One coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{3}$.
- The second coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{4}$.
- The third coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{5}$.

The random variable X is the number of heads that Harry obtains when he throws all three coins together.

(a) Find the probability generating function of X. [3]

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Isaac has two fair coins. The random variable Y is the number of heads that Isaac obtains when he throws both of his coins together. The random variable Z is the total number of heads obtained when Harry throws his three coins and Isaac throws his two coins.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [4]

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(c) Use the probability generating function of Z to find $E(Z)$. [2]

6 The continuous random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \frac{3}{28}\left(e^{\frac{1}{2}x} + 4e^{-\frac{1}{2}x}\right) & 0 \leqslant x \leqslant 2\ln 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function of X. [3]

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The random variable Y is defined by $Y = e^{\frac{1}{2}(X)}$.

(b) Find the probability density function of Y. [3]

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(c) Find the 30th percentile of Y. [3]

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(d) Find $\mathrm{E}(Y^4)$. [2]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/41**

Paper 4 Further Probability & Statistics 41 **May/June 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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Question	Answer	Marks	Guidance
1	$\overline{x} = \frac{91.2}{12}[= 7.6]$ $\quad s_x{}^2 = \frac{1}{11}\left(695.8 - \frac{91.2^2}{12}\right)$ $\quad \left[= 0.2436\ \left(\frac{67}{275}\right)\right]$	**M1**	Both.
	CI: $7.6 \pm t_{11}(0.975) \times \sqrt{\frac{0.2436}{12}}$	**M1**	With a *t*-value.
	2.201 seen	**B1**	
	$7.6 \pm 0.3136 = [7.29, 7.91]$	**A1**	
		4	

Question	Answer	Marks	Guidance
2	$\bar{x} = \frac{99.6}{12}\ [= 8.3]\quad s_x^{\ 2} = \frac{21.5}{11}\left[= 1.9545 \text{ or } \frac{43}{22}\right]$	**M1**	Both SOI.
	H_0: $\mu_P = \mu_Q$ H_1: $\mu_P > \mu_Q$	**B1**	
	Pooled estimate = $\frac{9 \times 2.16 + 11 \times 1.9545}{10 + 12 - 2}$	**M1 FT**	FT *their* 1.9545 only
	2.047 (2.05)	**A1**	May be implied, allow 2.04
	$t = \frac{9.12 - 8.3}{\sqrt{2.047} \times \sqrt{\frac{1}{10} + \frac{1}{12}}}$	**M1 FT**	FT their pooled estimate
	1.339 (or 1.338)	**A1**	Accept 1.34
	$'1.339' < 1.725$ Accept H_0 / not significant	**M1**	Compare with correct tabular value 1.725 and conclusion without context. Condone 'reject H_1'.
	Insufficient evidence to suggest that the (mean) time taken at *P* is greater than the (mean) time taken at *Q*.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language (not 'prove', not 'there is no evidence'), no contradictions. A0 if H_0 and H_1 reversed.
		8	

Question	Answer	Marks	Guidance
3(a)	$\int_{1.5}^{2} \frac{1}{24}\left(\frac{4}{x^2}+x^2\right)\mathrm{d}x = \frac{1}{24}\left[-\frac{4}{x}+\frac{x^3}{3}\right]$	**M1**	Integration with correct powers and correct limits seen.
	$a = 50\times 0.092014 = 4.6007$	**A1**	AG $\frac{53}{576}$ or $\frac{1325}{288}$ or 0.092014 or 4.60069 seen
	$b = 11.4211$	**B1**	Or 11.421(0)
		3	
3(b)	Combine first two columns: 6, 9.0278	**M1**	Must be seen, or implied by 1.0155.
	Calculate value of chi-squared: 1.0155 + 0.5715 + 0.7661 + 0.2183 + 0.5885	**M1**	At least 2 correct values (at least 3sf) or expressions seen. Allow columns not combined or three columns combined.
	3.16	**A1**	CWO. Dependent on M1M1 scored. SCB1 for 3.16 with no working
	H_0: f is a good fit for the data H_1: f is not a good fit for the data	**B1**	
	Tabular value of chi-squared: 7.78 '3.16' < 7.78 and accept H_0	**M1**	Correct tabular value: allow correct FT value if columns not combined (9.236) or three columns combined (6.251).
	Insufficient evidence to suggest that f is not a good fit for the data Condone: sufficient evidence to suggest that f is a good fit for the data.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language (not 'prove', not 'there is no evidence'), no contradictions. A0 if hypotheses reversed
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="7">4(a)</td><td rowspan="2"><table><tr><td>2</td><td>5</td><td>1</td><td>–6</td><td>42</td><td>11</td><td>–3</td><td>7</td><td>20</td><td>12</td><td>–8</td><td>–10</td><td>50</td></tr><tr><td>2</td><td>4</td><td>1</td><td>–5</td><td>12</td><td>9</td><td>–3</td><td>6</td><td>11</td><td>10</td><td>–7</td><td>–8</td><td>13</td></tr></table></td><td>M1</td><td>Differences (allow up to 3 errors)</td></tr>
<tr><td>A1</td><td>Correct rank order, ignore signs</td></tr>
<tr><td>$[P = 68,]\quad Q = 23$</td><td>A1</td><td>CWO</td></tr>
<tr><td>H_0: difference in population medians = 0
H_1: population median in 2022 > population median in 2018</td><td>B1</td><td>‘Population’ required. Accept use of m, not μ.
Do not accept ‘difference between population medians > 0’ without 2022, 2018 OE specified</td></tr>
<tr><td>Critical value 21</td><td>B1</td><td></td></tr>
<tr><td>‘23’ > ‘21’ and accept H_0</td><td>M1</td><td>Compare their 23 with their 21 and FT conclusion. Their 23 must be less than 46.</td></tr>
<tr><td>Insufficient evidence to support researcher’s belief / insufficient evidence that the (median) number of employees in 2022 is greater than the (median) number of employees in 2018</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language (not ‘prove’, not ‘there is no evidence’), no contradictions.
A0 if hypotheses reversed</td></tr>
<tr><td></td><td></td><td>7</td><td></td></tr>
<tr><td rowspan="3">4(b)</td><td>Rank for G will now be +3, giving Q = 20 which is < 21</td><td>M1</td><td>Must include numbers, ft their 21 from part (a) and ft their 20 (23 – 3)</td></tr>
<tr><td>Change in conclusion</td><td>A1</td><td>CWO
Condone ‘reject H_0’ OE.</td></tr>
<tr><td></td><td>2</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
5(a)	$P(3H) = \frac{1}{60}$ $P(2H) = \frac{9}{60}$ $P(1H) = \frac{26}{60}$ $P(0H) = \frac{24}{60}$	**B1**	Two probabilities correct, seen anywhere.
	$G_X(t) = \frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3$	**M1**	Cubic polynomial with 4 probabilities as coefficients with the correct powers of *t* from their working. Equivalent forms are acceptable.
		A1	Correct, AEF.
		3	
5(b)	$G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$	**B1**	
	$G_Z(t) = \left(\frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3\right)\left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)$	**M1**	Attempt to multiply their two PGFs.
	$\frac{1}{240}\left(24 + 74t + 85t^2 + 45t^3 + 11t^4 + t^5\right)$	**M1**	Obtain quintic expression and collect terms
		A1	Correct, AEF.
		4	
5(c)	Differentiate: $G'_Z(t) = \frac{1}{240}\left(74 + 170t + 135t^2 + 44t^3 + 5t^4\right)$	**M1**	Differentiate *their* G.
	$G'_Z(1) = \frac{428}{240} = \frac{107}{60} = 1.78$	**A1**	Any correct form.
		2	

Question	Answer	Marks	Guidance
6(a)	$\mathrm{F}(x)=\frac{3}{28}\left(2\mathrm{e}^{\frac{1}{2}x}-8\mathrm{e}^{-\frac{1}{2}x}(+c)\right)$	**M1**	+ c not required.
	$\mathrm{F}(x)=\frac{3}{28}\left(2\mathrm{e}^{\frac{1}{2}x}-8\mathrm{e}^{-\frac{1}{2}x}\right)+\frac{9}{14}$	**A1**	
	Correct ranges including $0 \leqslant x \leqslant 2\ln 3$ associated with *their* $\mathrm{F}(x)$ $\mathrm{F}(x)=0$ for $x<0$ and 1 for $x>2\ln 3$	**B1**	No gaps in range.
		3	
6(b)	$\mathrm{G}(y)=\frac{3}{28}\left(2y-\frac{8}{y}+6\right)$	**M1**	y substituted into *their* F.
	$1 \le y \le 3$	**B1**	Seen anywhere, condone $1 \leqslant y \leqslant \mathrm{e}^{\ln 3}$.
	$\mathrm{g}(y)=\frac{3}{28}\left(2+\frac{8}{y^2}\right)$	**A1**	Correct expression, not containing logs.
		3	
6(c)	$\frac{3}{28}\left(2y-\frac{8}{y}+6\right)=\frac{3}{10}$	**M1**	*Their* G(y) $=\frac{3}{10}$
	$5y^2+8y-20=0$	**M1**	Obtain from G(y) and solve quadratic equation to find y.
	$y=1.35$	**A1**	Single correct answer.
		3	

Question	Answer	Marks	Guidance
6(d)	$\text{E}(Y^4) = \int_1^3 \frac{3}{28}\left(2+\frac{8}{y^2}\right)y^4\text{d}y = \int_1^3 \frac{3}{28}\left(2y^4+8y^2\right)\text{d}y = \frac{3}{28}\left[\frac{2}{5}y^5+\frac{8}{3}y^3\right]$	**1**	Integrate $y^4 \times$ *their* g(y). Limits must be 1 and 3.
	17.8	**A1**	$\frac{89}{5}$
		2	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

3670167567

FURTHER MATHEMATICS **9231/42**

Paper 4 Further Probability & Statistics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **12** pages.

DC (CJ) 316279/3

1 The lengths of the leaves of a particular type of tree are normally distributed with mean μ cm. The lengths, x cm, of a random sample of 12 leaves of this type are recorded. The results are summarised as follows.

$$\sum x = 91.2 \qquad \sum x^2 = 695.8$$

Find a 95% confidence interval for μ. [4]

2 The children at two large schools, P and Q, are all given the same puzzle to solve. A random sample of size 10 is taken from the children at school P. Their individual times to complete the puzzle give a sample mean of 9.12 minutes and an unbiased variance estimate of 2.16 minutes^2. A random sample of size 12 is taken from the children at school Q. Their individual times, x minutes, to complete the puzzle are summarised by

$$\Sigma x = 99.6 \qquad \Sigma(x-\overline{x})^2 = 21.5,$$

where $\overline{x}$ is the sample mean. Times to complete the puzzle are assumed to be normally distributed with the same population variance.

Test at the 5% significance level whether the population mean time taken to complete the puzzle by children at school P is greater than the population mean time taken to complete the puzzle by children at school Q. [8]

3 A random sample of 50 values of the continuous random variable X was taken. These values are summarised in the following table.

Interval	$1 \leqslant x < 1.5$	$1.5 \leqslant x < 2$	$2 \leqslant x < 2.5$	$2.5 \leqslant x < 3$	$3 \leqslant x < 3.5$	$3.5 \leqslant x \leqslant 4$
Observed frequency	3	3	8	11	13	12

It is required to test the goodness of fit of the distribution with probability density function f given by

$$f(x) = \begin{cases} \dfrac{1}{24}\left(\dfrac{4}{x^2} + x^2\right) & 1 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

The expected frequencies, correct to 4 decimal places, are given in the following table.

Interval	$1 \leqslant x < 1.5$	$1.5 \leqslant x < 2$	$2 \leqslant x < 2.5$	$2.5 \leqslant x < 3$	$3 \leqslant x < 3.5$	$3.5 \leqslant x \leqslant 4$
Expected frequency	4.4271	a	6.1285	8.4549	b	14.9678

(a) Show that $a = 4.6007$ and find the value of b. [3]

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(b) Carry out a goodness of fit test, at the 10% significance level, to test whether f is a satisfactory model for the data. [6]

4 A random sample of 13 technology companies is chosen and the numbers of employees in 2018 and in 2022 are recorded.

Company	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*
Number in 2018	104	19	126	234	970	514	35	149	429	12	86	304	1104
Number in 2022	106	24	127	228	1012	525	32	156	449	24	78	294	1154

A researcher claims that there has been an increase in the median number of employees at technology companies between 2018 and 2022.

(a) Carry out a Wilcoxon matched-pairs signed-rank test, at the 5% significance level, to test whether the data supports this claim. [7]

The researcher notices that the figures for company G have been recorded incorrectly. In fact, the number of employees in 2018 was 32 and the number of employees in 2022 was 35.

(b) Explain, with numerical justification, whether or not the conclusion of the test in part **(a)** remains the same. [2]

5 Harry has three coins.

- One coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{3}$.
- The second coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{4}$.
- The third coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{5}$.

The random variable X is the number of heads that Harry obtains when he throws all three coins together.

(a) Find the probability generating function of X. [3]

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Isaac has two fair coins. The random variable Y is the number of heads that Isaac obtains when he throws both of his coins together. The random variable Z is the total number of heads obtained when Harry throws his three coins and Isaac throws his two coins.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [4]

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(c) Use the probability generating function of Z to find $E(Z)$. [2]

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6 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \frac{3}{28}\left(e^{\frac{1}{2}x} + 4e^{-\frac{1}{2}x}\right) & 0 \leqslant x \leqslant 2\ln 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function of X. [3]

The random variable Y is defined by $Y = e^{\frac{1}{2}(X)}$.

(b) Find the probability density function of Y. [3]

(c) Find the 30th percentile of Y. [3]

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(d) Find $\mathrm{E}(Y^4)$. [2]

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FURTHER MATHEMATICS **9231/42**

Paper 4 Further Probability & Statistics 42 **May/June 2023**

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WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$\bar{x} = \frac{91.2}{12}\left[= 7.6\right] \qquad s_x{}^2 = \frac{1}{11}\left(695.8 - \frac{91.2^2}{12}\right) \quad \left[= 0.2436\ \left(\frac{67}{275}\right)\right]$	**M1**	Both.
	CI: $7.6 \pm t_{11}(0.975) \times \sqrt{\frac{0.2436}{12}}$	**M1**	With a *t*-value.
	2.201 seen	**B1**	
	$7.6 \pm 0.3136 = [7.29, 7.91]$	**A1**	
		4	

Question	Answer	Marks	Guidance
2	$\bar{x} = \frac{99.6}{12}\ [= 8.3]\quad s_x^{\,2} = \frac{21.5}{11}\left[= 1.9545 \text{ or } \frac{43}{22}\right]$	**M1**	Both SOI.
	H_0: $\mu_P = \mu_Q$ H_1: $\mu_P > \mu_Q$	**B1**	
	Pooled estimate = $\frac{9 \times 2.16 + 11 \times 1.9545}{10 + 12 - 2}$	**M1 FT**	FT *their* 1.9545 only
	2.047 (2.05)	**A1**	May be implied, allow 2.04
	$t = \frac{9.12 - 8.3}{\sqrt{2.047} \times \sqrt{\frac{1}{10} + \frac{1}{12}}}$	**M1 FT**	FT their pooled estimate
	1.339 (or 1.338)	**A1**	Accept 1.34
	$'1.339' < 1.725$ Accept H_0 / not significant	**M1**	Compare with correct tabular value 1.725 and conclusion without context. Condone 'reject H_1'.
	Insufficient evidence to suggest that the (mean) time taken at *P* is greater than the (mean) time taken at *Q*.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language (not 'prove', not 'there is no evidence'), no contradictions. A0 if H_0 and H_1 reversed.
		8	

Question	Answer	Marks	Guidance
3(a)	$\int_{1.5}^{2} \frac{1}{24}\left(\frac{4}{x^2}+x^2\right)dx = \frac{1}{24}\left[-\frac{4}{x}+\frac{x^3}{3}\right]$	**M1**	Integration with correct powers and correct limits seen.
	$a = 50 \times 0.092014 = 4.6007$	**A1**	AG $\frac{53}{576}$ or $\frac{1325}{288}$ or 0.092014 or 4.60069 seen
	$b = 11.4211$	**B1**	Or 11.421(0)
		3	
3(b)	Combine first two columns: 6, 9.0278	**M1**	Must be seen, or implied by 1.0155.
	Calculate value of chi-squared: 1.0155 + 0.5715 + 0.7661 + 0.2183 + 0.5885	**M1**	At least 2 correct values (at least 3sf) or expressions seen. Allow columns not combined or three columns combined.
	3.16	**A1**	CWO. Dependent on M1M1 scored. SCB1 for 3.16 with no working
	H_0: f is a good fit for the data H_1: f is not a good fit for the data	**B1**	
	Tabular value of chi-squared: 7.78 ‘3.16’ < 7.78 and accept H_0	**M1**	Correct tabular value: allow correct FT value if columns not combined (9.236) or three columns combined (6.251).
	Insufficient evidence to suggest that f is not a good fit for the data Condone: sufficient evidence to suggest that f is a good fit for the data.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language (not ‘prove’, not ‘there is no evidence’), no contradictions. A0 if hypotheses reversed
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="7">4(a)</td><td rowspan="2"><table><tr><td>2</td><td>5</td><td>1</td><td>–6</td><td>42</td><td>11</td><td>–3</td><td>7</td><td>20</td><td>12</td><td>–8</td><td>–10</td><td>50</td></tr><tr><td>2</td><td>4</td><td>1</td><td>–5</td><td>12</td><td>9</td><td>–3</td><td>6</td><td>11</td><td>10</td><td>–7</td><td>–8</td><td>13</td></tr></table></td><td>M1</td><td>Differences (allow up to 3 errors)</td></tr>
<tr><td>A1</td><td>Correct rank order, ignore signs</td></tr>
<tr><td>$[P = 68,]\quad Q = 23$</td><td>A1</td><td>CWO</td></tr>
<tr><td>H_0: difference in population medians = 0
H_1: population median in 2022 > population median in 2018</td><td>B1</td><td>‘Population’ required. Accept use of m, not μ.
Do not accept ‘difference between population medians > 0’ without 2022, 2018 OE specified</td></tr>
<tr><td>Critical value 21</td><td>B1</td><td></td></tr>
<tr><td>‘23’ > ‘21’ and accept H_0</td><td>M1</td><td>Compare their 23 with their 21 and FT conclusion. Their 23 must be less than 46.</td></tr>
<tr><td>Insufficient evidence to support researcher’s belief / insufficient evidence that the (median) number of employees in 2022 is greater than the (median) number of employees in 2018</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language (not ‘prove’, not ‘there is no evidence’), no contradictions.
A0 if hypotheses reversed</td></tr>
<tr><td></td><td></td><td>7</td><td></td></tr>
<tr><td rowspan="3">4(b)</td><td>Rank for G will now be +3, giving $Q = 20$ which is < 21</td><td>M1</td><td>Must include numbers, ft their 21 from part (a) and ft their 20 (23 – 3)</td></tr>
<tr><td>Change in conclusion</td><td>A1</td><td>CWO
Condone ‘reject H_0’ OE.</td></tr>
<tr><td></td><td>2</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
5(a)	P(3H) = $\frac{1}{60}$ P(2H) = $\frac{9}{60}$ P(1H) = $\frac{26}{60}$ P(0H) = $\frac{24}{60}$	**B1**	Two probabilities correct, seen anywhere.
	$G_X(t) = \frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3$	**M1**	Cubic polynomial with 4 probabilities as coefficients with the correct powers of *t* from their working. Equivalent forms are acceptable.
		A1	Correct, AEF.
		3	
5(b)	$G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$	**B1**	
	$G_Z(t) = \left(\frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3\right)\left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)$	**M1**	Attempt to multiply their two PGFs.
	$\frac{1}{240}\left(24 + 74t + 85t^2 + 45t^3 + 11t^4 + t^5\right)$	**M1**	Obtain quintic expression and collect terms
		A1	Correct, AEF.
		4	
5(c)	Differentiate: $G'_Z(t) = \frac{1}{240}\left(74 + 170t + 135t^2 + 44t^3 + 5t^4\right)$	**M1**	Differentiate *their* G.
	$G'_Z(1) = \frac{428}{240} = \frac{107}{60} = 1.78$	**A1**	Any correct form.
		2	

Question	Answer	Marks	Guidance
6(a)	$\mathrm{F}(x) = \frac{3}{28}\left(2e^{\frac{1}{2}x} - 8e^{-\frac{1}{2}x} (+c)\right)$	**M1**	$+ c$ not required.
	$\mathrm{F}(x) = \frac{3}{28}\left(2e^{\frac{1}{2}x} - 8e^{-\frac{1}{2}x}\right) + \frac{9}{14}$	**A1**	
	Correct ranges including $0 \leqslant x \leqslant 2\ln 3$ associated with *their* $\mathrm{F}(x)$ $\mathrm{F}(x) = 0$ for $x < 0$ and 1 for $x > 2\ln 3$	**B1**	No gaps in range.
		3	
6(b)	$\mathrm{G}(y) = \frac{3}{28}\left(2y - \frac{8}{y} + 6\right)$	**M1**	y substituted into *their* F.
	$1 \le y \le 3$	**B1**	Seen anywhere, condone $1 \leqslant y \leqslant e^{\ln 3}$.
	$\mathrm{g}(y) = \frac{3}{28}\left(2 + \frac{8}{y^2}\right)$	**A1**	Correct expression, not containing logs.
		3	
6(c)	$\frac{3}{28}\left(2y - \frac{8}{y} + 6\right) = \frac{3}{10}$	**M1**	*Their* $\mathrm{G}(y) = \frac{3}{10}$
	$5y^2 + 8y - 20 = 0$	**M1**	Obtain from $\mathrm{G}(y)$ and solve quadratic equation to find y.
	$y = 1.35$	**A1**	Single correct answer.
		3	

Question	Answer	Marks	Guidance
6(d)	$\mathrm{E}(Y^4) = \int_1^3 \frac{3}{28}\left(2+\frac{8}{y^2}\right)y^4\mathrm{d}y = \int_1^3 \frac{3}{28}\left(2y^4+8y^2\right)\mathrm{d}y = \frac{3}{28}\left[\frac{2}{5}y^5+\frac{8}{3}y^3\right]$	1	Integrate $y^4 \times$ *their* g(y). Limits must be 1 and 3.
	17.8	**A1**	$\frac{89}{5}$
		2	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

3811535200

FURTHER MATHEMATICS **9231/43**

Paper 4 Further Probability & Statistics **May/June 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \frac{1}{6}\left(x^{-\frac{1}{3}} - x^{-\frac{2}{3}}\right) & 1 \leqslant x \leqslant 27, \\ 0 & \text{otherwise}. \end{cases}$$

(a) Find the cumulative distribution function of X. [3]

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The random variable Y is defined by $Y = X^{\frac{1}{3}}$.

(b) Find the probability density function of Y. [3]

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(c) Find the exact value of the median of Y. [2]

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2 Shane is studying the lengths of the tails of male red kangaroos. He takes a random sample of 14 male red kangaroos and measures the length of the tail, x m, for each kangaroo. He then calculates a 90% confidence interval for the population mean tail length, μ m, of male red kangaroos. He assumes that the tail lengths are normally distributed and finds that $1.11 \leqslant \mu \leqslant 1.14$.

Find the values of $\sum x$ and $\sum x^2$ for this sample. [6]

3 A large number of students took two test papers in mathematics. The teacher believes that the marks obtained in Paper 1 will be higher than the marks obtained in Paper 2. She chooses a random sample of 9 students and compares their marks. The marks are shown in the table.

Student	A	B	C	D	E	F	G	H	I
Paper 1	46	73	55	64	86	42	66	68	60
Paper 2	41	66	61	63	90	40	58	42	70

(a) Carry out a Wilcoxon matched-pairs signed-rank test, at the 5% significance level, to test whether the data supports the teacher's belief. [7]

(b) State an assumption that you have made in carrying out the test in part **(a)**. [1]

4 An inspector is checking the lengths of metal rods produced by two machines, X and Y. These rods should be of the same length, but the inspector suspects that those made by machine X are shorter, on average, than those made by machine Y. The inspector chooses a random sample of 80 rods made by machine X and a random sample of 60 rods made by machine Y. The lengths of these rods are x cm and y cm respectively. Her results are summarised as follows.

$$\sum x = 164.0 \quad \sum x^2 = 338.1 \quad \sum y = 124.8 \quad \sum y^2 = 261.1$$

(a) Test at the 10% significance level whether the data supports the inspector's suspicion. [8]

(b) Give a reason why it is not necessary to make any assumption about the distributions of the lengths of the rods. [1]

5 The random variable X has probability generating function $\mathrm{G}_X(t)$ given by

$$\mathrm{G}_X(t) = k(1+3t+4t^2),$$

where k is a constant.

(a) Show that $\mathrm{E}(X) = \frac{11}{8}$. [3]

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The random variable Y has probability generating function $\mathrm{G}_Y(t)$ given by

$$\mathrm{G}_Y(t) = \tfrac{1}{3}t^2(1+2t).$$

The random variables X and Y are independent and $Z = X+Y$.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [2]

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(c) Use your answer to part **(b)** to find the value of Var(Z). [3]

(d) Write down the most probable value of Z. [1]

6 A scientist is investigating whether the ability to remember depends on age. A random sample of 150 students in different age groups is chosen. Each student is shown a set of 20 objects for thirty seconds and then asked to list as many as they can remember. The students are graded *A* or *B* according to how many objects they remembered correctly: grade *A* for 16 or more correct and grade *B* for fewer than 16 correct. The results are shown in the table.

	Age of students		
	11−12 years	13−14 years	15−16 years
Grade *A*	25	16	19
Grade *B*	28	45	17

(a) Carry out a χ^2-test at the 2.5% significance level to test whether grade is independent of age of student. [7]

The scientist decides instead to use three grades: grade A for 16 or more correct, grade B for 10 to 15 correct and grade C for fewer than 10 correct. The results are shown in the following table.

	Age of students		
	11−12 years	13−14 years	15−16 years
Grade A	25	16	19
Grade B	12	27	11
Grade C	16	18	6

With this second set of data, the test statistic is calculated as 10.91.

(b) Complete the χ^2-test at the 2.5% significance level for this second set of data. [2]

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(c) State, with a reason, whether you would prefer to use the result from part **(a)** or part **(b)** to investigate whether the ability to remember depends on age. [1]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/43**

Paper 4 Further Probability & Statistics 43 **May/June 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$F(x)=\frac{1}{6}\left[\frac{3}{2}x^{\frac{2}{3}}-3x^{\frac{1}{3}}\right]+c$	**M1**	Integrate f(x), + c not required, at least one power correct.
	$\frac{1}{4}x^{\frac{2}{3}}-\frac{1}{2}x^{\frac{1}{3}}+\frac{1}{4}$	**A1**	AEF
	Correct ranges including $1 \leqslant x \leqslant 27$ associated with their $F(x)$ $F(x)=0$ for $x<1$ and $F(x)=1$ for $x>27$	**A1**	No gaps.
		3	
1(b)	$G(y)=\frac{1}{4}\left(y^2-2y+1\right)$	**M1**	Using $y=x^{\frac{1}{3}}$ in *their* F(x).
	$g(y)=\frac{1}{2}(y-1)$	**A1**	Correct, AEF, 0 otherwise not required.
	For $1 \leqslant y \leqslant 3$	**B1**	Seen anywhere, correct variable.
		3	
1(c)	$\frac{1}{4}\left(y^2-2y+1\right)=\frac{1}{2}, \quad (y-1)^2=2$ OE	**M1**	Equate their G(y) to ½ and solve to find y.
	$[y=] \quad 1+\sqrt{2}$	**A1**	Only.
		2	

Question	Answer	Marks	Guidance
2	$\bar{x}+1.771\sqrt{\frac{s^2}{14}}=1.14$ or $\bar{x}-1.771\sqrt{\frac{s^2}{14}}=1.11$	**M1**	SOI Allow incorrect t-value, not z-value.
	[Add: $\bar{x}=\frac{1}{2}(1.14+1.11)$ $=1.125$] $\sum x=15.75$	**B1**	Does not depend on use of a t-value.
	Subtract or substitute: $\sqrt{\frac{s^2}{14}}=\frac{1}{2}\left(\frac{1.14-1.11}{1.771}\right)$	**M1**	Allow incorrect t-value, but not a z-value.
	$s^2=0.00100[4]$ or $s=0.0316[9]$	**A1**	$\frac{450}{448063}$, implied by correct final answer.
	$\sum x^2=13s^2+\frac{\left(\sum x\right)^2}{14}$	**M1**	OE
	$\sum x^2=17.7(3)$	**A1**	CWO
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="2">3(a)</td><td rowspan="2"><table><tr><td>5</td><td>7</td><td>-6</td><td>1</td><td>–4</td><td>2</td><td>8</td><td>26</td><td>–10</td></tr><tr><td>4</td><td>6</td><td>-5</td><td>1</td><td>–3</td><td>2</td><td>7</td><td>9</td><td>–8</td></tr></table></td><td>M1</td><td>Differences, allow at most 2 errors.</td></tr>
<tr><td>A1</td><td>Correct rank order, ignore signs.</td></tr>
<tr><td></td><td>[P = 29] Q = 16</td><td>A1</td><td>Condone P not excluded.</td></tr>
<tr><td></td><td>H_0: population medians equal or $m_1 = m_2$
H_1: population median X > population median Y or $m_1 > m_2$</td><td>B1</td><td>‘Population’ required. Accept use of m, not μ.
Do not accept ‘difference between population medians > 0’ without X, Y OE specified.</td></tr>
<tr><td></td><td>Critical value = 8</td><td>B1</td><td></td></tr>
<tr><td></td><td>16 > 8 and accept H_o / not significant</td><td>M1</td><td>Compare their ‘16’ with their ‘8’ and conclusion.
Their ‘16’ must be less than 23. Ignore their hypotheses.
Condone ‘reject H_1’.</td></tr>
<tr><td></td><td>Insufficient evidence to support teacher’s belief
or
insufficient evidence that the marks/median in Paper 1 are/is higher than the marks/median in Paper 2</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language (not ‘prove’, not ‘there is no evidence’), no contradictions.
e.g. Proves that the teacher is incorrect scores A0. A0 if hypotheses wrong way round.</td></tr>
<tr><td></td><td></td><td>7</td><td></td></tr>
<tr><td>3(b)</td><td>The population differences are symmetrical (about the median difference)</td><td>B1</td><td>Words in bold, or their equivalent, are required.</td></tr>
<tr><td></td><td></td><td>1</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
4(a)	H_0: $\mu_x = \mu_y$ H_1: $\mu_x < \mu_y$	B1	
	$s_x^{\ 2} = \frac{1}{79}\left(338.1 - \frac{164^2}{80}\right)[= 0.02405]$ and $s_y^{\ 2} = \frac{1}{59}\left(261.1 - \frac{124.8^2}{60}\right)[= 0.02569]$	B1	Both. Implied by $\frac{19}{790}$, $\frac{379}{14750}$ or 3sf.
	$s^2 = \frac{0.02405}{80} + \frac{0.02569}{60}$	M1	
	$s^2 = 0.0007289$ or $s = 0.026998$ or 0.0270	A1	Implied by $z = -1.11$
	$z = \dfrac{\frac{164.0}{80} - \frac{124.8}{60}}{s}$	M1	FT *their* value for *s*.
	-1.11	A1	
	'1.11' < 1.282 Accept H_0/ not significant	M1	Compare with correct *z*-value 1.282 and consistent signs. Condone 'reject H_1'. Using probabilities, P(Z >1.11) = 0.1333 > 0.1.
	Insufficient evidence to support the inspector's suspicion/ Insufficient evidence that the (mean) lengths of rods from machine *X* are shorter than the (mean) lengths of rods from machine *Y*	A1	Correct conclusion in context, following correct work, level of uncertainty in language (not 'prove', not 'there is no evidence'), no contradictions. e.g. proves that the inspector is incorrect scores A0 A0 if hypotheses wrong way round.
		8	

Question	Answer	Marks	Guidance
4(b)	Large sample sizes OR central limit theorem applies.	**B1**	
		1	

Question	Answer	Marks	Guidance
5(a)	$\left[G_X(t)=k\left(1+3t+4t^2\right)\right]$ $k\left(1+3+4\right)=1,\ k=\frac{1}{8}$	**B1**	Working required.
	$G'_X(t) = k\left(3+8t\right)$,	**M1**	Or $\sum px = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{4}{8}\left[=\frac{11}{8}\right]$
	$E(X) = G'_X(1) = \frac{11}{8}$	**A1**	AG Evidence of using $t=1$ or $\sum px$ required. CWO
		3	
5(b)	$G_Z(t) = \frac{1}{8}\left(1+3t+4t^2\right)\times\frac{1}{3}t^2\left(1+2t\right)$	**M1**	Multiply the two PGFs to obtain a single polynomial of degree 5.
	$\frac{1}{24}\left(t^2+5t^3+10t^4+8t^5\right)$	**A1**	May have t^2 as a factor.
		2	

Question	Answer	Marks	Guidance
5(c)	$G'_Z(t) = \frac{1}{24}\left(2t + 15t^2 + 40t^3 + 40t^4\right)$ $G''_Z(t) = \frac{1}{24}\left(2 + 30t + 120t^2 + 160t^3\right)$	**M1**	Differentiate twice, allow one slip.
	$\text{Var}(X) = G''_Z(1) + G'_Z(1) - (G'_Z(1))^2 = \frac{1}{24}(312) + \frac{97}{24} - \left(\frac{97}{24}\right)^2$	**M1**	Use correct formula.
	0.707	**A1**	$\frac{407}{576}$
		3	
5(d)	$(Z =)4$	**B1 FT**	FT *their* final polynomial in part **(b)**
		1	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="7">6(a)</td><td>H_0: grade is independent of age
H_1: grade is not independent of age</td><td>B1</td><td>Condone ‘Ability to remember’ instead of ‘grade’.</td></tr>
<tr><td>Calculate expected values, shown in table in bold
<table>
<tr><td>25 21.2</td><td>16 24.4</td><td>19 14.4</td><td>60</td></tr>
<tr><td>28 31.8</td><td>45 36.6</td><td>17 21.6</td><td>90</td></tr>
<tr><td>53</td><td>61</td><td>36</td><td>150</td></tr>
</table></td><td>M1

A1</td><td>At least 2 correct values or expressions.

6 correct values or expressions.</td></tr>
<tr><td>Calculate chi-squared values:
0.6811 + 2.8918 + 1.4694 + 0.4541 + 1.9279 + 0.9796</td><td>M1</td><td>At least 2 correct values (at least 3sf) or expressions seen.</td></tr>
<tr><td>8.40[4]</td><td>A1</td><td></td></tr>
<tr><td>Tabular value, 2 degrees of freedom = 7.378
‘8.404’ > 7.378 and reject H_0/significant</td><td>M1</td><td>Compare their value with 7.378 and conclusion without context.
Condone ‘accept H_1’.</td></tr>
<tr><td>Sufficient evidence to suggest that grade is not independent of age.</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language (not ‘prove’, not ‘there is no evidence’), no contradictions.
A0 if hypotheses wrong way round.</td></tr>
<tr><td></td><td>7</td><td></td></tr>
<tr><td rowspan="3">6(b)</td><td>Value, 4 degrees of freedom = 11.14
10.91 < 11.14 Accept H_0/not significant</td><td>M1</td><td>Condone ‘reject H_1’.</td></tr>
<tr><td>Insufficient evidence to suggest that grade is not independent of age.</td><td>A1</td><td>CAO</td></tr>
<tr><td></td><td>2</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
6(c)	For example, result in part **(b)** because the table contains more information e.g. More degrees of freedom, more groups, more detail	**B1**	Any appropriate comment to support part **(a)** or part **(b)**. Allow ‘more specific’. Not ‘more data’ or ‘more accurate’ on its own.
		1	

Cambridge Assessment International Education

Grade thresholds – November 2023

Cambridge International A Level Further Mathematics (9231)

Grade thresholds taken for Syllabus 9231 (Further Mathematics) in the November 2023 examination.

		Minimum raw mark required for grade:				
	Maximum raw mark available	A	B	C	D	E
Component 11	75	50	40	34	28	22
Component 12	75	50	40	34	28	22
Component 13	75	50	40	34	28	22
Component 21	75	62	54	45	37	27
Component 22	75	66	58	50	43	34
Component 23	75	62	54	45	37	27
Component 31	50	27	21	18	15	12
Component 32	50	31	23	20	17	14
Component 33	50	27	21	18	15	12
Component 41	50	38	34	27	20	14
Component 42	50	42	38	32	26	20
Component 43	50	38	34	27	20	14

Grade A* does not exist at the level of an individual component.

The overall thresholds for the different grades were set as follows.

Option	Maximum mark after weighting	Combination of components	A*	A	B	C	D	E
AX	250	11, 21, 31, 41	205	177	149	124	99	75
AY	250	12, 22, 32, 42	219	189	159	136	113	90
AZ	250	13, 23, 33, 43	205	177	149	124	99	75
BQ	250	21, 41, 85	220	191	162	133	104	76
BR	250	22, 42, 86	222	195	166	139	112	85
BY	250	22, 42, 85	224	199	170	143	116	89
CQ	250	21, 31, 88	215	187	159	132	105	79
CR	250	22, 32, 89	220	191	161	135	109	84
CY	250	22, 32, 88	222	195	165	139	113	88
DX	250	21, 41, 94	198	172	146	120	95	70

Grade thresholds continued

Cambridge A Level Further Mathematics (9231)

Option	Maximum mark after weighting	Combination of components	A*	A	B	C	D	E
DY	250	22, 42, 95	206	180	154	130	106	83
DZ	250	23, 43, 96	198	172	146	120	95	70
EX	250	21, 31, 97	201	173	145	119	94	69
EY	250	22, 32, 98	211	181	151	126	102	78
EZ	250	23, 33, 99	201	173	145	119	94	69
FY	250	12, 22, 32, 42	219	189	159	136	113	90
FZ	250	13, 23, 33, 43	205	177	149	124	99	75
HQ	250	11, 21, 31, 41	205	177	149	124	99	75
S1	125	11, 31	–	77	61	52	43	34
S2	125	11, 41	–	88	74	61	48	36
S3	125	12, 32	–	81	63	54	45	36
S4	125	12, 42	–	92	78	66	54	42
S5	125	13, 33	–	77	61	52	43	34
S6	125	13, 43	–	88	74	61	48	36

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

4639031271

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (LK) 311425/1

1 **(a)** By considering $(r+1)^2 - r^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$

[4]

(b) Given that $\sum_{r=1}^{n}(r+a) = n$, find a in terms of n. [3]

2 Prove by mathematical induction that, for all positive integers n,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$ [6]

3 The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots α, β, γ, δ. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

4 The lines l_1 and l_2 have equations

$$\mathbf{r} = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and the point with position vector $-\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

(b) Find an equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

5 Let k be a constant. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & k & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

It is given that $\mathbf{A}$ is singular.

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [5]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by $\mathbf{CAB}$. [5]

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(c) The matrices **D**, **E** and **F** represent geometrical transformations in the $x-y$ plane.

- **D** represents an enlargement, centre the origin.
- **E** represents a stretch parallel to the x-axis.
- **F** represents a reflection in the line $y = x$.

Given that $\mathbf{CAB} = \mathbf{D} - 9\mathbf{EF}$, find **D**, **E** and **F**. [5]

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6 **(a)** Show that the curve with Cartesian equation

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

has polar equation $r=\cos\theta$. [3]

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The curves C_1 and C_2 have polar equations

$$r=\cos\theta \quad \text{and} \quad r=\sin 2\theta$$

respectively, where $0\leqslant\theta\leqslant\frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole and at another point P.

(b) Find the polar coordinates of P. [3]

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(c) In a single diagram sketch C_1 and C_2, clearly identifying each curve, and mark the point P. [3]

(d) The region R is enclosed by C_1 and C_2 and includes the line OP.

Find, in exact form, the area of R. [6]

7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2+2}{x^2-x-2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C, giving your answers correct to 1 decimal place. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

(e) Find the set of values for which $\dfrac{1}{\mathrm{f}(x)} < \mathrm{f}(x)$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

BLANK PAGE

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/11**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

This document consists of **15** printed pages.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate’s intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$r^2+2r+1-r^2=2r+1$	**B1**	Expands
	$2\sum_{r=1}^{n} r+n=(n+1)^2-1^2$	**M1 A1**	Uses method of differences and sums both sides.
	$\Rightarrow 2\sum_{r=1}^{n} r=n^2+n=n(n+1)$	**A1**	AG.
		4	
1(b)	$\sum_{r=1}^{n}(r+a)=\sum_{r=1}^{n} r+an$	**M1**	Relates with $\sum r$.
	$\frac{1}{2}n(n+1)+an=n$	**M1**	Applies $\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$.
	$a=\frac{1}{2}(1-n)$	**A1**	
		3	

Question	Answer	Marks	Guidance
2	$1=\dfrac{1-2x+x^2}{(1-x)^2}=\dfrac{(1-x)^2}{(1-x)^2}$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\displaystyle\sum_{r=1}^{k} rx^{r-1}=\dfrac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$.	**B1**	States inductive hypothesis.
	$\displaystyle\sum_{r=1}^{k+1} rx^{r-1}=\dfrac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}+(k+1)x^k$	**M1**	Considers sum to $k+1$.
	$\dfrac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k\left(1-2x+x^2\right)}{(1-x)^2}$	**M1**	Puts over a common denominator.
	$\dfrac{1+kx^{k+1}+(k+1)x^k\left(-2x+x^2\right)}{(1-x)^2}=\dfrac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$	**A1**	
	So H_{k+1} is true. By induction, H_n is true for all positive integers *n*.	**A1**	States conclusion.
		6	

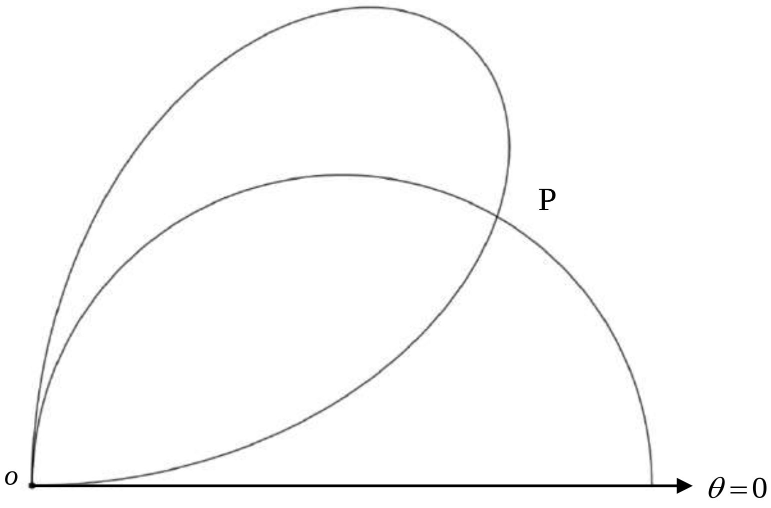
Question	Answer	Marks	Guidance
3(a)	$b=-(\alpha+\beta+\gamma+\delta)=-3$	**B1**	
	$5=(-3)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	**M1 A1**	Uses formula for sum of squares.
	$c=2$	**A1**	
	$6=\dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}=\dfrac{-d}{-2}$	**M1**	Uses $\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}=\dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$.
	$d=12$	**A1**	Equation is $x^4-3x^3+2x^2+12x-2=0$.
		6	
3(b)	$\alpha^4+\beta^4+\gamma^4+\delta^4=3(-27)-2(5)-12(3)+2(4)$	**M1**	Uses *their* quartic equation derived in **(a)**.
	–119	**A1**	
		2	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}2\\-2\\3\end{pmatrix}-\begin{pmatrix}-2\\-3\\-5\end{pmatrix}=\begin{pmatrix}4\\1\\8\end{pmatrix}$	**B1**	Finds direction of one line to another.
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-4 & 3 & 5\\2 & -3 & 1\end{vmatrix}=\begin{pmatrix}18\\14\\6\end{pmatrix}\sim\begin{pmatrix}9\\7\\3\end{pmatrix}$	**M1 A1**	Find common perpendicular.
	$\frac{1}{\sqrt{139}}\left\|\begin{pmatrix}4\\1\\8\end{pmatrix}\bullet\begin{pmatrix}9\\7\\3\end{pmatrix}\right\|=\frac{67}{\sqrt{139}}(=5.68)$	**M1 A1**	Uses formula for shortest distance.
		5	
4(b)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 0 & 1\\-4 & 3 & 5\end{vmatrix}=\begin{pmatrix}3\\9\\-3\end{pmatrix}\sim\begin{pmatrix}1\\3\\-1\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to the plane.
	$1(-1)+3(-3)-1(-4)=-6\Rightarrow x+3y-z=-6$	**M1 A1**	Uses point in the plane.
		4	

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix}1&3\\2&5\end{vmatrix}-k\begin{vmatrix}2&3\\3&5\end{vmatrix}+3\begin{vmatrix}2&1\\3&2\end{vmatrix}=0\Rightarrow -1-k+3=0\Rightarrow k=2$	**M1 A1**	Sets determinant of **A** equal to zero.
	$\begin{pmatrix}-2&-1&1\\1&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\\3&2&5\end{pmatrix}\begin{pmatrix}0&-2\\-1&3\\0&0\end{pmatrix}=\begin{pmatrix}-2&-1&1\\1&1&3\end{pmatrix}\begin{pmatrix}-2&4\\-1&-1\\-2&0\end{pmatrix}$	**M1**	Multiplying two matrices correctly, correct dimensions.
	$\begin{pmatrix}3&-7\\-9&3\end{pmatrix}$	**M1 A1**	Completing matrix multiplication, AG.
		5	
5(b)	$\begin{pmatrix}3&-7\\-9&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x-7y\\-9x+3y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}X\\Y\end{pmatrix}$.
	$-9x+3mx=m(3x-7mx)$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$-9+3m=3m-7m^2\Rightarrow 7m^2=9$	**A1**	
	$y=\frac{3}{\sqrt{7}}x$ and $y=-\frac{3}{\sqrt{7}}x$	**A1**	
		5	

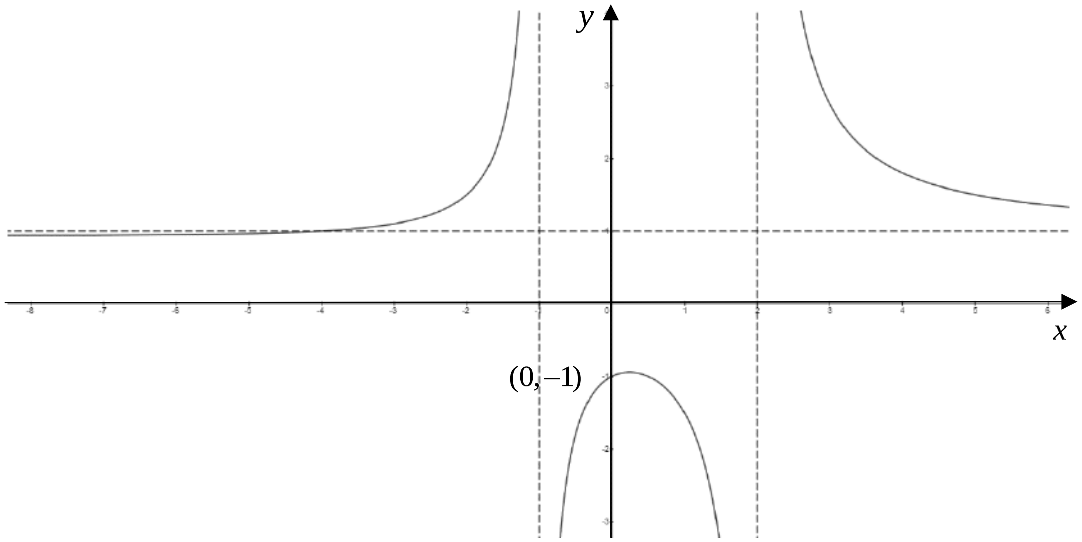
Question	Answer	Marks	Guidance
5(c)	$\mathbf{D}=\begin{pmatrix}\alpha & 0\\0 & \alpha\end{pmatrix}$	**B1**	
	$\mathbf{E}=\begin{pmatrix}\beta & 0\\0 & 1\end{pmatrix}$	**B1**	
	$\mathbf{F}=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	**B1**	
	$\begin{pmatrix}3 & -7\\-9 & 3\end{pmatrix}=\begin{pmatrix}\alpha & 0\\0 & \alpha\end{pmatrix}-9\begin{pmatrix}0 & \beta\\1 & 0\end{pmatrix}$	**M1**	Setting up simultaneous equations using their **D** and **E**.
	$\mathbf{D}=\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$ $\mathbf{E}=\begin{pmatrix}\frac{7}{9} & 0\\0 & 1\end{pmatrix}$	**A1**	Condone $\alpha=3, \beta=\frac{7}{9}$ if it is clear that they refer to the correct matrices.
		5	

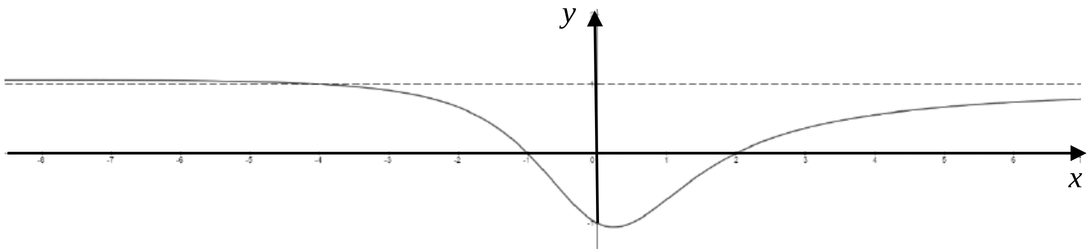
Question	Answer	Marks	Guidance
6(a)	$x^2-x+\frac{1}{4}+y^2=\frac{1}{4}\Rightarrow r^2-r\cos\theta+\frac{1}{4}=\frac{1}{4}$	**B1**	Uses $x^2+y^2=r^2$ and $x=r\cos\theta$.
	$r(r-\cos\theta)=0$	**M1**	Factorises.
	$[r\neq 0\Rightarrow]r=\cos\theta$	**A1**	AG.
		3	

Question	Answer	Marks	Guidance
6(b)	$\sin 2\theta = \cos\theta \Rightarrow 2\sin\theta\cos\theta = \cos\theta$	M1	Sets r values equal and uses $\sin 2\theta = 2\sin\theta\cos\theta$.
	$\cos\theta \neq 0 \Rightarrow \sin\theta = \frac{1}{2}$	A1	$\cos\theta \neq 0$ must be recognised.
	$\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$	A1	
		3	
6(c)		B1	Initial line drawn and one curve correct.
		B1	Other curve correct.
		B1	Intersection marked in correct position and both curves labelled.
		3	

Question	Answer	Marks	Guidance
6(d)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta+\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_0^{\frac{1}{6}\pi}1-\cos 4\theta\,\mathrm{d}\theta$	**M1**	Integrates $\sin^2 2\theta$ using identity.
	$=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{1}{6}\pi}$	**A1**	
	$\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}1+\cos 2\theta\,\mathrm{d}\theta$	**M1**	Integrates $\cos^2\theta$ using identity.
	$=\frac{1}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$	**A1**	
	$\frac{1}{4}\left(\frac{1}{6}\pi-\frac{1}{8}\sqrt{3}\right)+\frac{1}{4}\left(\frac{1}{2}\pi-\frac{1}{6}\pi-\frac{1}{4}\sqrt{3}\right)=\frac{1}{8}\left(\pi-\frac{3}{4}\sqrt{3}\right)$	**A1**	
		6	

Question	Answer	Marks	Guidance
7(a)	$x=-1,\quad x=2$	**B1**	Vertical asymptotes.
	$y=1$	**B1**	Horizontal asymptote.
		2	

Question	Answer	Marks	Guidance
7(b)	$\dfrac{dy}{dx} = \dfrac{(x^2 - x - 2)(2x) - (x^2 + 2)(2x - 1)}{\left(x^2 - x - 2\right)^2}$	**M1***	Finds $\dfrac{dy}{dx}$.
	$x^2 + 8x - 2 = 0$	**DM1**	Sets equal to 0 and forms equation.
	$(-8.2, 0.9)$, $(0.2, -0.9)$.	**A1 A1**	Condone $\left(-4 - 3\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$, $\left(-4 + 3\sqrt{2}, -\frac{2}{3}\sqrt{2}\right)$.
		4	
7(c)		**B1**	Axes and all three asymptotes.
		B1	Correct shape and position, crossing horizontal asymptote.
		B1	States $(0, -1)$ coordinates of intersection with axes, may be seen on diagram.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c)**
		B1	All correct.
		2	
7(e)	$\frac{x^2+2}{x^2-x-2}=1$ or $\frac{x^2+2}{x^2-x-2}=-1$ $x+4=0$ or $2x^2-x=0$	M2	Finds critical points, award M1 for each case.
	$x=-4$ or $x=0$, $x=\frac{1}{2}$	A1	
	$-4<x<-1$, $0<x<\frac{1}{2}$, $x>2$	B1	Must have three distinct regions. Condone $\leqslant -1$ and $\geqslant 2$.
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME										
CENTRE NUMBER						CANDIDATE NUMBER				

6289709443

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (PQ) 311424/2

1 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(3r^2+3r+1)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1}{r^3} - \frac{1}{(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [5]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{3r^2+3r+1}{r^3(r+1)^3}$. [1]

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2 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}\left(x^2 e^x\right) = \left(x^2 + 2nx + n(n-1)\right)e^x. \qquad [6]$$

3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ and $k \neq 1$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. [2]

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The unit square in the $x-y$ plane is transformed by $\mathbf{M}$ onto parallelogram $OPQR$.

(b) Find, in terms of k, the area of parallelogram $OPQR$ and the matrix which transforms $OPQR$ onto the unit square. [3]

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(c) Show that the line through the origin with gradient $\frac{1}{k-1}$ is invariant under the transformation in the $x-y$ plane represented by $\mathbf{M}$. [3]

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4 The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

The sum $(3\alpha+1)^n+(3\beta+1)^n+(3\gamma+1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

(c) Find the values of S_{-1} and S_{-2} . [3]

5 The plane Π_1 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

The line l, which does not lie in Π_1, has equation $\mathbf{r} = -3\mathbf{i} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) Show that l is parallel to Π_1. [2]

(c) Find the distance between l and Π_1. [3]

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(d) The plane Π_2 has equation $3x + 3y + 2z = 1$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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6 The curve C has polar equation $r = \mathrm{e}^{-\theta} - \mathrm{e}^{-\frac{1}{2}\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

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(b) Find the exact value of the area of the region bounded by C and the initial line. [5]

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(c) Show that, at the point on C furthest from the initial line,

$$1 - e^{\theta - \frac{1}{2}\pi} - \tan\theta = 0$$

and verify that this equation has a root between 0.56 and 0.57. [5]

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7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of any stationary points on C. [2]

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(c) Sketch C. [3]

(d) Find the coordinates of any stationary points on the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

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(e) Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$ and find, in exact form, the set of values for which $\dfrac{1}{\mathrm{f}(x)} > \mathrm{f}(x)$. [6]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/12**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1. Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2. Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3. Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5. Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6. Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\frac{1}{2}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+n$	**M1 A1**	Substitutes correct formulae from MF19.
	n^3+3n^2+3n	**A1**	Simplifies
		3	
1(b)	$\frac{1}{r^3}-\frac{1}{(r+1)^3}=\frac{(r+1)^3-r^3}{r^3(r+1)^3}=\frac{r^3+3r^2+3r+1-r^3}{r^3(r+1)^3}=\frac{3r^2+3r+1}{r^3(r+1)^3}$	**M1 A1**	Puts over a common denominator and expands, AG.
	$\sum_{r=1}^{n}\frac{3r^2+3r+1}{r^3(r+1)^3}=\sum_{r=1}^{n}\left(\frac{1}{r^3}-\frac{1}{(r+1)^3}\right)$ $=1-\frac{1}{2^3}+\frac{1}{2^3}-\frac{1}{3^3}+\ldots+\frac{1}{n^3}-\frac{1}{(n+1)^3}$	**M1 A1**	Shows three complete terms, including last.
	$1-\frac{1}{(n+1)^3}$	**A1**	
		5	
1(c)	1	**B1FT**	FT from *their* answer to part **(b)**.
		1	

Question	Answer	Marks	Guidance
2	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x=\left(x^2+2x\right)\mathrm{e}^x$ so true when $n=1$.	**M1 A1**	Differentiates once using the product rule.
	Assume that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ [for some value of k].	**B1**	States inductive hypothesis.
	$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x+\mathrm{e}^x\left(2x+2k\right)$	**M1**	Differentiates *k*th derivative.
	$\left(x^2+2(k+1)x+k(k+1)\right)\mathrm{e}^x$	**A1**	
	So true when $n=k+1$. By induction, true for all positive integers *n*.	**A1**	States conclusion.
		6	

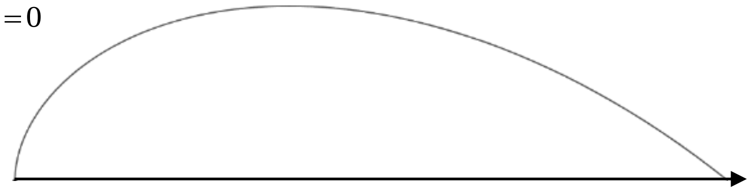
Question	Answer	Marks	Guidance
3(a)	Shear followed by a stretch.	**B2**	Award B1 if given in the wrong order.
		2	
3(b)	$\lvert OPQR\rvert=\lvert\det\mathbf{M}\rvert=\lvert k\rvert$	**B1**	
	$\mathbf{M}^{-1}=\frac{1}{k}\begin{pmatrix}1&0\\-1&k\end{pmatrix}$	**M1 A1**	
		3	

Question	Answer	Marks	Guidance
3(c)	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$	**B1**	Sets $y = \frac{1}{k-1}x$.
	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ x + \frac{1}{k-1}x \end{pmatrix} = \begin{pmatrix} kx \\ \frac{k}{k-1}x \end{pmatrix}$	**M1**	Shows that $Y = \frac{1}{k-1}X$.
	$k\begin{pmatrix} x \\ \frac{1}{k-1}x \end{pmatrix}$	**A1**	
	Alternative method for 3(c)		
	$\begin{pmatrix} k & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ x+y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$X = kx$ and $mX = x + y$ $mkx = x + mx$	**M1**	Uses $y = mx$ and $Y = mX$
	$m = \dfrac{1}{k-1}$ $y = \dfrac{1}{k-1}x$	**A1**	AG
		3	

Question	Answer	Marks	Guidance
4(a)	$y=3x+1 \Rightarrow x=\frac{1}{3}(y-1)$ $\Rightarrow 27\left(\frac{y-1}{3}\right)^3+18\left(\frac{y-1}{3}\right)^2+6\left(\frac{y-1}{3}\right)-1=0$	**B1**	Substitutes.
	$\Rightarrow (y-1)^3+2(y-1)^2+2(y-1)-1=0$ $\Rightarrow y^3-3y^2+3y-1+2y^2-4y+2+2y-2-1=0$	**M1**	Expands.
	$y^3-y^2+y-2=0$	**A1**	AG.
		3	
4(b)	$S_2=1^2-2(1)=-1$	**M1 A1**	Uses formula for sum of squares, AG.
	$S_3=(3\alpha+1)^3+(3\beta+1)^3+(3\gamma+1)^3=-1-(1)+6$	**M1**	Uses $y^3=y^2-y+2$ or expands and uses original equation.
	4	**A1**	
		4	
4(c)	$S_{-1}=\frac{(3\alpha+1)(3\beta+1)+(3\beta+1)(3\gamma+1)+(3\gamma+1)(3\alpha+1)}{(3\alpha+1)(3\beta+1)(3\gamma+1)}=\frac{1}{2}$	**B1**	
	$2S_{-2}=S_1-3+S_{-1}=1-3+\frac{1}{2}$	**M1**	Uses $2y^{-2}=y-1+y^{-1}$.
	$S_{-2}=-\frac{3}{4}$	**A1**	CAO
		3	

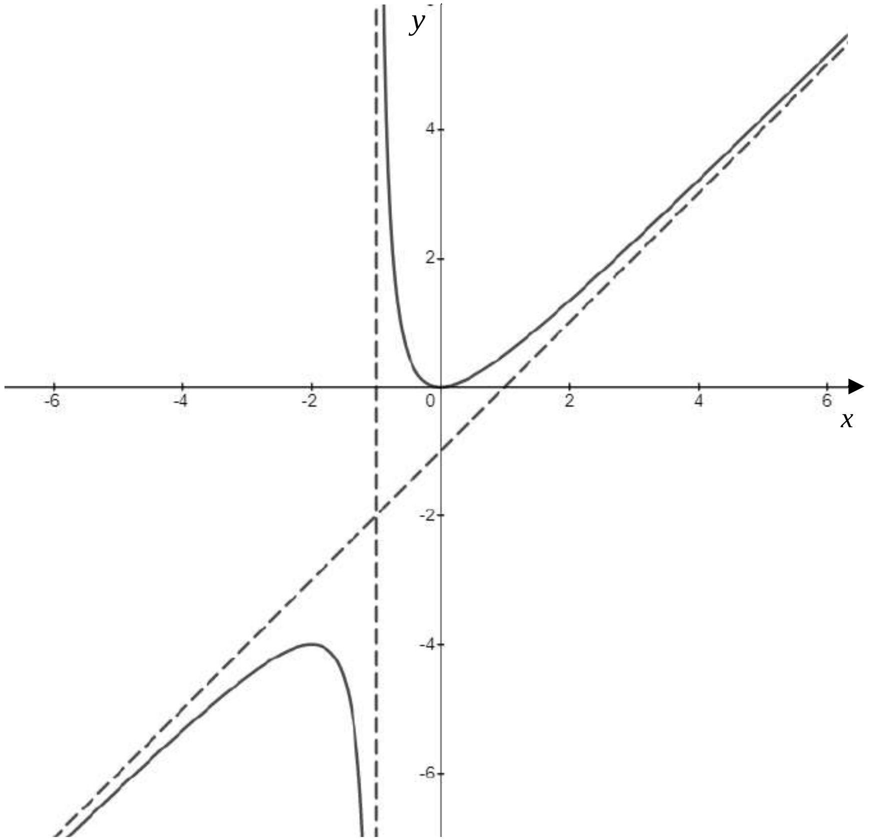
Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 3 & 0 & -1 \end{vmatrix} = \begin{pmatrix} 2 \\ -8 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$	**M1 A1**	Finds perpendicular to Π_1.
	$1(1) - 4(-1) + 3(-2) = -1$	**M1**	Uses point on Π_1.
	$x - 4y + 3z = -1$	**A1**	
		4	
5(b)	$\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 4 + 3 = 0$	**M1 A1**	Shows dot product with direction of line is 0.
		2	
5(c)	$\frac{1}{\sqrt{1^2+4^2+3^2}}\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ or $\frac{1}{\sqrt{1^2+4^2+3^2}}\left(\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + 1\right)$	**M1 A1**	Uses correct formula for distance from point on l to Π_1. $\frac{1}{\sqrt{1^2+4^2+3^2}}\left(-3.1 + 0. - 4 + 1.3 + 1\right)$
	$\frac{1}{\sqrt{26}} (= 0.196)$	**A1**	
		3	

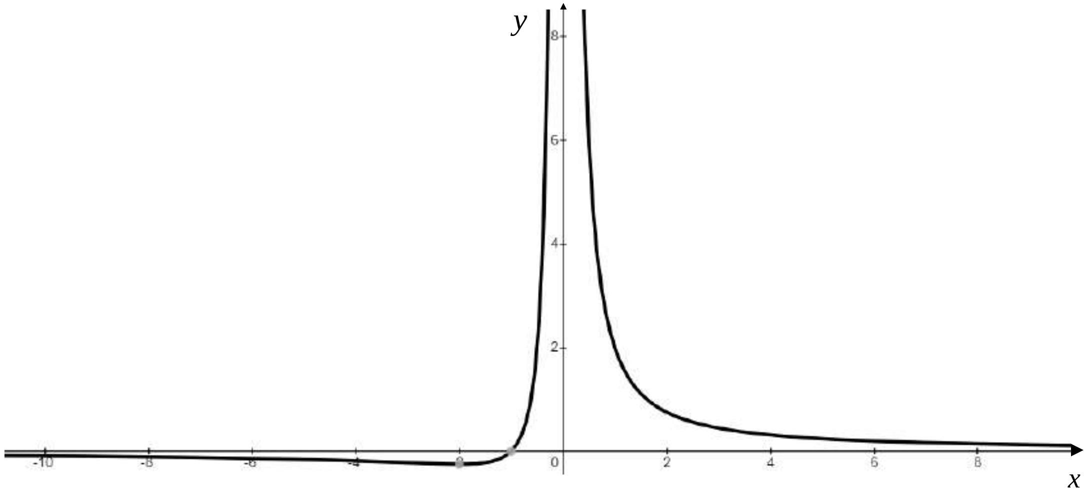
Question	Answer	Marks	Guidance
5(d)	States point common to both planes e.g. $\begin{pmatrix} \frac{1}{15} \\ \frac{4}{15} \\ 0 \end{pmatrix}$.	B1	$\begin{pmatrix} \frac{5}{7} \\ 0 \\ \frac{-4}{7} \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \frac{5}{17} \\ \frac{1}{17} \end{pmatrix}$ or alternative.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ 3 & 3 & 2 \end{vmatrix} = \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$	M1 A1	Finds direction of line.
	$\mathbf{r} = \begin{pmatrix} \frac{5}{7} \\ 0 \\ -\frac{4}{7} \end{pmatrix} + \lambda \begin{pmatrix} -17 \\ 7 \\ 15 \end{pmatrix}$	A1	OE.
		4	

Question	Answer	Marks	Guidance
6(a)	$\theta = 0$ 	**B1**	Initial line drawn. Correct shape, *r* strictly decreasing.
		B1	Correct shape at extremities.
	$1 - e^{-\frac{1}{2}\pi}$	**B1**	May be seen on *their* diagram.
		3	
6(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(e^{-\theta} - e^{-\frac{1}{2}\pi}\right)^2 d\theta$	**M1**	Uses correct formula with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} e^{-2\theta} - 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi}\, d\theta$	**A1**	
	$\frac{1}{2}\left[-\frac{1}{2}e^{-2\theta} + 2e^{-\theta-\frac{1}{2}\pi} + e^{-\pi}\theta\right]_0^{\frac{1}{2}\pi}$	**M1 A1**	Integrates.
	$\frac{1}{2}\left(-\frac{1}{2}e^{-\pi} + 2e^{-\pi} + \frac{1}{2}\pi e^{-\pi} + \frac{1}{2} - 2e^{-\frac{1}{2}\pi}\right) = \frac{3}{4}e^{-\pi} + \frac{1}{4}\pi e^{-\pi} - e^{-\frac{1}{2}\pi} + \frac{1}{4}$	**A1**	
		5	

Question	Answer	Marks	Guidance
6(c)	$y=\left(\mathrm{e}^{-\theta}-\mathrm{e}^{-\frac{1}{2}\pi}\right)\sin\theta$	B1	Uses $y=r\sin\theta$
	$\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\left(\mathrm{e}^{-\theta}-\mathrm{e}^{-\frac{1}{2}\pi}\right)\cos\theta+\sin\theta\left(-\mathrm{e}^{-\theta}\right)=0$	M1 A1	Sets derivative equal to zero.
	$\left[\theta\neq\frac{1}{2}\pi\Rightarrow\right]1+\left(\dfrac{-\mathrm{e}^{-\theta}}{\mathrm{e}^{-\theta}-\mathrm{e}^{-\frac{1}{2}\pi}}\right)\tan\theta=0\Rightarrow1-\mathrm{e}^{\theta-\frac{1}{2}\pi}-\tan\theta=0$	A1	AG.
	$1-\mathrm{e}^{0.56-\frac{1}{2}\pi}-\tan0.56=0.00912$ and $1-\mathrm{e}^{0.57-\frac{1}{2}\pi}-\tan0.57=-0.00856$	B1	Shows sign change.
		5	

Question	Answer	Marks	Guidance
7(a)	$x=-1$	B1	Vertical asymptote.
	$y=\dfrac{(x+1)(x-1)+1}{x+1}$	M1	Oblique asymptote.
	$y=x-1$	A1	
		3	
7(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x^2+2x}{(x+1)^2}=0$	M1	Sets $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$.
	$(0,0)$, $(-2,-4)$	A1	
		2	

Question	Answer	Marks	Guidance
7(c)		**B1**	Axes and asymptotes.
		B1	Left branch correct.
		B1	Right branch correct.
		3	
7(d)	$\left(-2,-\frac{1}{4}\right)$	**B1 B1**	B1 for each correct coordinate. SC B1 for $\left(-2,-\frac{1}{4}\right)$ and $(0,0)$.
		2	

Question	Answer	Marks	Guidance
7(e)		B1	Left branch correct.
		B1	Right branch correct.
	$\frac{x^2}{x+1}=1$ or $\frac{x^2}{x+1}=-1$ $x^2-x-1=0$	M2	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$ or $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$	A1	
	$x<-1,\ \frac{1}{2}-\frac{1}{2}\sqrt{5}<x<\frac{1}{2}+\frac{1}{2}\sqrt{5},\ \ x\neq 0$	B1	Condone missing $x\neq 0$.
		6	

Cambridge Assessment International Education

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CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

4244263062

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (SK) 333699

1 **(a)** By considering $(r+1)^2 - r^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [4]

(b) Given that $\sum_{r=1}^{n}(r+a) = n$, find a in terms of n. [3]

2 Prove by mathematical induction that, for all positive integers n,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

[6]

3 The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

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(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

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4 The lines l_1 and l_2 have equations

$$\mathbf{r} = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} + \lambda(-4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and the point with position vector $-\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

(b) Find an equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

5 Let k be a constant. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & k & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

It is given that $\mathbf{A}$ is singular.

(a) Show that $\mathbf{CAB} = \begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$. [5]

(b) Find the equations of the invariant lines, through the origin, of the transformation in the $x-y$ plane represented by $\mathbf{CAB}$. [5]

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(c) The matrices **D**, **E** and **F** represent geometrical transformations in the $x-y$ plane.

- **D** represents an enlargement, centre the origin.
- **E** represents a stretch parallel to the x-axis.
- **F** represents a reflection in the line $y = x$.

Given that $\mathbf{CAB} = \mathbf{D} - 9\mathbf{EF}$, find **D**, **E** and **F**. [5]

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6 **(a)** Show that the curve with Cartesian equation

$$\left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}$$

has polar equation $r=\cos\theta$. [3]

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The curves C_1 and C_2 have polar equations

$$r=\cos\theta \quad \text{and} \quad r=\sin 2\theta$$

respectively, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole and at another point P.

(b) Find the polar coordinates of P. [3]

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(c) In a single diagram sketch C_1 and C_2, clearly identifying each curve, and mark the point P. [3]

(d) The region R is enclosed by C_1 and C_2 and includes the line OP.

Find, in exact form, the area of R. [6]

7 The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2+2}{x^2-x-2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C, giving your answers correct to 1 decimal place. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

....

(d) Sketch the curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

(e) Find the set of values for which $\frac{1}{\mathrm{f}(x)} < \mathrm{f}(x)$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

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FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1. Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2. Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3. Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5. Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6. Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$r^2+2r+1-r^2=2r+1$	**B1**	Expands
	$2\sum_{r=1}^{n} r+n=(n+1)^2-1^2$	**M1 A1**	Uses method of differences and sums both sides.
	$\Rightarrow 2\sum_{r=1}^{n} r=n^2+n=n(n+1)$	**A1**	AG.
		4	
1(b)	$\sum_{r=1}^{n}(r+a)=\sum_{r=1}^{n} r+an$	**M1**	Relates with $\sum r$.
	$\frac{1}{2}n(n+1)+an=n$	**M1**	Applies $\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$.
	$a=\frac{1}{2}(1-n)$	**A1**	
		3	

Question	Answer	Marks	Guidance
2	$1=\dfrac{1-2x+x^2}{(1-x)^2}=\dfrac{(1-x)^2}{(1-x)^2}$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\sum_{r=1}^{k} rx^{r-1}=\dfrac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$.	**B1**	States inductive hypothesis.
	$\sum_{r=1}^{k+1} rx^{r-1}=\dfrac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}+(k+1)x^k$	**M1**	Considers sum to $k+1$.
	$\dfrac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k\left(1-2x+x^2\right)}{(1-x)^2}$	**M1**	Puts over a common denominator.
	$\dfrac{1+kx^{k+1}+(k+1)x^k\left(-2x+x^2\right)}{(1-x)^2}=\dfrac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$	**A1**	
	So H_{k+1} is true. By induction, H_n is true for all positive integers *n*.	**A1**	States conclusion.
		6	

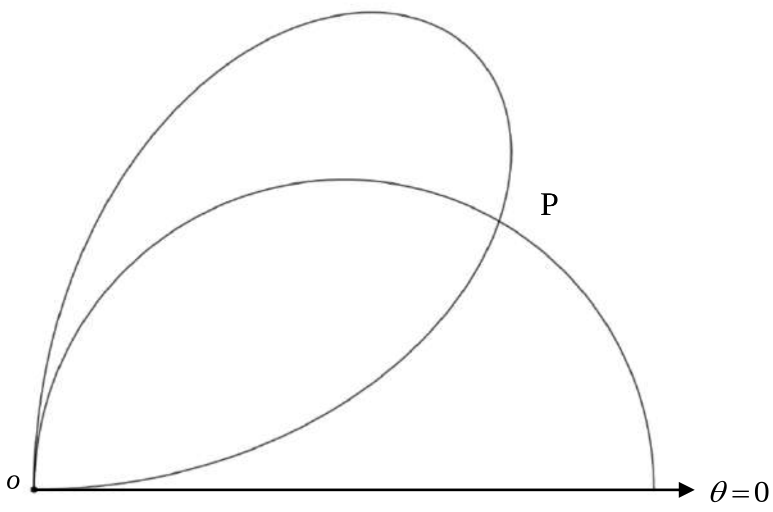
Question	Answer	Marks	Guidance
3(a)	$b = -(\alpha+\beta+\gamma+\delta) = -3$	B1	
	$5 = (-3)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	M1 A1	Uses formula for sum of squares.
	$c = 2$	A1	
	$6 = \dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta} = \dfrac{-d}{-2}$	M1	Uses $\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1} = \dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$.
	$d = 12$	A1	Equation is $x^4 - 3x^3 + 2x^2 + 12x - 2 = 0$.
		6	
3(b)	$\alpha^4+\beta^4+\gamma^4+\delta^4 = 3(-27)-2(5)-12(3)+2(4)$	M1	Uses *their* quartic equation derived in **(a)**.
	–119	A1	
		2	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}2\\-2\\3\end{pmatrix}-\begin{pmatrix}-2\\-3\\-5\end{pmatrix}=\begin{pmatrix}4\\1\\8\end{pmatrix}$	**B1**	Finds direction of one line to another.
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-4 & 3 & 5\\2 & -3 & 1\end{vmatrix}=\begin{pmatrix}18\\14\\6\end{pmatrix}\sim\begin{pmatrix}9\\7\\3\end{pmatrix}$	**M1 A1**	Find common perpendicular.
	$\frac{1}{\sqrt{139}}\left\|\begin{pmatrix}4\\1\\8\end{pmatrix}\bullet\begin{pmatrix}9\\7\\3\end{pmatrix}\right\|=\frac{67}{\sqrt{139}}(=5.68)$	**M1 A1**	Uses formula for shortest distance.
		5	
4(b)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 0 & 1\\-4 & 3 & 5\end{vmatrix}=\begin{pmatrix}3\\9\\-3\end{pmatrix}\sim\begin{pmatrix}1\\3\\-1\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to the plane.
	$1(-1)+3(-3)-1(-4)=-6\Rightarrow x+3y-z=-6$	**M1 A1**	Uses point in the plane.
		4	

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} - k\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0 \Rightarrow -1 - k + 3 = 0 \Rightarrow k = 2$	**M1 A1**	Sets determinant of **A** equal to zero.
	$\begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ -1 & -1 \\ -2 & 0 \end{pmatrix}$	**M1**	Multiplying two matrices correctly, correct dimensions.
	$\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}$	**M1 A1**	Completing matrix multiplication, AG.
		5	
5(b)	$\begin{pmatrix} 3 & -7 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 7y \\ -9x + 3y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$-9x + 3mx = m(3x - 7mx)$	**M1 A1**	Uses $y = mx$ and $Y = mX$.
	$-9 + 3m = 3m - 7m^2 \Rightarrow 7m^2 = 9$	**A1**	
	$y = \frac{3}{\sqrt{7}}x$ and $y = -\frac{3}{\sqrt{7}}x$	**A1**	
		5	

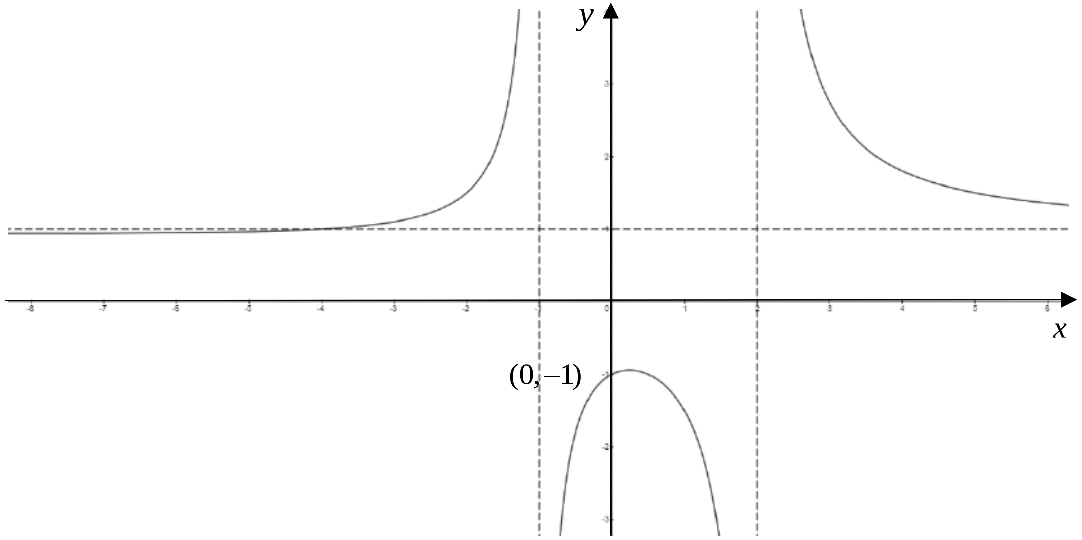
Question	Answer	Marks	Guidance
5(c)	$\mathbf{D}=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	**B1**	
	$\mathbf{E}=\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}$	**B1**	
	$\mathbf{F}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	**B1**	
	$\begin{pmatrix}3 & -7\\ -9 & 3\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}-9\begin{pmatrix}0 & \beta\\ 1 & 0\end{pmatrix}$	**M1**	Setting up simultaneous equations using their **D** and **E**.
	$\mathbf{D}=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$ $\mathbf{E}=\begin{pmatrix}\frac{7}{9} & 0\\ 0 & 1\end{pmatrix}$	**A1**	Condone $\alpha=3, \beta=\frac{7}{9}$ if it is clear that they refer to the correct matrices.
		5	

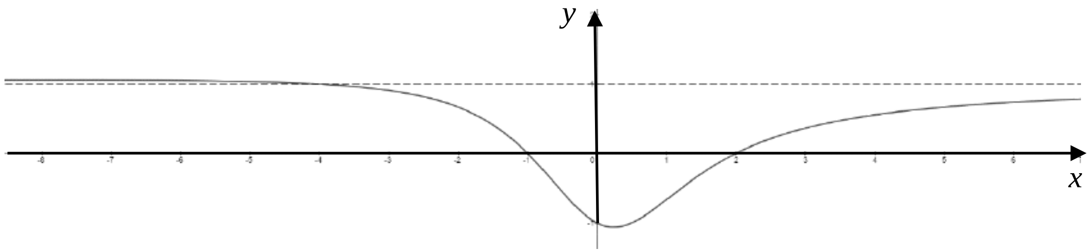
Question	Answer	Marks	Guidance
6(a)	$x^2-x+\frac{1}{4}+y^2=\frac{1}{4}\Rightarrow r^2-r\cos\theta+\frac{1}{4}=\frac{1}{4}$	**B1**	Uses $x^2+y^2=r^2$ and $x=r\cos\theta$.
	$r\left(r-\cos\theta\right)=0$	**M1**	Factorises.
	$\left[r\neq 0\Rightarrow\right]r=\cos\theta$	**A1**	AG.
		3	

Question	Answer	Marks	Guidance
6(b)	$\sin 2\theta = \cos\theta \Rightarrow 2\sin\theta\cos\theta = \cos\theta$	M1	Sets *r* values equal and uses $\sin 2\theta = 2\sin\theta\cos\theta$.
	$\cos\theta \neq 0 \Rightarrow \sin\theta = \frac{1}{2}$	A1	$\cos\theta \neq 0$ must be recognised.
	$\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$	A1	
		3	
6(c)		B1	Initial line drawn and one curve correct.
		B1	Other curve correct.
		B1	Intersection marked in correct position and both curves labelled.
		3	

Question	Answer	Marks	Guidance
6(d)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta+\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{6}\pi}\sin^2 2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_0^{\frac{1}{6}\pi}1-\cos 4\theta\,\mathrm{d}\theta$	**M1**	Integrates $\sin^2 2\theta$ using identity.
	$=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{1}{6}\pi}$	**A1**	
	$\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}1+\cos 2\theta\,\mathrm{d}\theta$	**M1**	Integrates $\cos^2\theta$ using identity.
	$=\frac{1}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$	**A1**	
	$\frac{1}{4}\left(\frac{1}{6}\pi-\frac{1}{8}\sqrt{3}\right)+\frac{1}{4}\left(\frac{1}{2}\pi-\frac{1}{6}\pi-\frac{1}{4}\sqrt{3}\right)=\frac{1}{8}\left(\pi-\frac{3}{4}\sqrt{3}\right)$	**A1**	
		6	

Question	Answer	Marks	Guidance
7(a)	$x=-1,\quad x=2$	**B1**	Vertical asymptotes.
	$y=1$	**B1**	Horizontal asymptote.
		2	

Question	Answer	Marks	Guidance
7(b)	$\frac{dy}{dx}=\frac{(x^2-x-2)(2x)-(x^2+2)(2x-1)}{\left(x^2-x-2\right)^2}$	M1*	Finds $\frac{dy}{dx}$.
	$x^2+8x-2=0$	DM1	Sets equal to 0 and forms equation.
	$(-8.2, 0.9)$, $(0.2, -0.9)$.	A1 A1	Condone $\left(-4-3\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$, $\left(-4+3\sqrt{2}, -\frac{2}{3}\sqrt{2}\right)$.
		4	
7(c)		B1	Axes and all three asymptotes.
		B1	Correct shape and position, crossing horizontal asymptote.
		B1	States $(0, -1)$ coordinates of intersection with axes, may be seen on diagram.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c)**
		B1	All correct.
		2	
7(e)	$\frac{x^2+2}{x^2-x-2}=1$ or $\frac{x^2+2}{x^2-x-2}=-1$ $x+4=0$ or $2x^2-x=0$	M2	Finds critical points, award M1 for each case.
	$x=-4$ or $x=0,\quad x=\frac{1}{2}$	A1	
	$-4<x<-1,\ 0<x<\frac{1}{2},\ x>2$	B1	Must have three distinct regions. Condone $\leqslant -1$ and $\geqslant 2$.
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

9340929644

FURTHER MATHEMATICS **9231/21**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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1 Show that the system of equations

$$\begin{aligned} 14x - 4y + 6z &= 5, \\ x + y + kz &= 3, \\ -21x + 6y - 9z &= 14, \end{aligned}$$

where k is a constant, does not have a unique solution and interpret this situation geometrically. [4]

2 Find the roots of the equation $(z+5\mathrm{i})^3 = 4+4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\cos\theta + \mathrm{i}(r\sin\theta - 5)$, where $r > 0$ and $0 < \theta < 2\pi$. [5]

3 Find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\frac{1}{2}e^x\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [6]

4 Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=27x^2,$$

given that, when $x=0$, $y=2$ and $\frac{dy}{dx}=-8$. [10]

5 The curve C has parametric equations

$$x = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}, \qquad y = 2t + 5, \qquad \text{for } 0 < t \leqslant 3.$$

(a) Find the exact length of C. [5]

(b) Find the set of values of t for which $\dfrac{d^2y}{dx^2} > 0$. [5]

6 **(a)** Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x. \qquad [3]$$

(b) Using the substitution $u = \sinh x$, find $\int \sinh^2 2x \cosh x \, \mathrm{d}x$. [4]

(c) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\tanh x = \sinh^2 2x,$$

given that $y = 4$ when $x = 0$. Give your answer in the form $y = f(x)$. [7]

7 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -6 & 2 & 13 \\ 0 & -2 & 5 \\ 0 & 0 & 8 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

8 **(a)** State the sum of the series $1+z+z^2+\ldots+z^{n-1}$, for $z \neq 1$. [1]

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(b) By letting $z = \cos\theta + \mathrm{i}\sin\theta$, where $\cos\theta \neq 1$, show that

$$1+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta = \frac{1}{2}\left(1-\cos n\theta+\frac{\sin n\theta\sin\theta}{1-\cos\theta}\right).$$ [7]

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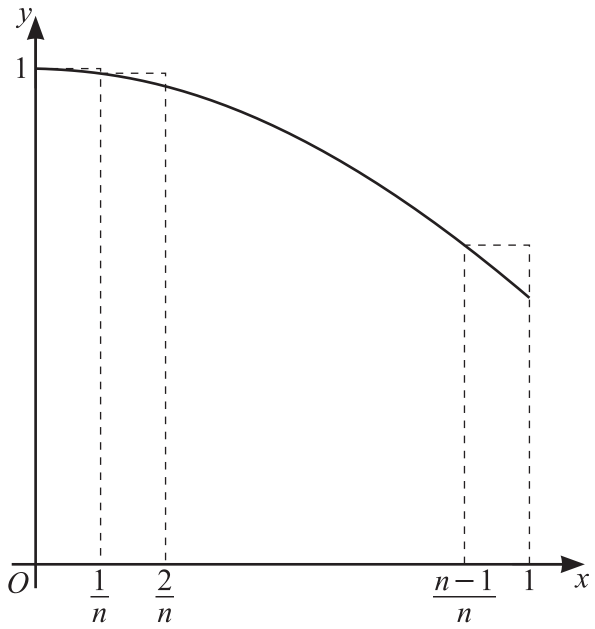

The diagram shows the curve with equation $y = \cos x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(c) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 \cos x \, dx < \frac{1}{2n}\left(1 - \cos 1 + \frac{\sin 1 \sin \frac{1}{n}}{1 - \cos \frac{1}{n}}\right). \qquad [4]$$

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(d) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 \cos x \, dx$. [3]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/21**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$\begin{vmatrix} 14 & -4 & 6 \\ 1 & 1 & k \\ -21 & 6 & -9 \end{vmatrix} = 14\begin{vmatrix} 1 & k \\ 6 & -9 \end{vmatrix} + 4\begin{vmatrix} 1 & k \\ -21 & -9 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ -21 & 6 \end{vmatrix}$ $= 14(-9-6k) + 4(-9+21k) + 6(6+21) = 0$	M1 A1	Evaluates determinant. Can expand along any row e.g. $-(36-36)+(-126+126)+k(84-84)$. If using row operations, they must show an inconsistent system for M1. All their row operations must be correct for A1.
	Two parallel planes, not identical.	B1	
	Other plane not parallel.	B1	
		4	

Question	Answer	Marks	Guidance
2	$(z+5\mathrm{i})^3 = 4+4\mathrm{i}\sqrt{3} = 8\mathrm{e}^{\frac{1}{3}\pi}$	B1	Finds modulus and argument of $4+4\mathrm{i}\sqrt{3}$.
	$z_1 = 2\left(\cos\frac{1}{9}\pi + \mathrm{i}\sin\frac{1}{9}\pi\right) - 5\mathrm{i} = 2\cos\frac{1}{9}\pi + \mathrm{i}\left(2\sin\frac{1}{9}\pi - 5\right)$	M1 A1	Finds one root.
	$z_2 = 2\cos\frac{7}{9}\pi + \mathrm{i}\left(2\sin\frac{7}{9}\pi - 5\right)$, $z_3 = 2\cos\frac{13}{9}\pi + \mathrm{i}\left(2\sin\frac{13}{9}\pi - 5\right)$	A1 FT	Finds other two roots. FT on their modulus.
		A1 FT	
		5	

Question	Answer	Marks	Guidance
3	$\dfrac{dy}{dx}=\dfrac{\frac{1}{2}e^{x}}{1-\frac{1}{4}e^{2x}}$	B1	
	$\dfrac{d^2y}{dx^2}=\dfrac{\left(1-\frac{1}{4}e^{2x}\right)\left(\frac{1}{2}e^{x}\right)-\frac{1}{2}e^{x}\left(-\frac{1}{2}e^{2x}\right)}{\left(1-\frac{1}{4}e^{2x}\right)^2}$	B1	
	$f'(0)=\dfrac{2}{3}\quad f''(0)=\dfrac{10}{9}$	M1	Evaluates derivatives at $x=0$.
	$f(0)=\tanh^{-1}\left(\dfrac{1}{2}\right)=\dfrac{1}{2}\ln\left(\dfrac{3}{2}\times 2\right)$	M1	Uses logarithmic form of $\tanh^{-1}$.
	$\dfrac{1}{2}\ln 3+\dfrac{2}{3}x+\dfrac{5}{9}x^2$	M1 A1	Applies $f(x)=f(0)+f'(0)x+\dfrac{1}{2!}f''(0)x^2$
		6	

Question	Answer	Marks	Guidance
4	$m^2+2m+3=0$	M1	Auxiliary equation.
	$[y=]e^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)$	A1	Complementary function. Allow with "$y=$" missing.
	$y=px^2+qx+r\Rightarrow y'=2px+q\Rightarrow y''=2p$	B1	Particular integral and its derivatives.
	$3p=27\quad 3q+4p=0\quad 2p+2q+3r=0$	M1	Substitutes and equates coefficients.

Question	Answer	Marks	Guidance
4	$p=9 \quad q=-12 \quad r=2$	**A1**	
	$y=\mathrm{e}^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)+9x^2-12x+2$	**A1**	General solution. Must have “$y=$”.
	$y'=\mathrm{e}^{-x}\left(-\sqrt{2}A\sin\sqrt{2}x+\sqrt{2}B\cos\sqrt{2}x\right)-\mathrm{e}^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)+18x-12$	**M1***	Differentiates. Must use product rule.
	$A+2=2 \quad \sqrt{2}B-A-12=-8$ $\Rightarrow A=0,\ B=2\sqrt{2}$	**DM1** **A1**	Uses initial conditions
	$y=2\sqrt{2}\mathrm{e}^{-x}\sin\sqrt{2}x+9x^2-12x+2$	**A1**	Must have “$y=$”.
		10	

Question	Answer	Marks	Guidance
5(a)	$\dot{x}=t^{\frac{1}{2}}-t^{-\frac{1}{2}}, \quad \dot{y}=2$	**B1**	Differentiates x and y with respect to t.
	$\dot{x}^2+\dot{y}^2=\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^2+4=t+2+t^{-1}=\left(t^{\frac{1}{2}}+t^{-\frac{1}{2}}\right)^2$	**M1 A1**	Factorises $\dot{x}^2+\dot{y}^2$.
	$\int_0^3 t^{\frac{1}{2}}+t^{-\frac{1}{2}}\,\mathrm{d}t=\left[\frac{2}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right]_0^3=4\sqrt{3}$	**M1 A1**	Applies correct formula for arc length. M1 for their $\int_0^3\sqrt{\dot{x}^2+\dot{y}^2}\,\mathrm{d}t$. Answer must be simplified to $4\sqrt{3}$ for A1.
		5	

Question	Answer	Marks	Guidance
5(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\dot{y}}{\dot{x}}=\frac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}$	B1	Finds first derivative.
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}\right)=\frac{-2\left(\frac{1}{2}t^{-\frac{1}{2}}+\frac{1}{2}t^{-\frac{3}{2}}\right)}{\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^2}$	B1	Differentiates $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to t.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}\right)\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{-2\left(\frac{1}{2}t^{-\frac{1}{2}}+\frac{1}{2}t^{-\frac{3}{2}}\right)}{\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^3}=-\frac{t+1}{(t-1)^3}$	M1 A1	Applies chain rule. OE. Does not have to be simplified for A1.
	$0<t<1$	A1	Accept $-1<t<1$. CWO.
		5	

Question	Answer	Marks	Guidance
6(a)	$\cosh x=\frac{1}{2}\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)\quad\sinh x=\frac{1}{2}\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)$	B1	
	$\frac{1}{2}\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)=\frac{1}{2}\left(\mathrm{e}^{2x}-\mathrm{e}^{-2x}\right)=\sinh 2x$	M1 A1	Expands, AG.
		3	

Question	Answer	Marks	Guidance
6(b)	$u = \sinh x \Rightarrow du = \cosh x\, dx$	B1	
	$\int \sinh^2 2x \cosh x\,\mathrm{d}x = 4\int \sinh^2 x \cosh^2 x\,\mathrm{d}u = 4\int \sinh^2 x\left(\sinh^2 x + 1\right)\mathrm{d}u$	M1	Applies identities to find integral in terms of *u*.
	$= 4\int u^2\left(u^2+1\right)du$	A1	
	$= 4\left(\frac{1}{5}u^5 + \frac{1}{3}u^3\right)(+C) = 4\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x\right)(+C)$	A1	
		4	
6(c)	$\mathrm{e}^{\int \tanh x\,dx} = \mathrm{e}^{\ln\cosh x} = \cosh x$	M1 A1	Finds integrating factor.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\cosh x\right) = \sinh^2 2x \cosh x$	M1	Correct form on LHS and attempt to integrate RHS.
	$y\cosh x = 4\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x\right) + C$	M1 A1	Integrates RHS using their part (b).
	$4 = C$	M1	Substitutes initial conditions.
	$y = 4\operatorname{sech} x\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x + 1\right)$	A1	
		7	

Question	Answer	Marks	Guidance
7(a)	$\lambda = -6,\quad \lambda = -2,\quad \lambda = 8$	B1	
	$\lambda = -6: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = -2: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 13 \\ 0 & 0 & 5 \end{vmatrix} = \begin{pmatrix} 10 \\ 20 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$	A1	
	$\lambda = 8: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -14 & 2 & 13 \\ 0 & -10 & 5 \end{vmatrix} = \begin{pmatrix} 140 \\ 70 \\ 140 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$	A1	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$	M1 A1	Or correctly matched permutations of columns. M1 for their (non-zero) eigenvectors matched to their eigenvalues.
		7	
7(b)	$\boldsymbol{A}^3 - 52\boldsymbol{A} - 96\mathbf{I} = 0$	M1	Substitutes **A** into characteristic equation.
	$96\mathbf{A}^{-1} = \mathbf{A}^2 - 52\mathbf{I}$	M1	Multiples through by $\mathbf{A}^{-1}$.
	$\mathbf{A}^2 = \begin{pmatrix} 36 & -16 & 36 \\ 0 & 4 & 30 \\ 0 & 0 & 64 \end{pmatrix}$	B1	

Question	Answer	Marks	Guidance
7(b)	$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{3}{8} \\ 0 & -\frac{1}{2} & \frac{5}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$	**A1**	
		4	

Question	Answer	Marks	Guidance
8(a)	$\dfrac{z^n - 1}{z - 1}$	**B1**	
		1	
8(b)	$\dfrac{z^n - 1}{z - 1} = \dfrac{\cos n\theta - 1 + \mathrm{i}\sin n\theta}{\cos\theta - 1 + \mathrm{i}\sin\theta}$	**B1**	
	$\dfrac{(\cos n\theta - 1 + \mathrm{i}\sin n\theta)(\cos\theta - 1 - \mathrm{i}\sin\theta)}{(\cos\theta - 1 + \mathrm{i}\sin\theta)(\cos\theta - 1 - \mathrm{i}\sin\theta)}$	**M1**	Multiplies numerator and denominator by complex conjugate.
	$\mathrm{Re}\left(\dfrac{z^n - 1}{z - 1}\right) = \dfrac{\cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos n\theta - \cos\theta + 1}{(\cos\theta - 1)^2 + \sin^2\theta}$	**M1**	Takes real part. $\cos(n-1)\theta = \cos n\theta\cos\theta + \sin n\theta\sin\theta$
	$= \dfrac{\cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos n\theta}{2(1 - \cos\theta)} + \dfrac{1}{2}$	**A1**	
	$= \dfrac{\cos n\theta(\cos\theta - 1) + \sin n\theta\sin\theta}{2(1 - \cos\theta)} + \dfrac{1}{2}$	**M1**	Factorises.

Question	Answer	Marks	Guidance
8(b)	$=\frac{1}{2}\left(1-\cos n\theta+\frac{\sin n\theta\sin\theta}{1-\cos\theta}\right)$	M1 A1	Divides through by denominator. AG.
	Alternative method for question 8(b)		
	$\frac{z^n-1}{z-1}=\frac{e^{in\theta}-1}{e^{i\theta}-1}$	B1	
	$\frac{e^{i\left(n-\frac{1}{2}\right)\theta}-e^{-i\frac{1}{2}\theta}}{e^{i\frac{1}{2}\theta}-e^{-i\frac{1}{2}\theta}}=\frac{\cos\left(n-\frac{1}{2}\right)\theta+i\sin\left(n-\frac{1}{2}\right)\theta-\cos\frac{1}{2}\theta+i\sin\left(\frac{1}{2}\theta\right)}{2i\sin\frac{1}{2}\theta}$	M1	
	$\text{Re}\left(\frac{z^n-1}{z-1}\right)=\frac{\sin(n-\frac{1}{2})\theta+\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	M1	Takes real part
	$=\frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}+\frac{1}{2}$	A1	
	$=\frac{\sin n\theta\cos\frac{1}{2}\theta-\cos n\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}+\frac{1}{2}$	M1	Uses compound angle identity
	$=\frac{\sin n\theta\cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}-\frac{1}{2}\cos n\theta+\frac{1}{2}$	M1	Divides through by denominator.
	$\frac{\sin n\theta\sin\theta}{4\sin^2\frac{1}{2}\theta}-\frac{1}{2}\cos n\theta+\frac{1}{2}=\frac{\sin n\theta\sin\theta}{2\left(1-\cos\theta\right)}-\frac{1}{2}\cos n\theta+\frac{1}{2}$	A1	AG. $\sin\theta=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $2\sin^2\frac{1}{2}\theta=1-\cos\theta$.
		7	

Question	Answer	Marks	Guidance
8(c)	$\int_0^1 \cos x \, dx < \frac{1}{n} + \frac{1}{n}\cos\frac{1}{n} + \frac{1}{n}\cos\frac{2}{n} + \ldots \frac{1}{n}\cos\frac{n-1}{n}$	**M1 A1**	Forms sum of areas of rectangles given in the diagram. A0 if comparison with integral missing or unclear.
	$= \frac{1}{n}\left(1 + \cos\frac{1}{n} + \cos\frac{2}{n} + \ldots + \cos\frac{n-1}{n}\right) = \frac{1}{2n}\left(1 - \cos\frac{n}{n} + \frac{\sin\frac{n}{n}\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$	**M1 A1**	Applies result from part (b) with $\theta = \frac{1}{n}$. AG.
		4	
8(d)	$\int_0^1 \cos x \, dx > \frac{1}{n}\cos\frac{1}{n} + \frac{1}{n}\cos\frac{2}{n} + \ldots \frac{1}{n}\cos\frac{n}{n}$	**M1 A1**	Forms sum of areas of rectangles. A0 if comparison with integral missing or unclear.
	$= \frac{1}{2n}\left(1 - \cos 1 + \frac{\sin 1 \sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right) + \frac{1}{n}\cos 1 - \frac{1}{n} = \frac{1}{2n}\left(\cos 1 - 1 + \frac{\sin 1 \sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$	**A1**	
		3	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

5192602583

FURTHER MATHEMATICS **9231/22**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION
- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (CE/JG) 311416/3

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1 Find the Maclaurin's series for $\ln(x+2)+\ln(x^2+5)$ up to and including the term in x^2. [5]

2 It is given that

$$x = 1 + \frac{1}{t} \qquad \text{and} \qquad y = te^t.$$

(a) Show that $\frac{dy}{dx} = -e^t(t^3 + t^2)$. [3]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t. [4]

3 **(a)** Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$ [4]

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(b) Hence obtain the roots of the equation

$$32x^5 - 40x^3 + 10x - \sqrt{2} = 0$$

in the form $\cos(q\pi)$, where q is a rational number. [4]

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4 Find the solution of the differential equation

$$\frac{dy}{dx} + 3y = \sin x$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. [9]

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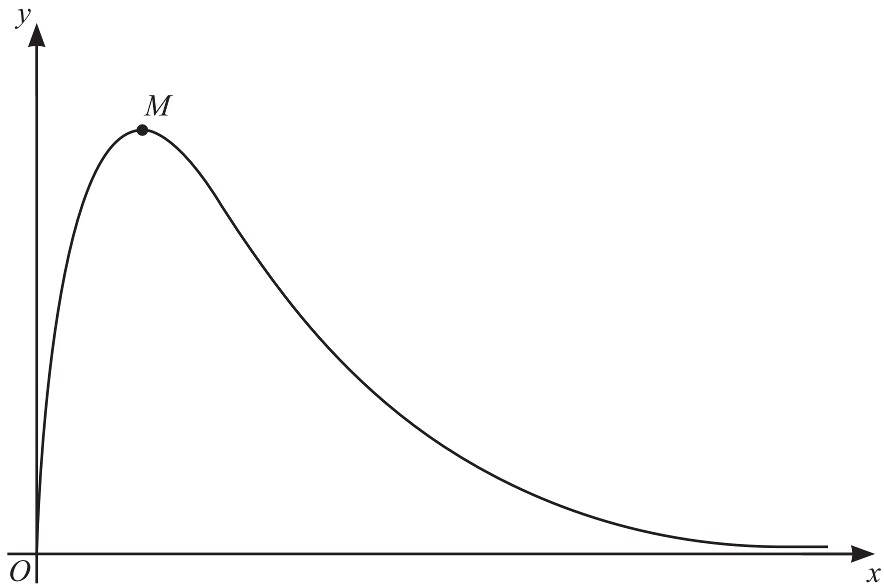

The diagram shows part of the curve $y = x\,\text{sech}^2 x$ and its maximum point M.

(a) Show that, at M,

$$2x\tanh x - 1 = 0$$

and verify that this equation has a root between 0.7 and 0.8. [4]

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(b) By considering a suitable set of rectangles, use the diagram to show that

$$\sum_{r=2}^{n} r\,\mathrm{sech}^2 r < n \tanh n + \ln \mathrm{sech}\, n - \tanh 1 - \ln \mathrm{sech}\, 1. \qquad [6]$$

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6 The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) State the eigenvalues of $\mathbf{P}$. [1]

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(b) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$. [4]

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The 3×3 matrix $\mathbf{A}$ has distinct non-zero eigenvalues a, $\frac{1}{2}$, 2 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}^{-1}$ in terms of a. [5]

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7 **(a)** Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\sinh^2 A = \cosh 2A - 1.$$ [3]

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(b) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by S.

Use the substitution $x = \frac{1}{2}\sinh u$ to show that $S = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$. [9]

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8 It is given that $v = y^4$ and

$$y^3 \frac{d^2y}{dx^2} + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3 \frac{dy}{dx} + y^4 = e^{-2x}.$$

(a) Show that

$$\frac{d^2v}{dx^2} + \frac{dv}{dx} + 4v = 4e^{-2x}.$$ [4]

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(b) Find y in terms of x, given that, when $x = 0$, $y = 1$ and $\frac{dy}{dx} = -\frac{3}{8}$. [10]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/22**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Due to a series-specific issue during the live exam series, all candidates were awarded full marks for question 6. This published mark scheme for these questions was created alongside the question paper but has not been used by examiners.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	**EITHER** Solution 1 $\ln(x+2)=\ln\left(2\left(1+\frac{1}{2}x\right)\right)=\ln 2+\ln\left(1+\frac{1}{2}x\right)$	**M1**	Changes $\ln(x+2)$ or $\ln\left(x^2+5\right)$ so that the formula given in the list of formulae (MF19) can be applied.
	$\ln(x+2)=\ln 2+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$	**A1**	
	$\ln\left(x^2+5\right)=\ln\left(5\left(1+\frac{1}{5}x^2\right)\right)=\ln 5+\frac{1}{5}x^2+\ldots$	**A1**	
	$\left(\ln 2+\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right)+\left(\ln 5+\frac{1}{5}x^2+\ldots\right)$	**M1**	Sums power series
	OR Solution 2 $f'(x)=(x+2)^{-1}+2x\left(x^2+5\right)^{-1}$	**(M1 A1)**	Finds first derivative.
	$f''(x)=-(x+2)^{-2}-(2x)^2\left(x^2+5\right)^{-2}+2\left(x^2+5\right)^{-1}$	**(A1)**	Finds second derivative.
	$f(0)=\ln 10 \quad f'(0)=\frac{1}{2} \quad f''(0)=\frac{3}{20}$	**(M1)**	Evaluates derivatives at zero.
	$\ln(x+2)+\ln\left(x^2+5\right)=\ln 10+\frac{1}{2}x+\frac{3}{40}x^2$	**A1**	Accept $\ln 10$ written as $\ln 2+\ln 5$ but do not accept decimals. WWW.
		5	

Question	Answer	Marks	Guidance
2(a)	$\frac{dy}{dt} = e^t(t+1)$, $\frac{dx}{dt} = -t^{-2}$	B1	
	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -t^2 e^t(t+1) = -e^t(t^3+t^2)$	M1 A1	Uses chain rule, AG.
		3	
2(b)	$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -e^t(3t^2+2t) - e^t(t^3+t^2) = -e^t(t^3+4t^2+2t)$	M1 A1	Applies product rule.
	$\frac{d^2y}{dx^2} = t^3e^t(t^2+4t+2)$	M1 A1	Uses chain rule, multiplies by *their* $\frac{dt}{dx}$ for M1.
	Alternative method for question 2(b)		
	$\frac{d^2y}{dx^2} = -e^t\left(3t^2\frac{dt}{dx} + 2t\frac{dt}{dx}\right) - e^t(t^3+t^2)\frac{dt}{dx}$	(M1 A1)	Differentiates $-e^t(t^3+t^2)$ implicitly.
	$\frac{d^2y}{dx^2} = e^t(-4t^2-2t-t^3)\frac{dt}{dx} = t^3e^t(t^2+4t+2)$	(M1 A1)	Substitutes $\frac{dt}{dx} = -t^2$.
		4	

Question	Answer	Marks	Guidance
3(a)	$\cos 5\theta = \mathrm{Re}\left(\cos\theta + \mathrm{i}\sin\theta\right)^5 = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta$	**M1 A1**	Expands and takes real part. Accept RHS to LHS using $2\cos\theta = z + \dfrac{1}{z}$.
	$= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right)$	**M1**	Applies $\sin^2\theta = 1-\cos^2\theta$.
	$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	**A1**	AG
		4	
3(b)	$x = \cos\theta, \quad \cos 5\theta = \frac{1}{2}\sqrt{2}$	**M1**	Applies identify given in **(b)**.
	$5\theta = \pm\frac{1}{4}\pi + 2k\pi$	**M1**	Solves $\cos 5\theta = \frac{1}{2}\sqrt{2}$
	$\cos\left(\frac{1}{20}\pi\right)$	**A1**	Gives one correct solution. Accept $q = \frac{1}{20}$.
	$\cos(\frac{9}{20}\pi), \cos(\frac{17}{20}\pi), \cos(\frac{25}{20}\pi), \cos(\frac{33}{20}\pi)$	**A1**	Gives other solutions. OE. A0 for repeated roots.
		4	

Question	Answer	Marks	Guidance
4	$e^{\int 3\mathrm{d}x} = e^{3x}$	**M1 A1**	Finds integrating factor.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{3x}\right) = e^{3x}\sin x$	**M1**	Correct form on LHS and attempt to integrate RHS.
	EITHER $\int e^{3x}\sin x\,\mathrm{d}x = -e^{3x}\cos x + 3\int e^{3x}\cos x\,\mathrm{d}x$ **OR** $\int e^{3x}\sin x\,\mathrm{d}x = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,\mathrm{d}x$	**M1 A1**	Integrates by parts once or uses $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.
	EITHER $\int e^{3x}\sin x\,\mathrm{d}x = -e^{3x}\cos x + 3\left(e^{3x}\sin x - 3\int e^{3x}\sin x\,\mathrm{d}x\right)$ **OR** $\int e^{3x}\sin x\,\mathrm{d}x = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\left(\frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x\,\mathrm{d}x\right)$	**M1**	Integrates by parts again or substitutes $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$ and $\frac{e^{ix} + e^{-ix}}{2} = \cos x$.
	$ye^{3x} = \frac{1}{10}e^{3x}(3\sin x - \cos x) + C$	**A1**	Must not see i.
	$1 = -\frac{1}{10} + C$	**M1**	Finds C. Substitutes into their expression (must be integrated).
	$y = \frac{3}{10}\sin x - \frac{1}{10}\cos x + \frac{11}{10}e^{-3x}$	**A1**	Divides through by coefficient of y.
		9	

Question	Answer	Marks	Guidance
5(a)	$\frac{dy}{dx} = -2x\operatorname{sech}^2 x \tanh x + \operatorname{sech}^2 x$ $\left[= 1 - \tanh^2 x - 2x\tanh x + 2x\tanh^3 x\right]$	M1 A1	Differentiating using product rule and chain rule for M1.
	$\operatorname{sech}^2 x \neq 0 \Rightarrow 2x\tanh x - 1 = 0$	A1	AG.
	$2(0.7)\tanh(0.7) - 1 = -0.154 < 0$ $2(0.8)\tanh(0.8) - 1 = 0.062 > 0$	B1	Shows sign change. Must write down values correct to at least 1dp for B1.
		4	
5(b)	$\sum_{r=2}^{n} r\operatorname{sech}^2 r < \int_1^n x\operatorname{sech}^2 x \, dx$	M1 A1	Compares sum with integral. Consistent limits for M1.
	$\int_1^n x\operatorname{sech}^2 x \, dx = \left[x\tanh x\right]_1^n - \int_1^n \tanh x \, dx$	M1 A1	Integrates by parts.
	$= \left[x\tanh x + \ln\operatorname{sech} x\right]_1^n$	A1	
	$= n\tanh n + \ln\operatorname{sech} n - \left(\tanh 1 + \ln\operatorname{sech} 1\right)$	A1	AG. Must have gained all previous marks.
		6	

Question	Answer	Marks	Guidance
6(a)	$1, \quad 2, \quad -1$	B1	
		1	

Question	Answer	Marks	Guidance
6(b)	$\mathbf{P}^3 - 2\mathbf{P}^2 - \mathbf{P} + 2\mathbf{I} = 0$	B1	States that **P** satisfies its characteristic equation.
	$2\mathbf{P}^{-1} = \mathbf{I} + 2\mathbf{P} - \mathbf{P}^2$	M1	Multiplies through by $\mathbf{P}^{-1}$.
	$\mathbf{P}^2 = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$	M1 A1	
		4	
6(c)	$\mathbf{D} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	B1	
	$\mathbf{A}^{-1} = \mathbf{P}\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\mathbf{P}^{-1}$	M1	Applies $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$.
	$= \begin{pmatrix} \frac{1}{a} & -2 & \frac{1}{2} \\ 0 & 4 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$	M1 A1	Multiplies two adjacent matrices.
	$\begin{pmatrix} \frac{1}{a} & \frac{1-2a}{2a} & \frac{3-3a}{2a} \\ 0 & 2 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	A1	
		5	

Question	Answer	Marks	Guidance
7(a)	$\cosh 2A = \frac{1}{2}\left(e^{2A} + e^{-2A}\right) \quad \sinh A = \frac{1}{2}\left(e^{A} - e^{-A}\right)$	**B1**	
	$2\sinh^2 A = \frac{1}{2}\left(e^{A} - e^{-A}\right)^2 = \frac{1}{2}\left(e^{2A} - 2 + e^{-2A}\right) = \cosh 2A - 1$	**M1 A1**	Expands, AG. A0 for mixing variables e.g. $\sinh A = \frac{1}{2}\left(e^{x} - e^{-x}\right)$.
		3	

Question	Answer	Marks	Guidance
7(b)	$S=\int_0^{\frac{2}{3}} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\pi\int_0^{\frac{2}{3}} x^2\sqrt{1+4x^2}\,\mathrm{d}x$	**M1 A1**	Correct formula with correct limits. Correct limits for M1. (Limits may be recovered.)
	$S=\pi\int_0^{\sinh^{-1}\frac{4}{3}} \frac{1}{4}\sinh^2 u\sqrt{1+\sinh^2 u}\cosh u\,\mathrm{d}u$	**M1**	Applies given substitution to their expression with correct limits. (Limits may be recovered.)
	$=\frac{1}{4}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 u\cosh^2 u\,\mathrm{d}u$	**A1**	Must be simplified.
	$=\frac{1}{16}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 2u\,\mathrm{d}u$	**M1**	Applies $\sinh 2u = 2\sinh u\cosh u$. May use double angle formulae for cosh and sinh instead.
	$=\frac{1}{32}\pi\int_0^{\sinh^{-1}\frac{4}{3}} \cosh 4u - 1\,\mathrm{d}u$	**M1**	Applies $\sinh^2 A = \frac{1}{2}(\cosh 2A - 1)$. S must have the form $a\int \sinh^2 u\cosh^2 u\,\mathrm{d}u$.
	$=\frac{1}{32}\pi\left[\frac{1}{4}\sinh 4u - u\right]_0^{\sinh^{-1}\frac{4}{3}}$	**A1**	
	$\sinh^{-1}\frac{4}{3} = \ln 3$	**B1**	
	$S=\frac{1}{32}\pi\left(\frac{1}{8}\left(e^{\ln 81} - e^{-\ln 81}\right) - \ln 3\right) = \frac{1}{32}\pi\left(\frac{1}{8}\left(81 - \frac{1}{81}\right) - \ln 3\right) = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$	**A1**	AG.
		9	

Question	Answer	Marks	Guidance
8(a)	$\dfrac{dv}{dx}=4y^3\dfrac{dy}{dx}$	**B1**	$v=y^4$
	$\dfrac{d^2v}{dx^2}=4y^3\dfrac{d^2y}{dx^2}+12y^2\left(\dfrac{dy}{dx}\right)^2$	**B1**	
	$\dfrac{d^2v}{dx^2}+\dfrac{dv}{dx}+4v=4y^3\dfrac{d^2y}{dx^2}+12y^2\left(\dfrac{dy}{dx}\right)^2+4y^3\dfrac{dy}{dx}+4y^4$	**M1**	Uses substitution to find *v*-*x* equation, AG.
	$=4\left(y^3\dfrac{d^2y}{dx^2}+3y^2\left(\dfrac{dy}{dx}\right)^2+y^3\dfrac{dy}{dx}+y^4\right)=4e^{-2x}$	**A1**	AG.
	Alternative method for question 8(a)		
	$\dfrac{dy}{dx}=\dfrac{1}{4y^3}\dfrac{dv}{dx}=\dfrac{1}{4v^{\frac{3}{4}}}\dfrac{dv}{dx}$	**(B1)**	$y=v^{\frac{1}{4}}$
	$\dfrac{d^2y}{dx^2}=\dfrac{1}{4v^{\frac{3}{4}}}\dfrac{d^2v}{dx^2}-\dfrac{3}{16v^{\frac{7}{4}}}\left(\dfrac{dv}{dx}\right)^2$	**(B1)**	
	$v^{\frac{3}{4}}\left(\dfrac{1}{4v^{\frac{3}{4}}}\dfrac{d^2v}{dx^2}-\dfrac{3}{16v^{\frac{7}{4}}}\left(\dfrac{dv}{dx}\right)^2\right)+3v^{\frac{1}{2}}\left(\dfrac{1}{4v^{\frac{3}{4}}}\dfrac{dv}{dx}\right)^2+v^{\frac{3}{4}}\left(\dfrac{1}{4v^{\frac{3}{4}}}\dfrac{dv}{dx}\right)+v=e^{-2x}$	**(M1)**	Uses substitution to find *v*-*x* equation, AG.
	$\dfrac{1}{4}\dfrac{d^2v}{dx^2}-\dfrac{3}{16v}\left(\dfrac{dv}{dx}\right)^2+\dfrac{3}{16v}\left(\dfrac{dv}{dx}\right)^2+\dfrac{1}{4}\dfrac{dv}{dx}+v=e^{-2x}$	**(A1)**	
		4	

Question	Answer	Marks	Guidance
8(b)	$m^2 + m + 4 = 0$	**M1**	Auxiliary equation.
	$[v=]\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right)$	**A1**	Complimentary function. Allow '$v=$' missing.
	$v = k\mathrm{e}^{-2x} \Rightarrow v' = -2k\mathrm{e}^{-2x} \Rightarrow v'' = 4k\mathrm{e}^{-2x}$	**B1**	Particular integral and its derivatives.
	$4k\mathrm{e}^{-2x} - 2k\mathrm{e}^{-2x} + 4k\mathrm{e}^{-2x} = 4\mathrm{e}^{-2x}$	**M1**	Substitutes and equates coefficients.
	$k = \frac{2}{3}$	**A1**	
	$v = y^4 = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right) + \frac{2}{3}\mathrm{e}^{-2x}$	**A1**	Substitutes for y and find v in terms of x. Must not see i.
	$v' = 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-\frac{1}{2}x}\left(-\frac{\sqrt{15}}{2}A\sin\frac{\sqrt{15}}{2}x + \frac{\sqrt{15}}{2}B\cos\frac{\sqrt{15}}{2}x\right) - \frac{1}{2}\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right) - \frac{4}{3}\mathrm{e}^{-2x}$	**B1**	
	$1 = A + \frac{2}{3} \qquad -\frac{3}{2} = \frac{\sqrt{15}}{2}B - \frac{1}{2}A - \frac{4}{3} \qquad \Rightarrow A = \frac{1}{3},\quad B = 0$	**M1 A1**	Substitutes initial conditions and forms simultaneous equations. Must have used the product rule when differentiating *their* v for M1.
	$y = \left(\frac{1}{3}\mathrm{e}^{-\frac{1}{2}x}\cos\frac{\sqrt{15}}{2}x + \frac{2}{3}\mathrm{e}^{-2x}\right)^{\frac{1}{4}}$	**A1**	Accept $y = \pm\left(\frac{1}{3}\mathrm{e}^{-\frac{1}{2}x}\cos\frac{\sqrt{15}}{2}x + \frac{2}{3}\mathrm{e}^{-2x}\right)^{\frac{1}{4}}$
		10	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME			
CENTRE NUMBER		CANDIDATE NUMBER	

9662100742

FURTHER MATHEMATICS **9231/23**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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1 Show that the system of equations

$$\begin{aligned} 14x - 4y + 6z &= 5, \\ x + y + kz &= 3, \\ -21x + 6y - 9z &= 14, \end{aligned}$$

where k is a constant, does not have a unique solution and interpret this situation geometrically. [4]

2 Find the roots of the equation $(z+5\mathrm{i})^3 = 4+4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\cos\theta + \mathrm{i}(r\sin\theta - 5)$, where $r > 0$ and $0 < \theta < 2\pi$. [5]

3 Find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\frac{1}{2}e^x\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [6]

4 Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 27x^2,$$

given that, when $x = 0$, $y = 2$ and $\frac{dy}{dx} = -8$. [10]

5 The curve C has parametric equations

$$x = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}, \qquad y = 2t + 5, \qquad \text{for } 0 < t \leqslant 3.$$

(a) Find the exact length of C. [5]

(b) Find the set of values of t for which $\dfrac{d^2y}{dx^2} > 0$. [5]

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6 **(a)** Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x. \qquad [3]$$

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(b) Using the substitution $u = \sinh x$, find $\int \sinh^2 2x \cosh x \, dx$. [4]

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(c) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\tanh x = \sinh^2 2x,$$

given that $y = 4$ when $x = 0$. Give your answer in the form $y = f(x)$. [7]

7 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -6 & 2 & 13 \\ 0 & -2 & 5 \\ 0 & 0 & 8 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

8 **(a)** State the sum of the series $1+z+z^2+\ldots+z^{n-1}$, for $z \neq 1$. [1]

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(b) By letting $z = \cos\theta + \mathrm{i}\sin\theta$, where $\cos\theta \neq 1$, show that

$$1+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta = \frac{1}{2}\left(1-\cos n\theta+\frac{\sin n\theta \sin\theta}{1-\cos\theta}\right).$$ [7]

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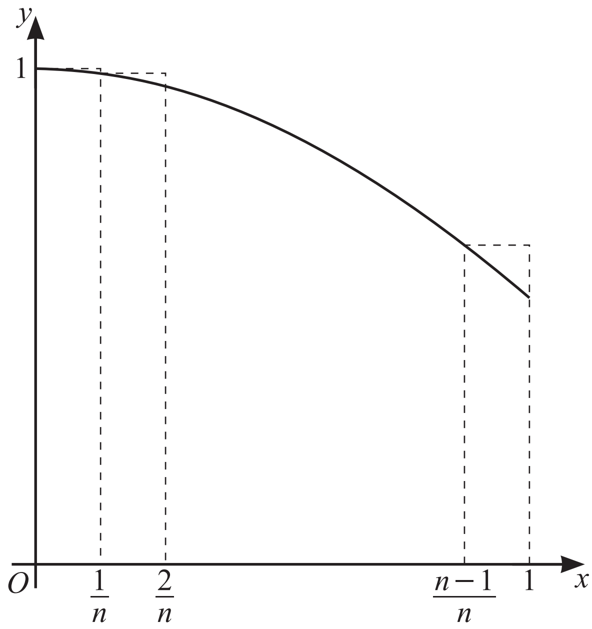

The diagram shows the curve with equation $y = \cos x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(c) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 \cos x \, dx < \frac{1}{2n}\left(1 - \cos 1 + \frac{\sin 1 \sin \frac{1}{n}}{1 - \cos \frac{1}{n}}\right).$$ [4]

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(d) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 \cos x \, dx$. [3]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/23**

Paper 2 Further Pure Mathematics 2 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics-Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate’s intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$\begin{vmatrix} 14 & -4 & 6 \\ 1 & 1 & k \\ -21 & 6 & -9 \end{vmatrix} = 14\begin{vmatrix} 1 & k \\ 6 & -9 \end{vmatrix} + 4\begin{vmatrix} 1 & k \\ -21 & -9 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ -21 & 6 \end{vmatrix}$ $= 14(-9-6k)+4(-9+21k)+6(6+21)=0$	M1 A1	Evaluates determinant. Can expand along any row e.g. $-(36-36)+(-126+126)+k(84-84)$. If using row operations, they must show an inconsistent system for M1. All their row operations must be correct for A1.
	Two parallel planes, not identical.	B1	
	Other plane not parallel.	B1	
		4	

Question	Answer	Marks	Guidance
2	$(z+5\mathrm{i})^3 = 4+4\mathrm{i}\sqrt{3} = 8\mathrm{e}^{\frac{1}{3}\pi}$	B1	Finds modulus and argument of $4+4\mathrm{i}\sqrt{3}$.
	$z_1 = 2\left(\cos\frac{1}{9}\pi + \mathrm{i}\sin\frac{1}{9}\pi\right) - 5\mathrm{i} = 2\cos\frac{1}{9}\pi + \mathrm{i}\left(2\sin\frac{1}{9}\pi - 5\right)$	M1 A1	Finds one root.
	$z_2 = 2\cos\frac{7}{9}\pi + \mathrm{i}\left(2\sin\frac{7}{9}\pi - 5\right)$, $z_3 = 2\cos\frac{13}{9}\pi + \mathrm{i}\left(2\sin\frac{13}{9}\pi - 5\right)$	A1 FT	Finds other two roots. FT on their modulus.
		A1 FT	
		5	

Question	Answer	Marks	Guidance
3	$\dfrac{dy}{dx}=\dfrac{\frac{1}{2}e^x}{1-\frac{1}{4}e^{2x}}$	B1	
	$\dfrac{d^2y}{dx^2}=\dfrac{\left(1-\frac{1}{4}e^{2x}\right)\left(\frac{1}{2}e^x\right)-\frac{1}{2}e^x\left(-\frac{1}{2}e^{2x}\right)}{\left(1-\frac{1}{4}e^{2x}\right)^2}$	B1	
	$f'(0)=\dfrac{2}{3} \quad f''(0)=\dfrac{10}{9}$	M1	Evaluates derivatives at $x=0$.
	$f(0)=\tanh^{-1}\left(\dfrac{1}{2}\right)=\dfrac{1}{2}\ln\left(\dfrac{3}{2}\times 2\right)$	M1	Uses logarithmic form of $\tanh^{-1}$.
	$\dfrac{1}{2}\ln 3+\dfrac{2}{3}x+\dfrac{5}{9}x^2$	M1 A1	Applies $f(x)=f(0)+f'(0)x+\dfrac{1}{2!}f''(0)x^2$
		6	

Question	Answer	Marks	Guidance
4	$m^2+2m+3=0$	M1	Auxiliary equation.
	$[y=]e^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)$	A1	Complementary function. Allow with “$y=$” missing.
	$y=px^2+qx+r\Rightarrow y'=2px+q\Rightarrow y''=2p$	B1	Particular integral and its derivatives.
	$3p=27 \quad 3q+4p=0 \quad 2p+2q+3r=0$	M1	Substitutes and equates coefficients.

Question	Answer	Marks	Guidance
4	$p=9 \quad q=-12 \quad r=2$	**A1**	
	$y=\mathrm{e}^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)+9x^2-12x+2$	**A1**	General solution. Must have " $y=$ ".
	$y'=\mathrm{e}^{-x}\left(-\sqrt{2}A\sin\sqrt{2}x+\sqrt{2}B\cos\sqrt{2}x\right)-\mathrm{e}^{-x}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)+18x-12$	**M1***	Differentiates. Must use product rule.
	$A+2=2 \quad \sqrt{2}B-A-12=-8$ $\Rightarrow A=0,\ B=2\sqrt{2}$	**DM1** **A1**	Uses initial conditions
	$y=2\sqrt{2}\mathrm{e}^{-x}\sin\sqrt{2}x+9x^2-12x+2$	**A1**	Must have " $y=$ ".
		10	

Question	Answer	Marks	Guidance
5(a)	$\dot{x}=t^{\frac{1}{2}}-t^{-\frac{1}{2}}, \quad \dot{y}=2$	**B1**	Differentiates x and y with respect to t.
	$\dot{x}^2+\dot{y}^2=\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^2+4=t+2+t^{-1}=\left(t^{\frac{1}{2}}+t^{-\frac{1}{2}}\right)^2$	**M1 A1**	Factorises $\dot{x}^2+\dot{y}^2$.
	$\int_0^3 t^{\frac{1}{2}}+t^{-\frac{1}{2}}\,\mathrm{d}t=\left[\frac{2}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right]_0^3=4\sqrt{3}$	**M1 A1**	Applies correct formula for arc length. M1 for their $\int_0^3\sqrt{\dot{x}^2+\dot{y}^2}\,\mathrm{d}t$. Answer must be simplified to $4\sqrt{3}$ for A1.
		5	

Question	Answer	Marks	Guidance
5(b)	$\dfrac{dy}{dx}=\dfrac{\dot{y}}{\dot{x}}=\dfrac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}$	B1	Finds first derivative.
	$\dfrac{d}{dt}\left(\dfrac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}\right)=\dfrac{-2\left(\frac{1}{2}t^{-\frac{1}{2}}+\frac{1}{2}t^{-\frac{3}{2}}\right)}{\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^2}$	B1	Differentiates $\dfrac{dy}{dx}$ with respect to t.
	$\dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(\dfrac{2}{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}\right)\times\dfrac{dt}{dx}=\dfrac{-2\left(\frac{1}{2}t^{-\frac{1}{2}}+\frac{1}{2}t^{-\frac{3}{2}}\right)}{\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^3}=-\dfrac{t+1}{(t-1)^3}$	M1 A1	Applies chain rule. OE. Does not have to be simplified for A1.
	$0<t<1$	A1	Accept $-1<t<1$. CWO.
		5	

Question	Answer	Marks	Guidance
6(a)	$\cosh x=\dfrac{1}{2}\left(e^x+e^{-x}\right)\quad \sinh x=\dfrac{1}{2}\left(e^x-e^{-x}\right)$	B1	
	$\dfrac{1}{2}\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)=\dfrac{1}{2}\left(e^{2x}-e^{-2x}\right)=\sinh 2x$	M1 A1	Expands, AG.
		3	

Question	Answer	Marks	Guidance
6(b)	$u = \sinh x \Rightarrow du = \cosh x\, dx$	B1	
	$\int \sinh^2 2x \cosh x\, \mathrm{d}x = 4\int \sinh^2 x \cosh^2 x\, \mathrm{d}u = 4\int \sinh^2 x\left(\sinh^2 x + 1\right)\mathrm{d}u$	M1	Applies identities to find integral in terms of u.
	$= 4\int u^2\left(u^2+1\right)du$	A1	
	$= 4\left(\frac{1}{5}u^5 + \frac{1}{3}u^3\right)(+C) = 4\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x\right)(+C)$	A1	
		4	
6(c)	$\mathrm{e}^{\int \tanh x\, dx} = \mathrm{e}^{\ln\cosh x} = \cosh x$	M1 A1	Finds integrating factor.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\cosh x\right) = \sinh^2 2x \cosh x$	M1	Correct form on LHS and attempt to integrate RHS.
	$y\cosh x = 4\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x\right) + C$	M1 A1	Integrates RHS using their part (b).
	$4 = C$	M1	Substitutes initial conditions.
	$y = 4\,\mathrm{sech}\, x\left(\frac{1}{5}\sinh^5 x + \frac{1}{3}\sinh^3 x + 1\right)$	A1	
		7	

Question	Answer	Marks	Guidance
7(a)	$\lambda = -6,\quad \lambda = -2,\quad \lambda = 8$	B1	
	$\lambda = -6:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = -2:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 13 \\ 0 & 0 & 5 \end{vmatrix} = \begin{pmatrix} 10 \\ 20 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$	A1	
	$\lambda = 8:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -14 & 2 & 13 \\ 0 & -10 & 5 \end{vmatrix} = \begin{pmatrix} 140 \\ 70 \\ 140 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$	A1	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$	M1 A1	Or correctly matched permutations of columns. M1 for their (non-zero) eigenvectors matched to their eigenvalues.
		7	
7(b)	$\boldsymbol{A}^3 - 52\boldsymbol{A} - 96\mathbf{I} = 0$	M1	Substitutes **A** into characteristic equation.
	$96\mathbf{A}^{-1} = \mathbf{A}^2 - 52\mathbf{I}$	M1	Multiples through by $\mathbf{A}^{-1}$.
	$\mathbf{A}^2 = \begin{pmatrix} 36 & -16 & 36 \\ 0 & 4 & 30 \\ 0 & 0 & 64 \end{pmatrix}$	B1	

Question	Answer	Marks	Guidance
7(b)	$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{3}{8} \\ 0 & -\frac{1}{2} & \frac{5}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$	**A1**	
		4	

Question	Answer	Marks	Guidance
8(a)	$\dfrac{z^n - 1}{z - 1}$	**B1**	
		1	
8(b)	$\dfrac{z^n - 1}{z - 1} = \dfrac{\cos n\theta - 1 + \mathrm{i}\sin n\theta}{\cos\theta - 1 + \mathrm{i}\sin\theta}$	**B1**	
	$\dfrac{(\cos n\theta - 1 + \mathrm{i}\sin n\theta)(\cos\theta - 1 - \mathrm{i}\sin\theta)}{(\cos\theta - 1 + \mathrm{i}\sin\theta)(\cos\theta - 1 - \mathrm{i}\sin\theta)}$	**M1**	Multiplies numerator and denominator by complex conjugate.
	$\mathrm{Re}\left(\dfrac{z^n - 1}{z - 1}\right) = \dfrac{\cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos n\theta - \cos\theta + 1}{(\cos\theta - 1)^2 + \sin^2\theta}$	**M1**	Takes real part. $\cos(n-1)\theta = \cos n\theta\cos\theta + \sin n\theta\sin\theta$
	$= \dfrac{\cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos n\theta}{2(1 - \cos\theta)} + \dfrac{1}{2}$	**A1**	
	$= \dfrac{\cos n\theta(\cos\theta - 1) + \sin n\theta\sin\theta}{2(1 - \cos\theta)} + \dfrac{1}{2}$	**M1**	Factorises.

Question	Answer	Marks	Guidance
8(b)	$=\frac{1}{2}\left(1-\cos n\theta+\frac{\sin n\theta\sin\theta}{1-\cos\theta}\right)$	M1 A1	Divides through by denominator. AG.
	Alternative method for question 8(b)		
	$\frac{z^n-1}{z-1}=\frac{e^{in\theta}-1}{e^{i\theta}-1}$	B1	
	$\frac{e^{i\left(n-\frac{1}{2}\right)\theta}-e^{-i\frac{1}{2}\theta}}{e^{i\frac{1}{2}\theta}-e^{-i\frac{1}{2}\theta}}=\frac{\cos\left(n-\frac{1}{2}\right)\theta+i\sin\left(n-\frac{1}{2}\right)\theta-\cos\frac{1}{2}\theta+i\sin\left(\frac{1}{2}\theta\right)}{2i\sin\frac{1}{2}\theta}$	M1	
	$\mathrm{Re}\left(\frac{z^n-1}{z-1}\right)=\frac{\sin(n-\frac{1}{2})\theta+\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	M1	Takes real part
	$=\frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}+\frac{1}{2}$	A1	
	$=\frac{\sin n\theta\cos\frac{1}{2}\theta-\cos n\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}+\frac{1}{2}$	M1	Uses compound angle identity
	$=\frac{\sin n\theta\cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}-\frac{1}{2}\cos n\theta+\frac{1}{2}$	M1	Divides through by denominator.
	$\frac{\sin n\theta\sin\theta}{4\sin^2\frac{1}{2}\theta}-\frac{1}{2}\cos n\theta+\frac{1}{2}=\frac{\sin n\theta\sin\theta}{2\left(1-\cos\theta\right)}-\frac{1}{2}\cos n\theta+\frac{1}{2}$	A1	AG. $\sin\theta=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $2\sin^2\frac{1}{2}\theta=1-\cos\theta$.
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Question	Answer	Marks	Guidance
8(c)	$\int_0^1 \cos x \, dx < \frac{1}{n} + \frac{1}{n}\cos\frac{1}{n} + \frac{1}{n}\cos\frac{2}{n} + \ldots \frac{1}{n}\cos\frac{n-1}{n}$	**M1 A1**	Forms sum of areas of rectangles given in the diagram. A0 if comparison with integral missing or unclear.
	$= \frac{1}{n}\left(1 + \cos\frac{1}{n} + \cos\frac{2}{n} + \ldots + \cos\frac{n-1}{n}\right) = \frac{1}{2n}\left(1 - \cos\frac{n}{n} + \frac{\sin\frac{n}{n}\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$	**M1 A1**	Applies result from part (b) with $\theta = \frac{1}{n}$. AG.
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8(d)	$\int_0^1 \cos x \, dx > \frac{1}{n}\cos\frac{1}{n} + \frac{1}{n}\cos\frac{2}{n} + \ldots \frac{1}{n}\cos\frac{n}{n}$	**M1 A1**	Forms sum of areas of rectangles. A0 if comparison with integral missing or unclear.
	$= \frac{1}{2n}\left(1 - \cos 1 + \frac{\sin 1 \sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right) + \frac{1}{n}\cos 1 - \frac{1}{n} = \frac{1}{2n}\left(\cos 1 - 1 + \frac{\sin 1 \sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$	**A1**	
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Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

8795483048

FURTHER MATHEMATICS **9231/31**

Paper 3 Further Mechanics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\mathrm{m\,s^{-2}}$.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

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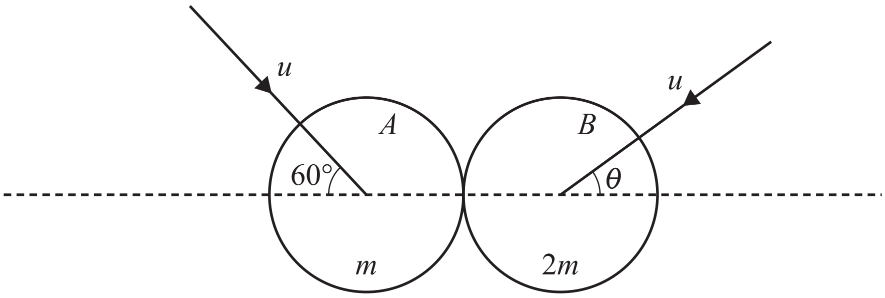

Two uniform smooth spheres A and B of equal radii have masses m and $2m$ respectively. The two spheres are moving with equal speeds u on a smooth horizontal surface when they collide. Immediately before the collision, A's direction of motion makes an angle of $60°$ with the line of centres, and B's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is e.

After the collision, the component of the velocity of A along the line of centres is v and B moves perpendicular to the line of centres. Sphere A now has twice as much kinetic energy as sphere B.

(a) Show that $v = \frac{1}{2}u(4\cos\theta - 1)$. [1]

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(b) Find the value of $\cos\theta$. [4]

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(c) Find the value of e. [2]

2 A ball of mass 2 kg is projected vertically downwards with speed $5\,\mathrm{m\,s^{-1}}$ through a liquid. At time t s after projection, the velocity of the ball is $v\,\mathrm{m\,s^{-1}}$ and its displacement from its starting point is x m. The forces acting on the ball are its weight and a resistive force of magnitude $0.2v^2$ N.

(a) Find an expression for v in terms of t. [6]

(b) Deduce what happens to v for large values of t. [1]

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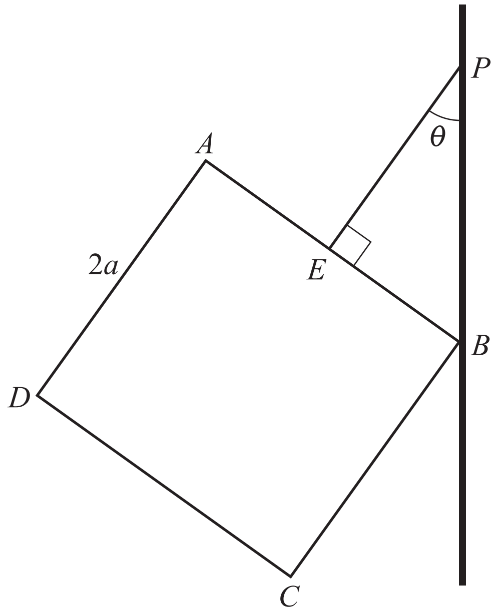

A uniform square lamina of side $2a$ and weight W is suspended from a light inextensible string attached to the midpoint E of the side AB. The other end of the string is attached to a fixed point P on a rough vertical wall. The vertex B of the lamina is in contact with the wall. The string EP is perpendicular to the side AB and makes an angle θ with the wall (see diagram). The string and the lamina are in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the lamina is $\frac{1}{2}$.

Given that the vertex B is about to slip up the wall, find the value of $\tan\theta$. [8]

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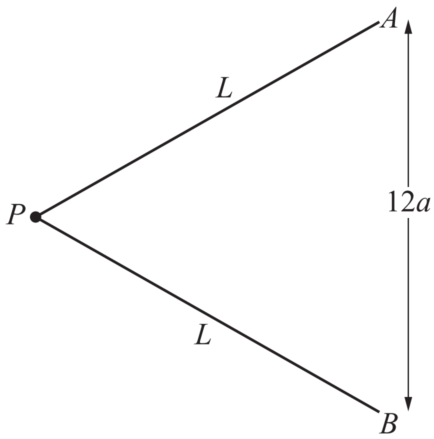

A light elastic string has natural length $8a$ and modulus of elasticity $5mg$. A particle P of mass m is attached to the midpoint of the string. The ends of the string are attached to points A and B which are a distance $12a$ apart on a smooth horizontal table. The particle P is held on the table so that $AP = BP = L$ (see diagram). The particle P is released from rest. When P is at the midpoint of AB it has speed $\sqrt{80ag}$.

(a) Find L in terms of a. [5]

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(b) Find the initial acceleration of P in terms of g. [3]

5 A particle P is projected with speed $u\,\text{m}\,\text{s}^{-1}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. During its flight P passes through the point which is a horizontal distance $3a$ from O and a vertical distance $\frac{3}{8}a$ above the horizontal plane. It is given that $\tan\theta = \frac{1}{3}$.

(a) Show that $u^2 = 8ag$. [2]

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A particle Q is projected with speed $V\,\text{m}\,\text{s}^{-1}$ at an angle α above the horizontal from O at the instant when P is at its highest point. Particles P and Q both land at the same point on the horizontal plane at the same time.

(b) Find V in terms of a and g. [7]

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6 A particle P of mass m is attached to one end of a light inextensible rod of length $3a$. An identical particle Q is attached to the other end of the rod. The rod is smoothly pivoted at a point O on the rod, where $OQ = x$. The system, of rod and particles, rotates about O in a vertical plane.

At an instant when the rod is vertical, with P above Q, the particle P is moving horizontally with speed u. When the rod has turned through an angle of 60° from the vertical, the speed of P is $2\sqrt{ag}$, and the tensions in the two parts of the rod, OP and OQ, have equal magnitudes.

(a) Show that the speed of Q when the rod has turned through an angle of 60° from the vertical is $\frac{2x}{3a-x}\sqrt{ag}$. [2]

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(b) Find x in terms of a. [5]

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(c) Find u in terms of a and g. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/31**

Paper 3 Further Mechanics **October/November 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1. Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2. Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3. Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5. Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6. Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	PCLM: $mv = -mu\cos 60° + 2mu\cos\theta$ $v = -\frac{1}{2}u + 2u\cos\theta = v = \frac{1}{2}u(4\cos\theta - 1)$	**B1**	First line must be seen. AG
		1	
1(b)	KE of $A = \frac{1}{2}m\left(v^2 + (u\sin 60°)^2\right)$	**B1**	
	KE of A = 2 × KE of B, so $\frac{1}{2}m\left(v^2 + (u\sin 60°)^2\right) = 2 \times \frac{1}{2} \times 2m(u\sin\theta)^2$	**M1**	
	$\left(\frac{1}{2}u(4\cos\theta - 1)\right)^2 + \frac{3}{4}u^2 = 4u^2(\sin\theta)^2$ $8\cos^2\theta - 2\cos\theta - 3 = 0$	**M1**	Use result of **(a)** and rearrange. Obtain 3-term quadratic.
	$\cos\theta = \frac{3}{4}$, $-\frac{1}{2}$ but angle is acute, so $\cos\theta = \frac{3}{4}$	**A1**	
		4	
1(c)	NEL: $v = e(u\cos 60° + u\cos\theta)$	**M1**	
	$v = u$, $u = e\left(\frac{1}{2}u + \frac{3}{4}u\right)$ $e = \frac{4}{5}$	**A1**	
		2	

Question	Answer	Marks	Guidance
2(a)	$2\frac{\mathrm{d}v}{\mathrm{d}t} = 2g - 0.2v^2$	**B1**	
	Separate variables and attempt to integrate $\frac{\mathrm{d}v}{0.1\left(100 - v^2\right)} = \mathrm{d}t$	**M1**	Integrate to a ln term of the correct form.
	$\frac{1}{20}\ln\left(\frac{10+v}{10-v}\right) = 0.1t + c$	**A1**	
	$t = 0,\ \ v = 5,\ \ \ c = \frac{1}{20}\ln 3$	**M1**	Use initial condition.
	$2t = \ln\frac{10+v}{3(10-v)},\ \mathrm{e}^{2t} = \frac{10+v}{3(10-v)}$	**M1**	Rearrange, removing ln.
	$v = \frac{30 - 10\mathrm{e}^{-2t}}{3 + \mathrm{e}^{-2t}}$	**A1**	AEF
		6	
2(b)	$v \to 10$	**B1FT**	FT from expression of correct form.
		1	

Question	Answer	Marks	Guidance
3	Let N be normal reaction at B and F the frictional force acting downwards $\uparrow T\cos\theta = F + W$	**B1**	
	$\rightarrow T\sin\theta = N$	**B1**	
	Moments about B: $Ta = W\sin\theta \times a + W\cos\theta \times a$	**M1A1**	A moments equation with all relevant forces.
	$F = \frac{1}{2}N$ used	**M1**	
	$(\cos\theta + \sin\theta)\left(\cos\theta - \frac{1}{2}\sin\theta\right) = 1$ oe	**M1**	Combine to obtain equation in θ. Equation in trigonometric functions only.
	$\frac{1}{2}\cos\theta\sin\theta = \frac{3}{2}(\sin\theta)^2$ $\sin\theta(\cos\theta - 3\sin\theta) = 0$	**M1**	Solve trigonometric equation.
	$\tan\theta = \frac{1}{3}$	**A1**	
		8	

Question	Answer	Marks	Guidance
4(a)	Energy equation: gain in KE = loss in EPE $\frac{1}{2}m\times 80ag=\frac{1}{2}\times\frac{5mg}{8a}\left((2L-8a)^2-(4a)^2\right)$	B1 M1 A1	At least one correct EPE. 3-term energy equation. All correct.
	$4L^2-8aL-20a^2=0$	M1	Simplify to quadratic in L.
	$L=10a$	A1	Correct single answer.
		5	
4(b)	At P: $2T\cos\theta=m\times\text{acceleration}$	M1	
	Hooke's law: $T=\frac{5mg}{8a}\times 12a=\frac{15}{2}mg$	M1	
	Solve, acceleration = $12g$	A1	
		3	

Question	Answer	Marks	Guidance
5(a)	Use equation of trajectory: $y = x\tan\alpha - \dfrac{gx^2}{2u^2}\sec^2\alpha$: $\dfrac{3}{8}a = 3a \times \dfrac{1}{3} - g \times \dfrac{(3a)^2}{2u^2}\left(1+\dfrac{1}{9}\right)$	**M1**	
	$\dfrac{5}{8}a = \dfrac{5ga}{u^2}$, $\quad u^2 = 8ga$	**A1**	At least one step of working. AG
		2	

Question	Answer	Marks	Guidance
5(b)	For P, time of flight $T = \dfrac{2u\sin\theta}{g}$	B1	
	For P, range = $Tu\cos\theta$	B1	
	For Q, time of flight = $\frac{1}{2}T$, so range = $\frac{1}{2}TV\cos\alpha$ Equate: $V\cos\alpha = 2u\cos\theta$ (1)	B1	
	For Q vertically: $0 = V\sin\alpha \times \frac{1}{2}T - \dfrac{\frac{1}{2}gT^2}{4}$, so $T = \dfrac{4V\sin\alpha}{g}$	M1	
	Equate with result for P: $\dfrac{4V\sin\alpha}{g} = \dfrac{2u\sin\theta}{g}$ so $V\sin\alpha = \frac{1}{2}u\sin\theta$ (2)	A1	
	From (1) and (2): $V^2 = u^2\left(4(\cos\theta)^2 + \frac{1}{4}(\sin\theta)^2\right)$	M1	
	$V^2 = u^2 \times \left(4\times\frac{9}{10} + \frac{1}{4}\times\frac{1}{10}\right) = \frac{145}{40}u^2$ $V = \sqrt{29ag}$	A1	
		7	

Question	Answer	Marks	Guidance
6(a)	Angular speeds of P and Q are equal, so $\frac{v_Q}{x} = \frac{v_P}{3a - x}$	M1	
	$v_Q = \frac{2x\sqrt{ag}}{3a - x}$	A1	Shown convincingly: angular speeds equal stated. AG
		2	
6(b)	For P: $T + mg\cos 60° = \frac{m \times 4ag}{3a - x}$	B1	
	For Q: $T - mg\cos 60° = \frac{mv_Q^2}{x}$	B1	
	Eliminate T: $-mg\cos 60° + \frac{m.4ag}{3a - x} = mg\cos 60° + \frac{mv_Q^2}{x}$	M1	
	$\frac{m \times 4ag}{3a - x} = 1 + \frac{mx4ag}{(3a - x)^2}$ $4a(3a - x) = (3a - x)^2 + 4ax$, $x^2 + 2ax - 3a^2 = 0$	M1	Solve to find x. Obtain 3-term quadratic equation.
	$(x - a)(x + 3a) = 0$, $x = a$	A1	
		5	

Question	Answer	Marks	Guidance
6(c)	Energy changes from initial position:	B1	KEs correct.
	Gain in KE of P: $\frac{1}{2}m\left(4ag-u^2\right)$ Loss in KE of Q: $\frac{1}{2}m\left(\left(\frac{u}{2}\right)^2-v_Q^2\right)$ Loss in GPE of $P = mg(3a-x)(1-\cos 60°)$ $(=mga)$ Gain in GPE of $Q = mgx(1-\cos 60°)$ $\left(=\frac{1}{2}mga\right)$	B1FT	GPEs correct.
	$\frac{1}{2}m\left(4ag-u^2\right)-\frac{1}{2}m\left(\left(\frac{u}{2}\right)^2-v_Q^2\right)=-mgx(1-\cos 60°)+mg(3a-x)(1-\cos 60°)$	M1	Energy equation.
	Simplify: $4ag-\frac{5}{4}u^2+ag=ag$ $u^2=\frac{16}{5}ag,\quad u=\frac{4}{5}\sqrt{5ag}$	A1	AEF
		4	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

1394054656

FURTHER MATHEMATICS **9231/32**

Paper 3 Further Mechanics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\text{m}\,\text{s}^{-2}$.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

1 One end of a light inextensible string of length a is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The string is taut and makes an angle θ with the downward vertical through O, where $\cos\theta = \frac{2}{3}$. The particle moves in a horizontal circle with speed v.

Find v in terms of a and g. [4]

2 A particle P of mass $0.5\,\mathrm{kg}$ moves in a straight line. At time $t\,\mathrm{s}$ the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from a fixed point O on the line is $x\,\mathrm{m}$. The only forces acting on P are a force of magnitude $\dfrac{150}{(x+1)^2}\,\mathrm{N}$ in the direction of increasing displacement and a resistive force of magnitude $\dfrac{450}{(x+1)^3}\,\mathrm{N}$. When $t = 0$, $x = 0$ and $v = 20$.

Find v in terms of x, giving your answer in the form $v = \dfrac{Ax+B}{(x+1)}$, where A and B are constants to be determined. [6]

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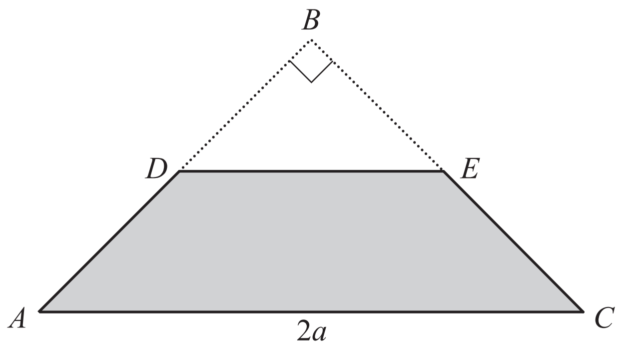

A uniform lamina is in the form of an isosceles triangle ABC in which $AC = 2a$ and angle $ABC = 90°$. The point D on AB is such that the ratio $DB:AB = 1:k$. The point E on CB is such that DE is parallel to AC. The triangle DBE is removed from the lamina (see diagram).

(a) Find, in terms of k, the distance of the centre of mass of the remaining lamina $ADEC$ from the midpoint of AC. [4]

When the lamina $ADEC$ is freely suspended from the vertex A, the edge AC makes an angle θ with the downward vertical, where $\tan\theta = \frac{5}{18}$.

(b) Find the value of k. [3]

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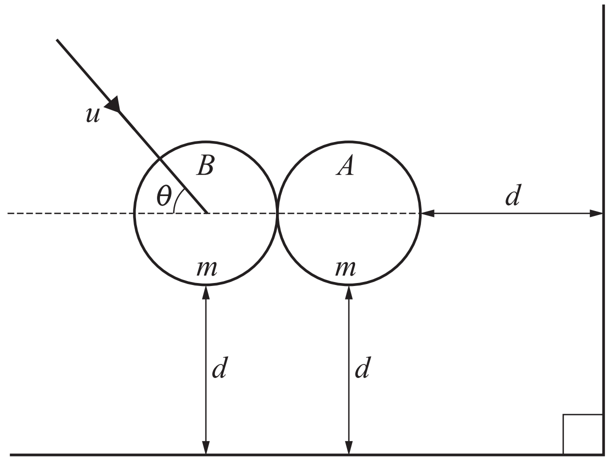

Two smooth vertical walls meet at right angles. The smooth sphere A, with mass m, is at rest on a smooth horizontal surface and is at a distance d from each wall. An identical smooth sphere B is moving on the horizontal surface with speed u at an angle θ with the line of centres when the spheres collide (see diagram). After the collision, the spheres take the same time to reach a wall. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(a) Find the value of $\tan\theta$. [4]

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(b) Find the percentage loss in the total kinetic energy of the spheres as a result of this collision. [3]

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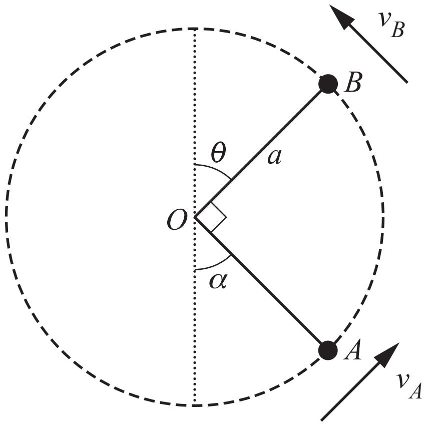

A bead of mass m moves on a smooth circular wire, with centre O and radius a, in a vertical plane. The bead has speed v_A when it is at the point A where OA makes an angle α with the downward vertical through O, and $\cos\alpha = \frac{3}{5}$. Subsequently the bead has speed v_B at the point B, where OB makes an angle θ with the upward vertical through O. Angle AOB is a right angle (see diagram). The reaction of the wire on the bead at B is in the direction OB and has magnitude equal to $\frac{1}{6}$ of the magnitude of the reaction when the bead is at A.

(a) Find, in terms of m and g, the magnitude of the reaction at B. [6]

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(b) Given that $v_A = \sqrt{kag}$, find the value of k. [2]

6 A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha .$$ [3]

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During its flight, P must clear an obstacle of height h m that is at a horizontal distance of 32 m from the point of projection. When $u = 40\sqrt{2}\,\mathrm{m\,s^{-1}}$, P just clears the obstacle. When $u = 40\,\mathrm{m\,s^{-1}}$, P only achieves 80% of the height required to clear the obstacle.

(b) Find the two possible values of h. [6]

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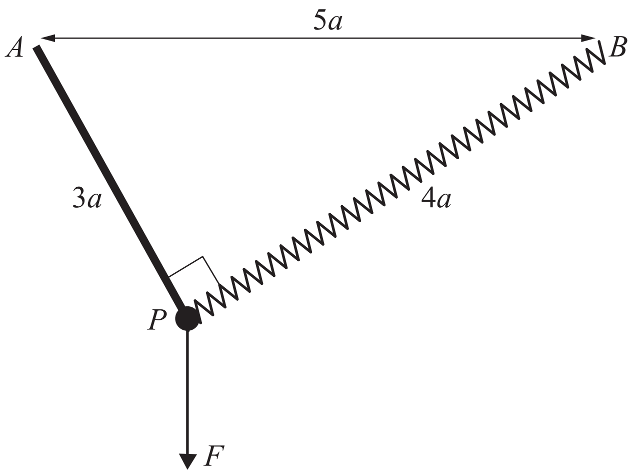

A particle P of mass m is attached to one end of a light rod of length $3a$. The other end of the rod is able to pivot smoothly about the fixed point A. The particle is also attached to one end of a light spring of natural length a and modulus of elasticity kmg. The other end of the spring is attached to a fixed point B. The points A and B are in a horizontal line, a distance $5a$ apart, and these two points and the rod are in a vertical plane.

Initially, P is held in equilibrium by a vertical force F with the stretched length of the spring equal to $4a$ (see diagram). The particle is released from rest in this position and has a speed of $\frac{6}{5}\sqrt{2ag}$ when the rod becomes horizontal.

(a) Find the value of k. [5]

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(b) Find F in terms of m and g. [2]

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(c) Find, in terms of m and g, the tension in the rod immediately before it is released. [2]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/32**

Paper 3 Further Mechanics **October/November 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Due to a series-specific issue during the live exam series, all candidates were awarded full marks for questions 1 and 2. This published mark scheme for these questions was created alongside the question paper, but has not been used by examiners.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1. Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2. Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3. Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5. Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6. Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$\uparrow T\cos\theta = mg$	**B1**	
	$\rightarrow\ T\sin\theta = \dfrac{mv^2}{a\sin\theta}$	**B1**	
	Eliminate T and substitute for θ	**M1**	
	$v = \sqrt{\frac{5}{6}ag}$	**A1**	
		4	

Question	Answer	Marks	Guidance
2	$0.5v\dfrac{dv}{dx} = \dfrac{150}{(x+1)^2} - \dfrac{450}{(x+1)^3}$	**M1**	Allow sign errors.
	Integrate: $0.5v^2 = -\dfrac{300}{x+1} + \dfrac{450}{(x+1)^2} + A$	**M1A1**	Correct powers, allow sign errors.
	$x = 0,\ v = 20;\ A = 50$	**M1**	Use initial condition.
	Rearrange: $v^2 = \dfrac{100(x^2 - 4x + 4)}{(x+1)^2}$	**A1**	AEF
	$v^2 = \dfrac{100(x-2)^2}{(x+1)^2}$ so $v = \pm\dfrac{10(x-2)}{(x+1)}$ From initial condition, sign must be negative, $v = \dfrac{20-10x}{x+1}$	**A1**	Signs dealt with convincingly.
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>3(a)</td><td>Mass is proportional to area
<table><tr><th></th><th>Area</th><th>Centre of mass from AC</th></tr><tr><td>ABC</td><td>$\frac{1}{2}\times 2a\times a$</td><td>$\frac{1}{3}a$</td></tr><tr><td>BDE</td><td>$\frac{1}{2}\times 2\left(\frac{a}{k}\right)^2$</td><td>$a-\frac{2}{3}\left(\frac{a}{k}\right)$</td></tr><tr><td>ADEC</td><td>$-\frac{1}{2}\times 2\left(\frac{a}{k}\right)^2+a^2$</td><td>$\bar{x}$</td></tr></table></td><td>B1</td><td>At least two areas correct, at least one distance correct</td></tr>
<tr><td></td><td>$\bar{x}\left(-\frac{1}{2}\times 2\left(\frac{a}{k}\right)^2+a^2\right)=\frac{1}{3}a\times a^2-\left(a-\frac{2}{3}\left(\frac{a}{k}\right)\right)\times\left(\frac{a}{k}\right)^2$</td><td>M1A1</td><td>Moments equation, dimensionally correct, correct number of terms.</td></tr>
<tr><td></td><td>$\bar{x}=\frac{a\left(k^2+k-2\right)}{3k\left(k+1\right)}$</td><td>A1</td><td>Allow unsimplified single fraction $\frac{a\left(k^3-3k+2\right)}{3k\left(k^2-1\right)}$.</td></tr>
<tr><td></td><td></td><td>4</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
3(b)	$\tan\theta = \dfrac{\bar{x}}{a}$ $\left[= \dfrac{5}{18}\right]$	**B1**	With *their* $\bar{x}$ from part **(a)**.
	So $18\left(k^2 + k - 2\right) = 5\left(3k^2 + 3k\right)$ $k^2 + k - 12 = 0$	**M1**	Obtain a polynomial in k only, e.g. $k^3 - 13k + 12 = 0$, may be implied.
	$(k+4)(k-3) = 0$, $k = 3$ only	**A1**	CWO
		3	

Question	Answer	Marks	Guidance
4(a)	After collision, A has velocity v_A towards wall on right and B has component of velocity towards lower wall of $u\sin\theta$. Same distance and time, so $v_A = u\sin\theta$.	**B1**	
	Along line of centres: PCLM: $mv_A + mv_B = mu\cos\theta$ NEL: $v_A - v_B = eu\cos\theta$	**M1**	Both, consistent signs, must be $\cos\theta$.
	$2\sin\theta = (1+e)\cos\theta$	**M1**	Eliminating v_A and v_B to find an equation in θ. Condone common factor of u. Note: $v_A = \dfrac{3}{4}u\cos\theta$ and $v_B = \dfrac{1}{4}u\cos\theta$.
	$\tan\theta = \frac{3}{4}$	**A1**	
		4	

Question	Answer	Marks	Guidance
4(b)	Final KE = $\frac{1}{2}m\left(v_A^2+(u\sin\theta)^2+v_B^2\right)$	**B1**	Both components of velocity of *B* needed.
	Loss = $\frac{1}{2}mu^2-\frac{1}{2}m\left(v_A^2+(u\sin\theta)^2+v_B^2\right) = \frac{1}{2}mu^2\left(1-\frac{9}{25}-\frac{9}{25}-\frac{1}{25}\right)$	**M1**	v_A, v_B, θ substituted, ft *their* final KE with both components of velocity of B included.
	Loss = $\frac{3}{25}mu^2$ Percentage loss = 24%	**A1**	
		3	

Question	Answer	Marks	Guidance
5(a)	At A: $R_A - mg\cos\alpha = \frac{mv_A^2}{a}$	**B1**	
	At B: $-R_B + mg\cos\theta = \frac{mv_B^2}{a}$	**B1**	
	Energy: $\frac{1}{2}mv_A^2 - \frac{1}{2}mv_B^2 = mga(\cos\alpha + \cos\theta)$	**M1**	All terms present, allow sign errors, cos/sin mix.
		A1	AEF
	Eliminate velocities, use $R_B = \frac{1}{6}R_A$ and substitute angle values $\frac{1}{2}a(R_A - mg\cos\alpha) - \frac{1}{2}a(-R_B + mg\cos\theta) = mga(\cos\alpha + \cos\theta)$ $6R_B - \frac{3}{5}mg + R_B - \frac{4}{5}mg = 2mg \times \frac{7}{5}$	**M1**	Any equivalent working, leading to a dimensionally correct equation in R_B and mg.
	$R_B = \frac{3}{5}mg$	**A1**	
		6	
5(b)	From first equation in **(a)**, $6 \times \frac{3}{5}mg - \frac{3}{5}mg = \frac{mv_A^2}{a}$	**M1**	
	$v_A^2 = 3ag$, so $k = 3$	**A1**	CAO
		2	

Question	Answer	Marks	Guidance
6(a)	$\rightarrow x = u\cos\alpha t$ $\uparrow y = u\sin\alpha t - \frac{1}{2}gt^2$	**B1**	Both correct.
	Eliminate t: $y = u\sin\alpha \times \frac{x}{u\cos\alpha} - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$	**M1**	Eliminate.
	$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$ AG	**A1**	
		3	

Question	Answer	Marks	Guidance
6(b)	$\frac{4}{5}h = 32\tan\alpha - \frac{g32^2}{2\times 40^2}\sec^2\alpha$ $[= 32\tan\alpha - \frac{16}{5}\sec^2\alpha]$	**B1**	
	$h = 32\tan\alpha - \frac{g32^2}{2\left(40\sqrt{2}\right)^2}\sec^2\alpha$ $[= 32\tan\alpha - \frac{8}{5}\sec^2\alpha]$	**B1**	
	$32\tan\alpha - \frac{8}{5}\sec^2\alpha = \frac{5}{4}(32\tan\alpha - \frac{16}{5}\sec^2\alpha)$ $32t - \frac{8}{5}\left(1+t^2\right) = 40t - 4\left(1+t^2\right)$ $3t^2 - 10t + 3 = 0$	**M1**	Equate expressions for h and obtain a 3-term quadratic in $\tan\alpha$.
	$t = 3, \ \frac{1}{3}$	**A1**	Both correct
	$h = 80$ $h = \frac{80}{9}$	**M1**	For using their value of t to work out one value of h.
		A1	Both correct
		6	

Question	Answer	Marks	Guidance
7(a)	Gain in KE = $\frac{1}{2}m\left(\frac{6}{5}\sqrt{2ag}\right)^2$ and Gain in GPE = $mg \times 3a\sin\theta$	**B1**	
	Loss in EPE = $\frac{\frac{1}{2}kmg}{a}\left((3a)^2 - a^2\right)$	**B1**	
	Energy equation: $\frac{1}{2}m\left(\frac{6}{5}\sqrt{2ag}\right)^2 + mg \times 3a\sin\theta = \frac{\frac{1}{2}kmg}{a}\left((3a)^2 - a^2\right)$	**M1**	KE, GPE and at least one EPE present, allow sign errors, dimensionally correct.
		A1	All correct.
	$\frac{36}{25}mg + \frac{12}{5}mg = 4kmg$ $k = \frac{24}{25}$	**A1**	
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Question	Answer	Marks	Guidance
7(b)	In lower position, tension T in spring = $kmg \times \frac{3a}{a} = 3kmg = \frac{72}{25}mg$ Perpendicular to rod, $T = (F + mg)\cos\theta$	**M1**	Hooke's law.
	So, $(F + mg)\cos\theta = \frac{72}{25}mg$, $F = \frac{19}{5}mg$	**A1**	
	Alternative method for question 7(b)		
	In lower position, tension T in spring = $kmg \times \frac{3a}{a} = 3kmg = \frac{72}{25}mg$ $T'\cos\theta = Tsin\theta$ $T'\sin\theta + Tcos\theta = F + mg$	**M1**	Hooke's law. Eliminate T'.
	$F = \frac{19}{5}mg$	**A1**	
		2	

Question	Answer	Marks	Guidance
7(c)	Let tension in rod = T' Parallel to rod, $T' = (F + mg)\sin\theta$	**M1**	
	$T' = \frac{96}{25}mg$	**A1**	
	Alternative method for question 7(c)		
	$T'\cos\theta = T\sin\theta$ $T'\sin\theta + T\cos\theta = F + mg$	**M1**	At least one equation seen with their T and/or F.
	$T' = \frac{96}{25}mg$	**A1**	
		2	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

6541057198

FURTHER MATHEMATICS **9231/33**

Paper 3 Further Mechanics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.
- Where a numerical value for the acceleration due to gravity (g) is needed, use $10\,\text{m}\,\text{s}^{-2}$.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

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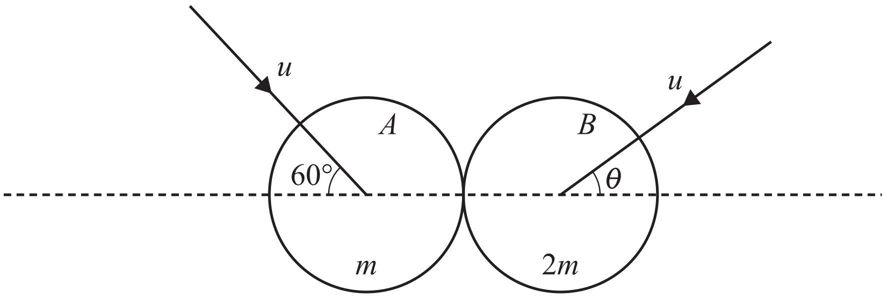

Two uniform smooth spheres A and B of equal radii have masses m and $2m$ respectively. The two spheres are moving with equal speeds u on a smooth horizontal surface when they collide. Immediately before the collision, A's direction of motion makes an angle of 60° with the line of centres, and B's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is e.

After the collision, the component of the velocity of A along the line of centres is v and B moves perpendicular to the line of centres. Sphere A now has twice as much kinetic energy as sphere B.

(a) Show that $v = \frac{1}{2}u(4\cos\theta - 1)$. [1]

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(b) Find the value of $\cos\theta$. [4]

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(c) Find the value of e. [2]

2 A ball of mass 2 kg is projected vertically downwards with speed $5\,\text{m}\,\text{s}^{-1}$ through a liquid. At time t s after projection, the velocity of the ball is $v\,\text{m}\,\text{s}^{-1}$ and its displacement from its starting point is x m. The forces acting on the ball are its weight and a resistive force of magnitude $0.2v^2$ N.

(a) Find an expression for v in terms of t. [6]

(b) Deduce what happens to v for large values of t. [1]

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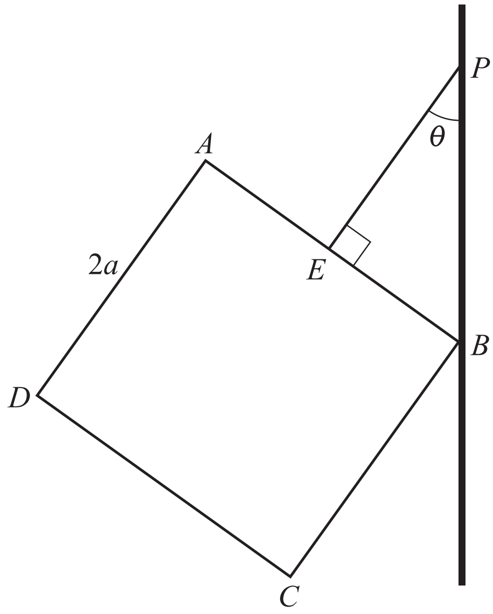

A uniform square lamina of side $2a$ and weight W is suspended from a light inextensible string attached to the midpoint E of the side AB. The other end of the string is attached to a fixed point P on a rough vertical wall. The vertex B of the lamina is in contact with the wall. The string EP is perpendicular to the side AB and makes an angle θ with the wall (see diagram). The string and the lamina are in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the lamina is $\frac{1}{2}$.

Given that the vertex B is about to slip up the wall, find the value of $\tan\theta$. [8]

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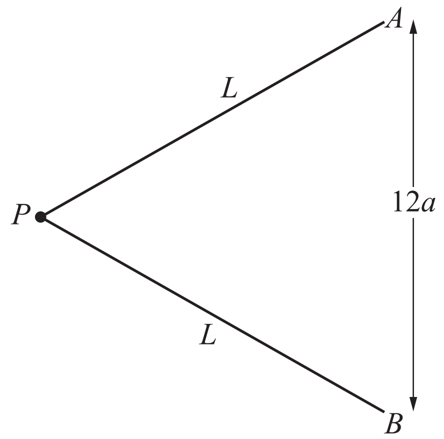

A light elastic string has natural length $8a$ and modulus of elasticity $5mg$. A particle P of mass m is attached to the midpoint of the string. The ends of the string are attached to points A and B which are a distance $12a$ apart on a smooth horizontal table. The particle P is held on the table so that $AP = BP = L$ (see diagram). The particle P is released from rest. When P is at the midpoint of AB it has speed $\sqrt{80ag}$.

(a) Find L in terms of a. [5]

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(b) Find the initial acceleration of P in terms of g. [3]

5 A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. During its flight P passes through the point which is a horizontal distance $3a$ from O and a vertical distance $\frac{3}{8}a$ above the horizontal plane. It is given that $\tan\theta = \frac{1}{3}$.

(a) Show that $u^2 = 8ag$. [2]

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A particle Q is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal from O at the instant when P is at its highest point. Particles P and Q both land at the same point on the horizontal plane at the same time.

(b) Find V in terms of a and g. [7]

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6 A particle P of mass m is attached to one end of a light inextensible rod of length $3a$. An identical particle Q is attached to the other end of the rod. The rod is smoothly pivoted at a point O on the rod, where $OQ = x$. The system, of rod and particles, rotates about O in a vertical plane.

At an instant when the rod is vertical, with P above Q, the particle P is moving horizontally with speed u. When the rod has turned through an angle of 60° from the vertical, the speed of P is $2\sqrt{ag}$, and the tensions in the two parts of the rod, OP and OQ, have equal magnitudes.

(a) Show that the speed of Q when the rod has turned through an angle of 60° from the vertical is $\dfrac{2x}{3a-x}\sqrt{ag}$. [2]

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(b) Find x in terms of a. [5]

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(c) Find u in terms of a and g. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/33**

Paper 3 Further Mechanics **October/November 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

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B Mark for a correct result or statement independent of method marks.

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FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

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WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	PCLM: $mv = -mu\cos 60° + 2mu\cos\theta$ $v = -\frac{1}{2}u + 2u\cos\theta = v = \frac{1}{2}u(4\cos\theta - 1)$	**B1**	First line must be seen. AG
		1	
1(b)	KE of A = $\frac{1}{2}m\left(v^2 + (u\sin 60°)^2\right)$	**B1**	
	KE of A = 2 × KE of B, so $\frac{1}{2}m\left(v^2 + (u\sin 60°)^2\right) = 2 \times \frac{1}{2} \times 2m(u\sin\theta)^2$	**M1**	
	$\left(\frac{1}{2}u(4\cos\theta - 1)\right)^2 + \frac{3}{4}u^2 = 4u^2(\sin\theta)^2$ $8\cos^2\theta - 2\cos\theta - 3 = 0$	**M1**	Use result of **(a)** and rearrange. Obtain 3-term quadratic.
	$\cos\theta = \frac{3}{4},\ -\frac{1}{2}$ but angle is acute, so $\cos\theta = \frac{3}{4}$	**A1**	
		4	
1(c)	NEL: $v = e(u\cos 60° + u\cos\theta)$	**M1**	
	$v = u,\ u = e\left(\frac{1}{2}u + \frac{3}{4}u\right)$ $e = \frac{4}{5}$	**A1**	
		2	

Question	Answer	Marks	Guidance
2(a)	$2\frac{dv}{dt} = 2g - 0.2v^2$	B1	
	Separate variables and attempt to integrate $\frac{dv}{0.1(100 - v^2)} = dt$	M1	Integrate to a ln term of the correct form.
	$\frac{1}{20}\ln\left(\frac{10+v}{10-v}\right) = 0.1t + c$	A1	
	$t = 0, \quad v = 5, \quad c = \frac{1}{20}\ln 3$	M1	Use initial condition.
	$2t = \ln\frac{10+v}{3(10-v)}, \; e^{2t} = \frac{10+v}{3(10-v)}$	M1	Rearrange, removing ln.
	$v = \frac{30 - 10e^{-2t}}{3 + e^{-2t}}$	A1	AEF
		6	
2(b)	$v \to 10$	B1FT	FT from expression of correct form.
		1	

Question	Answer	Marks	Guidance
3	Let N be normal reaction at B and F the frictional force acting downwards $\uparrow T\cos\theta = F + W$	**B1**	
	$\rightarrow T\sin\theta = N$	**B1**	
	Moments about B: $Ta = W\sin\theta \times a + W\cos\theta \times a$	**M1A1**	A moments equation with all relevant forces.
	$F = \frac{1}{2}N$ used	**M1**	
	$(\cos\theta + \sin\theta)\left(\cos\theta - \frac{1}{2}\sin\theta\right) = 1$ oe	**M1**	Combine to obtain equation in θ. Equation in trigonometric functions only.
	$\frac{1}{2}\cos\theta\sin\theta = \frac{3}{2}(\sin\theta)^2$ $\sin\theta(\cos\theta - 3\sin\theta) = 0$	**M1**	Solve trigonometric equation.
	$\tan\theta = \frac{1}{3}$	**A1**	
		8	

Question	Answer	Marks	Guidance
4(a)	Energy equation: gain in KE = loss in EPE $\frac{1}{2}m\times 80ag=\frac{1}{2}\times\frac{5mg}{8a}\left((2L-8a)^2-(4a)^2\right)$	**B1** **M1** **A1**	At least one correct EPE. 3-term energy equation. All correct.
	$4L^2-8aL-20a^2=0$	**M1**	Simplify to quadratic in L.
	$L=10a$	**A1**	Correct single answer.
		5	
4(b)	At P: $2T\cos\theta=m\times\text{acceleration}$	**M1**	
	Hooke's law: $T=\frac{5mg}{8a}\times 12a=\frac{15}{2}mg$	**M1**	
	Solve, acceleration = $12g$	**A1**	
		3	

Question	Answer	Marks	Guidance
5(a)	Use equation of trajectory: $y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$: $\frac{3}{8}a = 3a \times \frac{1}{3} - g \times \frac{(3a)^2}{2u^2}\left(1 + \frac{1}{9}\right)$	**M1**	
	$\frac{5}{8}a = \frac{5ga}{u^2}$, $u^2 = 8ga$	**A1**	At least one step of working. AG
		2	

Question	Answer	Marks	Guidance
5(b)	For P, time of flight $T = \dfrac{2u\sin\theta}{g}$	B1	
	For P, range = $Tu\cos\theta$	B1	
	For Q, time of flight = $\frac{1}{2}T$, so range = $\frac{1}{2}TV\cos\alpha$ Equate: $V\cos\alpha = 2u\cos\theta$ (1)	B1	
	For Q vertically: $0 = V\sin\alpha \times \frac{1}{2}T - \dfrac{\frac{1}{2}gT^2}{4}$, so $T = \dfrac{4V\sin\alpha}{g}$	M1	
	Equate with result for P: $\dfrac{4V\sin\alpha}{g} = \dfrac{2u\sin\theta}{g}$ so $V\sin\alpha = \frac{1}{2}u\sin\theta$ (2)	A1	
	From (1) and (2): $V^2 = u^2\left(4(\cos\theta)^2 + \frac{1}{4}(\sin\theta)^2\right)$	M1	
	$V^2 = u^2 \times \left(4\times\frac{9}{10} + \frac{1}{4}\times\frac{1}{10}\right) = \frac{145}{40}u^2$ $V = \sqrt{29ag}$	A1	
		7	

Question	Answer	Marks	Guidance
6(a)	Angular speeds of P and Q are equal, so $\frac{v_Q}{x} = \frac{v_P}{3a - x}$	M1	
	$v_Q = \frac{2x\sqrt{ag}}{3a - x}$	A1	Shown convincingly: angular speeds equal stated. AG
		2	
6(b)	For P: $T + mg\cos 60° = \frac{m \times 4ag}{3a - x}$	B1	
	For Q: $T - mg\cos 60° = \frac{mv_Q^2}{x}$	B1	
	Eliminate T: $-mg\cos 60° + \frac{m.4ag}{3a - x} = mg\cos 60° + \frac{mv_Q^2}{x}$	M1	
	$\frac{m \times 4ag}{3a - x} = 1 + \frac{mx4ag}{(3a - x)^2}$ $4a(3a - x) = (3a - x)^2 + 4ax$, $x^2 + 2ax - 3a^2 = 0$	M1	Solve to find x. Obtain 3-term quadratic equation.
	$(x - a)(x + 3a) = 0$, $x = a$	A1	
		5	

Question	Answer	Marks	Guidance
6(c)	Energy changes from initial position:	B1	KEs correct.
	Gain in KE of P: $\frac{1}{2}m\left(4ag - u^2\right)$ Loss in KE of Q: $\frac{1}{2}m\left(\left(\frac{u}{2}\right)^2 - v_Q^2\right)$ Loss in GPE of $P = mg(3a - x)(1 - \cos 60°)$ $(= mga)$ Gain in GPE of $Q = mgx(1 - \cos 60°)$ $\left(= \frac{1}{2}mga\right)$	B1FT	GPEs correct.
	$\frac{1}{2}m\left(4ag - u^2\right) - \frac{1}{2}m\left(\left(\frac{u}{2}\right)^2 - v_Q^2\right) = -mgx(1 - \cos 60°) + mg(3a - x)(1 - \cos 60°)$	M1	Energy equation.
	Simplify: $4ag - \frac{5}{4}u^2 + ag = ag$ $u^2 = \frac{16}{5}ag, \quad u = \frac{4}{5}\sqrt{5ag}$	A1	AEF
		4	

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

9469423944

FURTHER MATHEMATICS **9231/41**

Paper 4 Further Probability & Statistics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION
- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **12** pages. Any blank pages are indicated.

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1 Maya is an athlete who competes in 1500-metre races. Last summer her practice run times had mean 4.22 minutes. Over the winter she has done some intense training to try to improve her times. A random sample of 10 of her practice run times, x minutes, this summer are summarised as follows.

$$\sum x = 42.05 \qquad \sum x^2 = 176.83$$

Maya's new practice run times are normally distributed. She believes that on average her times have improved as a result of her training.

Test, at the 5% significance level, whether Maya's belief is supported by the data. [6]

2 A town council has published its plans for redeveloping the town centre and residents are being asked whether they approve or disapprove. A random sample of 250 responses has been selected from residents in the four main streets in the town: North, East, South and West Streets. The results are shown in the table.

	North Street	East Street	South Street	West Street
Approve	33	54	42	26
Disapprove	19	39	28	9

Test, at the 5% significance level, whether the opinions of the residents are independent of the streets on which they live. [7]

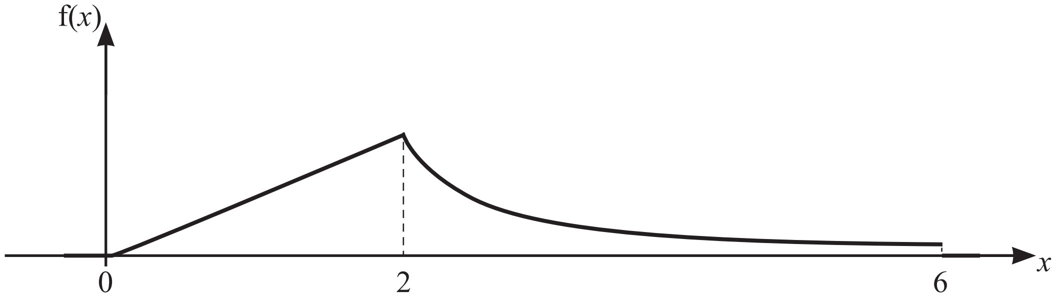
3 Scientists are studying the effects of exercise on LDL blood cholesterol levels. Over a three-month period, a large group of people exercised for 20 minutes each day. For a randomly chosen sample of 10 of these people, the LDL blood cholesterol levels were measured at the beginning and the end of the three-month period. The results, measured in suitable units, are as follows.

	Person	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Cholesterol level	Beginning	72	84	120	90	102	135	64	75	80	88
	End	64	76	105	92	105	115	67	75	75	84

(a) Test, at the 2.5% significance level, whether there is evidence that the population mean LDL blood cholesterol level has reduced by more than 2 units after the three-month period. [7]

(b) State any assumption that you have made in part **(a)**. [1]

4



As shown in the diagram, the continuous random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} mx & 0 \leqslant x \leqslant 2, \\ \dfrac{k}{x^2} + c & 2 \leqslant x \leqslant 6, \\ 0 & \text{otherwise,} \end{cases}$$

where m, k and c are constants.

(a) Given that $\mathrm{P}(X \leqslant 2) = \frac{1}{3}$, show that $m = \frac{1}{6}$ and find the values of k and c. [4]

(b) Find the exact numerical value of the interquartile range of X. [5]

5 The random variable X has the geometric distribution Geo(p).

(a) Show that the probability generating function of X is $\frac{pt}{1-qt}$, where $q = 1-p$. [3]

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(b) Use the probability generating function of X to show that $\mathrm{Var}(X) = \frac{q}{p^2}$. [5]

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Kenny throws an ordinary fair 6-sided dice repeatedly. The random variable X is the number of throws that Kenny takes in order to obtain a 6. The random variable Z denotes the sum of two independent values of X.

(c) Find the probability generating function of Z. [2]

6 A school is conducting an experiment to see whether the distance that children can throw a ball increases in hot weather. On a cold day, all the children at the school were asked to throw a ball as far as possible. The distances thrown were measured and recorded. The median distance thrown by a random sample of 25 of the children was 22.0 m. The children were asked to throw the ball again on a hot day. The distances thrown by the same 25 children were measured and recorded and these distances, in m, are shown below.

21.2	23.5	22.9	18.6	19.4
22.1	26.5	20.2	25.7	20.6
22.3	17.4	22.2	27.0	23.9
28.2	22.6	27.2	23.0	23.7
19.8	22.7	23.3	21.5	24.3

The teacher claims that on average the distances thrown will be further when it is hot.

Carry out a Wilcoxon signed-rank test, at the 5% significance level, to test whether the data supports the teacher's claim. [10]

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Additional Page

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[Turn over

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Question	Answer	Marks	Guidance
1	H_0: $\mu = 4.22$ H_1: $\mu < 4.22$	**B1**	
	$\bar{x} = \frac{42.05}{10}[= 4.205]$, $s^2 = \frac{1}{9}\left(176.83 - \frac{42.05^2}{10}\right)\left[= \frac{13}{12000} = 0.0010833\right]$	**M1**	
	$t = \frac{4.205 - 4.22}{\sqrt{\frac{s^2}{10}}}$	**M1**	*Their* 4.205, *their* s^2 .
	$t = -1.44$	**A1**	Condone sign.
	Tabular value = 1.833: ‘1.44’ < 1.833, accept H_0	**M1**	Compare *their* value with **correct** tabular value, **signs consistent**, allow ‘not significant’.
	There is insufficient evidence to support Maya’s belief.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language. A0 if hypotheses wrong way round or missing.
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="9">2</td><td>H_0: opinion is independent of street H_1: opinion is not independent of street</td><td>B1</td><td></td></tr>
<tr><td rowspan="2"><table>
<tr><td>33 32.24</td><td>54 57.66</td><td>42 43.4</td><td>26 21.7</td><td>155</td></tr>
<tr><td>19 19.76</td><td>39 35.34</td><td>28 26.6</td><td>9 13.3</td><td>95</td></tr>
<tr><td>52</td><td>93</td><td>70</td><td>35</td><td>250</td></tr>
</table></td><td>M1</td><td>At least 2 correct values or expressions.</td></tr>
<tr><td>A1</td><td>6 correct values or expressions.</td></tr>
<tr><td>Calculate chi-squared values:
0.0179 + 0.2323 + 0.0452 + 0.8521 + 0.0292 + 0.3790 + 0.0737 + 1.3902</td><td>M1</td><td>Correct to 3 decimal places.
At least 2 correct values or expressions.</td></tr>
<tr><td>3.02</td><td>A1</td><td>SC B1 3.02 following M1M0
SC B2 3.02 following M0M0</td></tr>
<tr><td>Tabular value, 3 degrees of freedom 7.815: ‘3.02’ < 7.815, accept H_0</td><td>M1</td><td>Allow ‘not significant’.</td></tr>
<tr><td>Insufficient evidence to suggest that opinion depends on street.</td><td>A1</td><td>CWO. Correct conclusion in context, following correct work, level of uncertainty in language.
A0 if hypotheses wrong way round or missing.</td></tr>
<tr><td></td><td>7</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
3(a)	H$_0$: $\mu_B - \mu_E = 2$ H$_1$: $\mu_B - \mu_E > 2$	**B1**	May use μ_d, but must be consistent with direction of differences found.
	8 | 8 | 15 | –2 | –3 | 20 | –3 | 0 | 5 | 4	**M1**	Differences, at most 2 errors.
	$\sum d = 52, \sum d^2 = 816$, $s^2 = \frac{1}{9}\left(816 - \frac{52^2}{10}\right)\left[= \frac{2728}{45} = 60.62\right]$	**M1**	*Their* values but must see 9 and 10 used correctly.
	$t = \dfrac{\frac{52}{10} - 2}{\sqrt{\frac{s^2}{10}}}$	**M1**	
	$t = 1.29967$, 1.30	**A1**	
	Tabular value = 2.262. '1.30' < 2.262, accept H$_0$	**M1**	Allow 'not significant'.
	Insufficient evidence to suggest that cholesterol level has reduced by more than 2.	**A1**	CWO. Correct conclusion in context, following correct work, level of uncertainty in language. A0 if hypotheses wrong way round or missing.
		7	
3(b)	**Population differences** are **normally** distributed	**B1**	
		1	

Question	Answer	Marks	Guidance
4(a)	$\mathrm{P}(X \leqslant 2) = \frac{1}{3}$, so $\left[\frac{mx^2}{2}\right]_0^2 = \frac{1}{3}$ $m\left(\frac{4}{2}\right) = \frac{1}{3}, \quad m = \frac{1}{6}$	**B1**	AG Reasoning required.
	$\int_2^6 \left(\frac{k}{x^2} + c\right) \mathrm{d}x = \frac{2}{3}$ so $\left[-\frac{k}{x} + cx\right]_2^6 = 4c + \frac{1}{3}k = \frac{2}{3}$ or $k + 12c = 2$	**M1**	Attempt to integrate and correct use of correct limits to form linear equation in *k* and *c*.
	Also, $2m = \frac{k}{4} + c$ or $\frac{1}{3} = \frac{k}{4} + c$ or $3k + 12c = 4$	**M1**	Matching at $x = 2$.
	Solve, $k = 1, \quad c = \frac{1}{12}$	**A1**	
		4	

Question	Answer	Marks	Guidance
4(b)	LQ: $\int_0^L \frac{1}{6} x\mathrm{d}x = \frac{1}{4}, \qquad L = \sqrt{3}$	**B1**	
	UQ: $\frac{1}{3} + \int_2^U \left(\frac{k}{x^2} + c\right)\mathrm{d}x = \frac{3}{4}$	**M1**	Or find and use CDF. May be in terms of *m*.
	$-\frac{k}{U} + Uc + \frac{k}{2} - 2c = \frac{5}{12}$ $U^2 - U - 12 = 0$	**M1**	Attempt at integral and correct use of correct limits. Simplify to quadratic equation, may be in terms of *k* and *c*.
	$U = 4$	**A1**	
	IQR = $4 - \sqrt{3}$	**A1 FT**	FT *their* UQ – *their* LQ, **exact values only**.
		5	

Question	Answer	Marks	Guidance
5(a)	P($X=r$) = $p(1-p)^{r-1} = pq^{r-1}$	**B1**	Implied by G(t) = $\sum pq^{r-1}\ t^r$.
	G(t) = $\sum pt(qt)^{r-1}$	**M1**	(Infinite summation) with common ratio (qt) and first term or common factor pt indicated **or** GP identified.
	G(t) = $pt\left(\frac{1}{1-qt}\right)=\frac{pt}{1-qt}$	**A1**	AG Convincingly obtained.
		3	
5(b)	G'(t) = $pqt(1-qt)^{-2}+p(1-qt)^{-1}$ [= $p(1-qt)^{-2}$]	**M1**	Attempt to differentiate as a product/quotient.
	G"(t) = $2pq^2t(1-qt)^{-3}+2pq(1-qt)^{-2}$ $[=2qp(1-qt)^{-3}]$	**M1**	Attempt to differentiate *their* G'(t).
	G'(1) = $p(1-q)^{-2}=\frac{1}{p}$, G"(1) = $\frac{2q}{P^2}$	**M1**	Substitute $t=1$ into their G'(t) and G"(t) and use formula for Var(X) (at some point).
	Var(X) = $\frac{2q}{P^2}+\frac{1}{p}-\left(\frac{1}{p}\right)^2$	**M1**	$(1-q)$ replaced by p in denominator throughout, and some cancellation seen (at some point).
	$\frac{q}{p^2}$	**A1**	AG Convincingly obtained.
		5	

Question	Answer	Marks	Guidance
5(c)	$G_X(t) = \dfrac{\frac{1}{6}t}{1-\frac{5}{6}t} \quad \left[= \dfrac{t}{6-5t}\right]$	**B1**	Implied by correct final answer.
	$G_Z(t) = (G_X(t))^2 = \left(\dfrac{t}{6-5t}\right)^2$	**B1**	Allow $\left(\dfrac{\frac{1}{6}t}{1-\frac{5}{6}t}\right)^2$.
		2	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>6</td><td>H_0: population medians are equal
H_1: population median after > population median before</td><td>B1</td><td>Do not accept ‘differences between population medians < 0’ unless difference defined.
Allow $H_0: m = 22, H_1: m > 22.0$</td></tr>
<tr><td></td><td rowspan="2"><table>
<tr><td>–0.8</td><td>-7</td><td>1.5</td><td>12</td><td>0.9</td><td>8</td><td>–3.4</td><td>–19</td><td>–2.6</td><td>–18</td></tr>
<tr><td>0.1</td><td>1</td><td>4.5</td><td>21</td><td>–1.8</td><td>–14</td><td>3.7</td><td>20</td><td>–1.4</td><td>–11</td></tr>
<tr><td>0.3</td><td>3</td><td>–4.6</td><td>–22</td><td>0.2</td><td>2</td><td>5</td><td>23</td><td>1.9</td><td>15</td></tr>
<tr><td>6.2</td><td>25</td><td>0.6</td><td>5</td><td>5.2</td><td>24</td><td>1</td><td>9</td><td>1.7</td><td>13</td></tr>
<tr><td>–2.2</td><td>-16</td><td>0.7</td><td>6</td><td>1.3</td><td>10</td><td>–0.5</td><td>–4</td><td>2.3</td><td>17</td></tr>
</table></td><td>M1</td><td>Signed differences, allow at most 4 errors</td></tr>
<tr><td></td><td>M1</td><td>Attempt at ranks (ignore signs).</td></tr>
<tr><td></td><td>($W_+ = 214$) $W_- = 111$</td><td>A1</td><td>Cao identified, or used, as test statistic.</td></tr>
<tr><td></td><td>Normal: mean = $\frac{1}{4}n(n+1) = \frac{1}{4} \times 25 \times 26 [= 162.5]$</td><td>B1</td><td></td></tr>
<tr><td></td><td>Variance = $\frac{1}{24}n(n+1)(2n+1) = \frac{1}{24} \times 25 \times 26 \times 51 [= 1381.25]$</td><td>B1</td><td></td></tr>
<tr><td></td><td>z- value: $\frac{111.5 - 162.5}{\sqrt{1381.25}}$</td><td>M1</td><td>Allow missing or incorrect continuity correction.
Their 111 must come from ranks.</td></tr>
<tr><td></td><td>–1.37</td><td>A1</td><td>CAO</td></tr>
<tr><td></td><td>Tabular value is -1.645: ‘-1.37’ > -1.645, or 0.915<0.95   oe, accept H_0,</td><td>M1</td><td>Consistent signs. Allow ‘not significant’.</td></tr>
<tr><td></td><td>Insufficient evidence to support the teacher’s claim
Insufficient evidence to suggest that the distances thrown are further when it is hot</td><td>A1</td><td>All correct. Correct conclusion in context, following correct work, level of uncertainty in language.

A0 if hypotheses wrong way round or missing</td></tr>
<tr><td></td><td></td><td>10</td><td></td></tr>
</table>

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME						
CENTRE NUMBER					CANDIDATE NUMBER	

0815246258

FURTHER MATHEMATICS **9231/42**

Paper 4 Further Probability & Statistics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **16** pages. Any blank pages are indicated.

DC (EF/FC) 316301/1

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1 A factory produces small bottles of natural spring water. Two different machines, X and Y, are used to fill empty bottles with the water. A quality control engineer checks the volumes of water in the bottles filled by each of the machines. He chooses a random sample of 60 bottles filled by machine X and a random sample of 75 bottles filled by machine Y. The volumes of water, x and y respectively, in millilitres, are summarised as follows.

$$\sum x = 6345 \quad \sum(x-\bar{x})^2 = 243.8 \quad \sum y = 7614 \quad \sum(y-\bar{y})^2 = 384.9$$

$\bar{x}$ and $\bar{y}$ are the sample means of the volume of water in the bottles filled by machines X and Y respectively.

Find a 95% confidence interval for the difference between the mean volume of water in bottles filled by machine X and the mean volume of water in bottles filled by machine Y. [6]

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2 The number of breakdowns on a particular section of road is recorded each day over a period of 90 days. It is suggested that the number of breakdowns follows a Poisson distribution with mean 3.5. The data is summarised in the table, together with some of the expected frequencies resulting from the suggested Poisson distribution.

Number of breakdowns per day	0	1	2	3	4	5	6	7	8 or more
Observed frequency	0	5	13	17	21	16	9	5	4
Expected frequency	2.718	9.512	16.646		16.993	11.895		3.469	2.407

(a) Complete the table. [2]

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(b) Carry out a goodness of fit test, at the 10% significance level, to determine whether or not Po(3.5) is a good fit to the data. [6]

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3 Toby has a bag which contains 6 red marbles and 3 green marbles. He randomly chooses 3 marbles from the bag, without replacement. The random variable X is the number of red marbles that Toby obtains.

(a) Find the probability generating function of X. [3]

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Ling also has a bag which contains 6 red marbles and 3 green marbles. He randomly chooses 2 marbles from his bag, without replacement. The random variable Y is the number of red marbles that Ling obtains. It is given that the probability generating function of Y is $\frac{1}{12}(1+6t+5t^2)$.

The random variable Z is the total number of red marbles that Toby and Ling obtain.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [3]

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(c) Use the probability generating function of Z to find $\mathrm{Var}(Z)$. [4]

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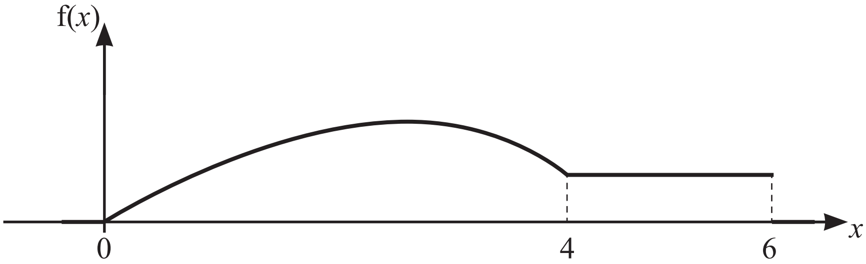

The diagram shows the continuous random variable X with probability density function f given by

$$f(x) = \begin{cases} \frac{1}{128}(4ax - bx^3) & 0 \leqslant x \leqslant 4, \\ c & 4 \leqslant x \leqslant 6, \\ 0 & \text{otherwise,} \end{cases}$$

where a, b and c are constants.

The upper quartile of X is equal to 4.

(a) Show that $c = \frac{1}{8}$ and find the values of a and b. [4]

(b) Find the exact value of the median of X. [3]

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(c) Find $E(\sqrt{X})$, giving your answer correct to 2 decimal places. [3]

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5 A company is deciding which of two machines, X and Y, can make a certain type of electrical component more quickly. The times taken, in minutes, to make one component of this type are recorded for a random sample of 8 components made by machine X and a random sample of 9 components made by machine Y. These times are as follows.

Machine X	4.0	4.6	4.7	4.8	5.0	5.2	5.6	5.8	
Machine Y	4.5	4.9	5.1	5.3	5.4	5.7	5.9	6.3	6.4

The manager claims that on average the time taken by machine X to make one component is less than that taken by machine Y.

(a) Carry out a Wilcoxon rank-sum test at the 5% significance level to test whether the manager's claim is supported by the data. [6]

(b) Assuming that the times taken to produce the components by the two machines are normally distributed with equal variances, carry out a *t*-test at the 5% significance level to test whether the manager's claim is supported by the data. [9]

Question 5(c) is printed on the next page.

(c) In general, would you expect the conclusions from the tests in parts **(a)** and **(b)** to be the same? Give a reason for your answer. [1]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/42**

Paper 4 Further Probability & Statistics **October/November 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Due to a series-specific issue during the live exam series, all candidates were awarded full marks for Question 5(b). This published mark scheme for this question was created alongside the question paper, but has not been used by examiners.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

This document consists of **13** printed pages.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$s_x^2 = \frac{243.8}{59}\left[=\frac{1219}{295}=4.132\right]$, $s_y^2 = \frac{384.9}{74}\left[=\frac{3849}{740}=5.201\right]$	**B1**	Implied by correct s or pooled estimate $\frac{243.8+384.9}{60+75-2}=4.727$
	$s^2 = \frac{4.132}{60}+\frac{5.201}{75}[=0.1382]$ **or** $s=0.3718$	**M1**	Using their sample variances. Pooled estimate M0.
		A1	
	CI: $\frac{6345}{60}-\frac{7614}{75}\pm 1.96\times '0.3718'$	**M1**	With a z-value
		A1	With 1.96 (with their s) Pooled $\frac{6345}{60}-\frac{7614}{75}\pm 1.96\times 2.174\sqrt{\frac{1}{60}+\frac{1}{75}}$
	$[3.5[0], 4.96]$ or $(3.5, 4.96)$	**A1**	4.23 ± 0.729 is A0, condone $[4.96, 3.5]$ etc.
		6	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="3">2(a)</td><td rowspan="2"><table><tr><td>0</td><td>5</td><td>13</td><td>17</td><td>21</td><td>16</td><td>9</td><td>5</td><td>4</td></tr><tr><td>2.718</td><td>9.512</td><td>16.646</td><td>19.421</td><td>16.993</td><td>11.895</td><td>6.939</td><td>3.469</td><td>2.407</td></tr></table></td><td>B1</td><td>Each.</td></tr>
<tr><td>B1</td><td></td></tr>
<tr><td></td><td>2</td><td></td></tr>
<tr><td rowspan="7">2(b)</td><td>H_0: Po(3.5) is a good fit to the data
H_1: Po(3.5) is not a good fit to the data</td><td>B1</td><td></td></tr>
<tr><td>Combine first 2 columns: 5, 12.23
And last 2 columns: 9, 5.876</td><td>M1</td><td>Both.</td></tr>
<tr><td>Chi-squared values:
4.274 + 0.799 + 0.302 + 0.945 + 1.417 + 0.612 + 1.661</td><td>M1</td><td>Allow if no or incorrect number of columns added.
At least two ‘correct’ values (3 sf) or expressions seen from their grouping (or lack of).
2.718 + 2.140 + … + 0.6757 + 1.054</td></tr>
<tr><td>10.0</td><td>A1</td><td>AWRT 10.0
If M0 awarded then SC B1 for 10.0.</td></tr>
<tr><td>Tabular value: 10.64 ‘10.0’ < 10.64, accept H_0 /not significant</td><td>M1</td><td>Allow equivalent to 10.64 if columns not combined or only one pair combined
(12.02 one pair combined, 13.36 none combined).</td></tr>
<tr><td>Insufficient evidence to suggest that Po(3.5) is not a good fit to the data</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language.
A0 if hypotheses the wrong way round or missing.</td></tr>
<tr><td></td><td>6</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
3(a)	P(3R) = $\frac{120}{504}$ P(2R) = $\frac{270}{504}$ P(1R) = $\frac{108}{504}$ P(0R) = $\frac{6}{504}$	**B1**	2 correct probabilities.
	$G_X(t) = \frac{1}{504}\left(6+108t+270t^2+120t^3\right)=\frac{1}{84}\left(1+18t+45t^2+20t^3\right)$	**M1**	Cubic polynomial with their probabilities.
	$\frac{1}{84}+\frac{3}{14}t+\frac{15}{28}t^2+\frac{5}{21}t^3$	**A1**	Correct.
		3	
3(b)	$G_Z(t) = \frac{1}{84}\left(1+18t+45t^2+20t^3\right)\times\frac{1}{12}\left(1+6t+5t^2\right)$	**M1**	Attempt to multiply out the brackets.
	$\frac{1}{1008}\left(1+24t+158t^2+380t^3+345t^4+100t^5\right)$	**M1**	Obtain quintic polynomial (may not be simplified).
	$\frac{1}{1008}+\frac{1}{42}t+\frac{79}{504}t^2+\frac{95}{252}t^3+\frac{115}{336}t^4+\frac{25}{252}t^5$	**A1**	Correct.
		3	

Question	Answer	Marks	Guidance
3(c)	$G' = \frac{1}{1008}\left(24+316t+1140t^2+1380t^3+500t^4\right)$ $G'' = \frac{1}{1008}\left(316+2280t+4140t^2+2000t^3\right)$	**M1**	Differentiate twice. May not see derivatives in terms of *t*.
	$\left[E(Z)=\frac{3360}{1008}\right]\ \frac{10}{3}$	**B1**	Seen or implied, NFWW (e.g. sampling with replacement).
	$\text{Var}(Z) = \frac{8736}{1008}+\frac{10}{3}-\left(\frac{10}{3}\right)^2\left[=\frac{26}{3}-\frac{10}{3}-\frac{100}{9}\right]$	**M1**	Use formula with their $G''(1)$ and $G'(1)$.
	$\frac{8}{9}$	**A1**	Or 0.889
		4	

Question	Answer	Marks	Guidance
4(a)	UQ = 4 so $\int_4^6 c\,dx = \frac{1}{4}$: $c[6-4]=\frac{1}{4}$, $c=\frac{1}{8}$	**B1**	AG. Some indication of reasoning required.
	$\int_0^4 \frac{1}{128}\left(4ax-bx^3\right)dx=\frac{3}{4}$, $\frac{1}{128}\left[2ax^2-\frac{b}{4}x^4\right]=\frac{3}{4}$, $32a-64b=96$	**M1**	Integrate and use correct limits to form equation in a and b.
	Equate at $x=4$: $a-4b=1$	**M1**	
	Solve: $a=5, b=1$	**A1**	
		4	
4(b)	$\int_0^m \frac{1}{128}\left(20x-x^3\right)dx=\frac{1}{2}$ OR $\mathrm{F}(x)=\frac{1}{128}\left(2ax^2-\frac{b}{4}x^4\right)=\frac{5}{64}x^2-\frac{1}{512}x^4$	**M1***	Attempt at integral and use of correct limits. Could be in terms of a,b,c Or attempt at relevant part of CDF and set $\mathrm{F}(m)=\frac{1}{2}$. Could be in terms of a,b,c.
	$m^4-40m^2+256=0$	**M1**	Simplify to quartic. Could be in terms of a,b,c.
	$m=2\sqrt{2}$	**A1**	Positive answer only.
		3	

Question	Answer	Marks	Guidance
4(c)	$\mathrm{E}\left(\sqrt{X}\right)=\frac{1}{128}\int_0^4 20x^{\frac{3}{2}}-x^{\frac{7}{2}}\,dx+\int_4^6\frac{1}{8}x^{\frac{1}{2}}dx$	**M1**	*Their* PDF multiplied by $\sqrt{x}$ with correct limits. Could be in terms of a,b,c .
	$\frac{1}{128}\left[8x^{\frac{5}{2}}-\frac{2}{9}x^{\frac{9}{2}}\right]_0^4+\frac{1}{12}\left[x^{\frac{3}{2}}\right]_4^6$	**M1**	Correct use of correct limits in *their* integral. Could be in terms of a,b,c. May be implied by correct final answer.
	$\frac{1}{128}\left(2^8-\frac{1}{9}2^{10}\right)+\frac{1}{12}\left(6\sqrt{6}-8\right)=\frac{4}{9}+\frac{1}{2}\sqrt{6}\ =1.67$	**A1**	CAO 1.67 following (first) M0 SC B2
		3	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="7">5(a)</td><td>H_0: population medians are equal or $m_x = m_y$
H_1: population median for X < population median for Y or $m_x < m_y$</td><td>B1</td><td>Do not accept ‘difference between population medians < 0’ without X or Y oe specified.</td></tr>
<tr><td><table><tr><th colspan="2">X</th><th colspan="2">Y</th></tr><tr><td>4.0</td><td>1</td><td>4.5</td><td>2</td></tr><tr><td>4.6</td><td>3</td><td>4.9</td><td>6</td></tr><tr><td>4.7</td><td>4</td><td>5.1</td><td>8</td></tr><tr><td>4.8</td><td>5</td><td>5.3</td><td>10</td></tr><tr><td>5.0</td><td>7</td><td>5.4</td><td>11</td></tr><tr><td>5.2</td><td>9</td><td>5.7</td><td>13</td></tr><tr><td>5.6</td><td>12</td><td>5.9</td><td>15</td></tr><tr><td>5.8</td><td>14</td><td>6.3</td><td>16</td></tr><tr><td></td><td></td><td>6.4</td><td>17</td></tr></table>Sum: 55 98</td><td>M1</td><td>Rankings, allow at most 3 errors.</td></tr>
<tr><td>Test statistic = 55</td><td>A1</td><td></td></tr>
<tr><td>Tabular value for $m = 8,\ n = 9$ is 54</td><td>B1</td><td></td></tr>
<tr><td>‘55’ > ‘54’, accept H_0 /not significant</td><td>M1</td><td>Ft their ‘55’ Must come from ranks.
Ft their ‘54’, must come from table.</td></tr>
<tr><td>Insufficient evidence to support manager’s claim.
Insufficient evidence to suggest that the median time of machine X is less than the median time of machine Y.</td><td>A1</td><td>Correct conclusion in context, following correct work, level of uncertainty in language.
A0 if hypotheses the wrong way round or missing.</td></tr>
<tr><td></td><td>6</td><td></td></tr>
</table>

Question	Answer	Marks	Guidance
5(b)	$H_0: \mu_x = \mu_y$ $H_1: \mu_x < \mu_y$	B1	
	$\sum x = 39.7$ $\sum x^2 = 199.33$ $\sum y = 49.5$ $\sum y^2 = 275.47$ $s_x^2 = \frac{1}{7}\left(199.33 - \frac{39.7^2}{8}\right) = 0.33125$	B1	
	$s_y^2 = \frac{1}{8}\left(275.47 - \frac{49.5^2}{9}\right) = 0.4025$	B1	
	Pooled variance $s^2 = \frac{7 \times 0.33125 + 8 \times 0.4025}{8+9-2}$	M1	
	0.36925	A1	
	$t = \dfrac{\frac{39.7}{8} - \frac{49.5}{9}}{s\sqrt{\frac{1}{8}+\frac{1}{9}}}$	M1	
	$t = -1.82$	A1	
	Tabular value = 1.753: 1.82 > 1.753	M1	
	Reject H_0, sufficient evidence that mean for machine X is less than mean for machine Y.	A1	CWO Correct conclusion in context, following correct work, level of uncertainty in language.
		9	
5(c)	t-test is assuming a normal distribution, and with equal variances. This may not be true. So, no reason to expect results to be the same. Outliers affect part **(b)** but not part **(a)**.	B1	Not specific to data in question. Mention of normal distribution is not enough.
		1	

Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME	

CENTRE NUMBER		CANDIDATE NUMBER	

5955151952

FURTHER MATHEMATICS **9231/43**

Paper 4 Further Probability & Statistics **October/November 2023**

1 hour 30 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS
- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION
- The total mark for this paper is 50.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **12** pages. Any blank pages are indicated.

BLANK PAGE

1 Maya is an athlete who competes in 1500-metre races. Last summer her practice run times had mean 4.22 minutes. Over the winter she has done some intense training to try to improve her times. A random sample of 10 of her practice run times, x minutes, this summer are summarised as follows.

$$\sum x = 42.05 \qquad \sum x^2 = 176.83$$

Maya's new practice run times are normally distributed. She believes that on average her times have improved as a result of her training.

Test, at the 5% significance level, whether Maya's belief is supported by the data. [6]

2 A town council has published its plans for redeveloping the town centre and residents are being asked whether they approve or disapprove. A random sample of 250 responses has been selected from residents in the four main streets in the town: North, East, South and West Streets. The results are shown in the table.

	North Street	East Street	South Street	West Street
Approve	33	54	42	26
Disapprove	19	39	28	9

Test, at the 5% significance level, whether the opinions of the residents are independent of the streets on which they live. [7]

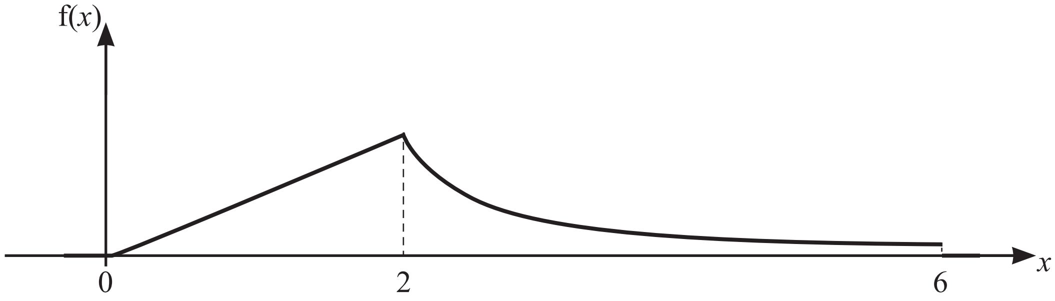
3 Scientists are studying the effects of exercise on LDL blood cholesterol levels. Over a three-month period, a large group of people exercised for 20 minutes each day. For a randomly chosen sample of 10 of these people, the LDL blood cholesterol levels were measured at the beginning and the end of the three-month period. The results, measured in suitable units, are as follows.

	Person	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Cholesterol level	Beginning	72	84	120	90	102	135	64	75	80	88
	End	64	76	105	92	105	115	67	75	75	84

(a) Test, at the 2.5% significance level, whether there is evidence that the population mean LDL blood cholesterol level has reduced by more than 2 units after the three-month period. [7]

(b) State any assumption that you have made in part **(a)**. [1]

4



As shown in the diagram, the continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} mx & 0 \leqslant x \leqslant 2, \\ \dfrac{k}{x^2} + c & 2 \leqslant x \leqslant 6, \\ 0 & \text{otherwise,} \end{cases}$$

where m, k and c are constants.

(a) Given that $P(X \leqslant 2) = \frac{1}{3}$, show that $m = \frac{1}{6}$ and find the values of k and c. [4]

(b) Find the exact numerical value of the interquartile range of X. [5]

5 The random variable X has the geometric distribution Geo(p).

(a) Show that the probability generating function of X is $\frac{pt}{1-qt}$, where $q = 1-p$. [3]

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(b) Use the probability generating function of X to show that $\mathrm{Var}(X) = \frac{q}{p^2}$. [5]

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Kenny throws an ordinary fair 6-sided dice repeatedly. The random variable X is the number of throws that Kenny takes in order to obtain a 6. The random variable Z denotes the sum of two independent values of X.

(c) Find the probability generating function of Z. [2]

6 A school is conducting an experiment to see whether the distance that children can throw a ball increases in hot weather. On a cold day, all the children at the school were asked to throw a ball as far as possible. The distances thrown were measured and recorded. The median distance thrown by a random sample of 25 of the children was 22.0 m. The children were asked to throw the ball again on a hot day. The distances thrown by the same 25 children were measured and recorded and these distances, in m, are shown below.

21.2	23.5	22.9	18.6	19.4
22.1	26.5	20.2	25.7	20.6
22.3	17.4	22.2	27.0	23.9
28.2	22.6	27.2	23.0	23.7
19.8	22.7	23.3	21.5	24.3

The teacher claims that on average the distances thrown will be further when it is hot.

Carry out a Wilcoxon signed-rank test, at the 5% significance level, to test whether the data supports the teacher's claim. [10]

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Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/43**

Paper 4 Further Probability & Statistics **October/November 2023**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

This document consists of **13** printed pages.

[Turn over

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	H_0: $\mu = 4.22$ H_1: $\mu < 4.22$	**B1**	
	$\bar{x} = \frac{42.05}{10}[= 4.205]$, $s^2 = \frac{1}{9}\left(176.83 - \frac{42.05^2}{10}\right)\left[= \frac{13}{12000} = 0.0010833\right]$	**M1**	
	$t = \frac{4.205 - 4.22}{\sqrt{\frac{s^2}{10}}}$	**M1**	*Their* 4.205, *their* s^2.
	$t = -1.44$	**A1**	Condone sign.
	Tabular value = 1.833: '1.44' < 1.833, accept H_0	**M1**	Compare *their* value with **correct** tabular value, **signs consistent**, allow 'not significant'.
	There is insufficient evidence to support Maya's belief.	**A1**	Correct conclusion in context, following correct work, level of uncertainty in language. A0 if hypotheses wrong way round or missing.
		6	

Question	Answer	Marks	Guidance
2	H_0: opinion is independent of street H_1: opinion is not independent of street	B1	
	33 **32.24** / 54 **57.66** / 42 **43.4** / 26 **21.7** / *155* 19 **19.76** / 39 **35.34** / 28 **26.6** / 9 **13.3** / *95* *52* / *93* / *70* / *35* / *250*	M1	At least 2 correct values or expressions.
		A1	6 correct values or expressions.
	Calculate chi-squared values: 0.0179 + 0.2323 + 0.0452 + 0.8521 + 0.0292 + 0.3790 + 0.0737 + 1.3902	M1	Correct to 3 decimal places. At least 2 correct values or expressions.
	3.02	A1	SC B1 3.02 following M1M0 SC B2 3.02 following M0M0
	Tabular value, 3 degrees of freedom 7.815: '3.02' < 7.815, accept H_0	M1	Allow 'not significant'.
	Insufficient evidence to suggest that opinion depends on street.	A1	CWO. Correct conclusion in context, following correct work, level of uncertainty in language. A0 if hypotheses wrong way round or missing.
		7	

Question	Answer	Marks	Guidance
3(a)	H0: $\mu_B - \mu_E = 2$ H1: $\mu_B - \mu_E > 2$	**B1**	May use μ_d, but must be consistent with direction of differences found.
	8 | 8 | 15 | –2 | –3 | 20 | –3 | 0 | 5 | 4	**M1**	Differences, at most 2 errors.
	$\sum d = 52, \sum d^2 = 816$, $s^2 = \frac{1}{9}\left(816 - \frac{52^2}{10}\right)\left[= \frac{2728}{45} = 60.62\right]$	**M1**	*Their* values but must see 9 and 10 used correctly.
	$t = \dfrac{\frac{52}{10} - 2}{\sqrt{\frac{s^2}{10}}}$	**M1**	
	$t = 1.29967$, 1.30	**A1**	
	Tabular value = 2.262. '1.30' < 2.262, accept H0	**M1**	Allow 'not significant'.
	Insufficient evidence to suggest that cholesterol level has reduced by more than 2.	**A1**	CWO. Correct conclusion in context, following correct work, level of uncertainty in language. A0 if hypotheses wrong way round or missing.
		7	
3(b)	**Population differences** are **normally** distributed	**B1**	
		1	

Question	Answer	Marks	Guidance
4(a)	$\mathrm{P}(X \leqslant 2) = \frac{1}{3}$, so $\left[\frac{mx^2}{2}\right]_0^2 = \frac{1}{3}$ $m\left(\frac{4}{2}\right) = \frac{1}{3}$, $m = \frac{1}{6}$	**B1**	AG Reasoning required.
	$\int_2^6 \left(\frac{k}{x^2} + c\right)\mathrm{d}x = \frac{2}{3}$ so $\left[-\frac{k}{x} + cx\right]_2^6 = 4c + \frac{1}{3}k = \frac{2}{3}$ or $k + 12c = 2$	**M1**	Attempt to integrate and correct use of correct limits to form linear equation in k and c.
	Also, $2m = \frac{k}{4} + c$ or $\frac{1}{3} = \frac{k}{4} + c$ or $3k + 12c = 4$	**M1**	Matching at $x = 2$.
	Solve, $k = 1$, $c = \frac{1}{12}$	**A1**	
		4	

Question	Answer	Marks	Guidance
4(b)	LQ: $\int_0^L \frac{1}{6}x\,dx = \frac{1}{4}$, $L = \sqrt{3}$	**B1**	
	UQ: $\frac{1}{3} + \int_2^U \left(\frac{k}{x^2} + c\right)dx = \frac{3}{4}$	**M1**	Or find and use CDF. May be in terms of *m*.
	$-\frac{k}{U} + Uc + \frac{k}{2} - 2c = \frac{5}{12}$ $U^2 - U - 12 = 0$	**M1**	Attempt at integral and correct use of correct limits. Simplify to quadratic equation, may be in terms of *k* and *c*.
	$U = 4$	**A1**	
	IQR = $4 - \sqrt{3}$	**A1 FT**	FT *their* UQ – *their* LQ, **exact values only**.
		5	

Question	Answer	Marks	Guidance
5(a)	P($X = r$) = $p(1-p)^{r-1}$ = pq^{r-1}	B1	Implied by G(t) = $\sum pq^{r-1}\ t^r$.
	G(t) = $\sum pt(qt)^{r-1}$	M1	(Infinite summation) with common ratio (qt) and first term or common factor pt indicated **or** GP identified.
	G(t) = $pt\left(\frac{1}{1-qt}\right) = \frac{pt}{1-qt}$	A1	AG Convincingly obtained.
		3	
5(b)	G'(t) = $pqt(1-qt)^{-2} + p(1-qt)^{-1}$ [= $p(1-qt)^{-2}$]	M1	Attempt to differentiate as a product/quotient.
	G"(t) = $2pq^2t(1-qt)^{-3} + 2pq(1-qt)^{-2}$ [$= 2qp(1-qt)^{-3}$]	M1	Attempt to differentiate *their* G'(t).
	G'(1) = $p(1-q)^{-2} = \frac{1}{p}$, G"(1) = $\frac{2q}{P^2}$	M1	Substitute $t = 1$ into their G'(t) and G"(t) and use formula for Var(X) (at some point).
	Var(X) = $\frac{2q}{P^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2$	M1	$(1-q)$ replaced by p in denominator throughout, and some cancellation seen (at some point).
	$\frac{q}{p^2}$	A1	AG Convincingly obtained.
		5	

Question	Answer	Marks	Guidance
5(c)	$\mathrm{G}_X(t) = \dfrac{\frac{1}{6}t}{1-\frac{5}{6}t}$ $\left[= \dfrac{t}{6-5t}\right]$	**B1**	Implied by correct final answer.
	$\mathrm{G}_Z(t) = (\mathrm{G}_X(t))^2 = \left(\dfrac{t}{6-5t}\right)^2$	**B1**	Allow $\left(\dfrac{\frac{1}{6}t}{1-\frac{5}{6}t}\right)^2$.
		2	

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>6</td><td>H_0: population medians are equal
H_1: population median after > population median before</td><td>B1</td><td>Do not accept ‘differences between population medians < 0’ unless difference defined.
Allow $H_0 : m = 22, H_1 : m > 22.0$</td></tr>
<tr><td></td><td rowspan="2"><table>
<tr><td>–0.8</td><td>-7</td><td>1.5</td><td>12</td><td>0.9</td><td>8</td><td>–3.4</td><td>–19</td><td>–2.6</td><td>–18</td></tr>
<tr><td>0.1</td><td>1</td><td>4.5</td><td>21</td><td>–1.8</td><td>–14</td><td>3.7</td><td>20</td><td>–1.4</td><td>–11</td></tr>
<tr><td>0.3</td><td>3</td><td>–4.6</td><td>–22</td><td>0.2</td><td>2</td><td>5</td><td>23</td><td>1.9</td><td>15</td></tr>
<tr><td>6.2</td><td>25</td><td>0.6</td><td>5</td><td>5.2</td><td>24</td><td>1</td><td>9</td><td>1.7</td><td>13</td></tr>
<tr><td>–2.2</td><td>-16</td><td>0.7</td><td>6</td><td>1.3</td><td>10</td><td>–0.5</td><td>–4</td><td>2.3</td><td>17</td></tr>
</table></td><td>M1</td><td>Signed differences, allow at most 4 errors</td></tr>
<tr><td></td><td>M1</td><td>Attempt at ranks (ignore signs).</td></tr>
<tr><td></td><td>($W_+ = 214$) $W_- = 111$</td><td>A1</td><td>Cao identified, or used, as test statistic.</td></tr>
<tr><td></td><td>Normal: mean = $\frac{1}{4}n(n+1) = \frac{1}{4}\times 25\times 26[=162.5]$</td><td>B1</td><td></td></tr>
<tr><td></td><td>Variance = $\frac{1}{24}n(n+1)(2n+1) = \frac{1}{24}\times 25\times 26\times 51[=1381.25]$</td><td>B1</td><td></td></tr>
<tr><td></td><td>z- value: $\frac{111.5-162.5}{\sqrt{1381.25}}$</td><td>M1</td><td>Allow missing or incorrect continuity correction.
Their 111 must come from ranks.</td></tr>
<tr><td></td><td>–1.37</td><td>A1</td><td>CAO</td></tr>
<tr><td></td><td>Tabular value is -1.645: ‘-1.37’ > -1.645, or 0.915<0.95 oe, accept H_0,</td><td>M1</td><td>Consistent signs. Allow ‘not significant’.</td></tr>
<tr><td></td><td>Insufficient evidence to support the teacher’s claim
Insufficient evidence to suggest that the distances thrown are further when it is hot</td><td>A1</td><td>All correct. Correct conclusion in context, following correct work, level of uncertainty in language.

A0 if hypotheses wrong way round or missing</td></tr>
<tr><td></td><td></td><td>10</td><td></td></tr>
</table>

