COMPENDIUM USAMO

Olimpiada Matemática USA

1987 – 2023

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TooMates Cooleccción

Toomates Coolección

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Ámbito Olímpico español: OME , OMEFL , OMEC , OMEA , OMEM
Ámbito Olímpico Internacional: IGO , IMO , OMI , SMT , USAMO , INMO , CMO , REOIM
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Presentación.

La prueba **United States of America Mathematical Olympiad** (USAMO) es la Fase Nacional de las Olimpiadas Matemáticas correspondiente a los Estados Unidos.

La prueba se desarrolla en dos jornadas, en cada una se proponen tres problemas a resolver en cuatro horas y media. Las respuestas deben ser tipo “ensayo”, es decir, se deben argumentar y se puntúa la claridad expositiva y la calidad matemática de los razonamientos. Cada problema se puntúa entre 0 y 7, haciendo un total de 42 puntos.

Fue creada en el año 1972 por Nura D. Turner y Samuel L.Greitzer como ronda final de las competiciones AMC. Los doce mejores clasificados en la USAMO son invitados a participar en el **Mathematical Olympiad Summer Program** (MOP) de donde se seleccionarán los seis componentes del equipo olímpico que representará a los Estados Unidos en las **Olimpiadas Matemáticas Internacionales** (IMO).

La prueba **America Junior Mathematical Olympiad** (USAJMO) fue introducida en el 2010 para reconocer a los mejores clasificados de la prueba AMC 10.

En el año 1983 se introdujo la prueba AIME (**American Invitational Mathematics Examination**) como puente entre las AMC y las USAMO.

Se pueden presentar todos los ciudadanos de los EEUU o Canadá, o con tarjeta de residencia de dichos países, seleccionados entre los mejores clasificados en las fases AMC y AIME, mediante el siguiente índice:

- Índice AMC 12:
  Puntuación de la prueba AMC 12 + 10*(Puntuación de la prueba AIME).
  Los mejores 260-270 clasificados se clasifican para la prueba USAMO.

- Índice AMC 10:
  Puntuación de la prueba AMC 10 + 10*(Puntuación de la prueba AIME).
  Los mejores 230-240 clasificados se clasifican para la prueba USAMO.

Si un estudiante se presenta a las dos pruebas (AMC 10 y AMC 12) y se clasifica por ambas, deberá optar obligatoriamente a la prueba USAMO.

Índice.

(*) https://web.evanchen.cc/

Temas tratados en las últimas pruebas:

2017
1. Teoría de números
2. Combinatoria
3. Geometría
4. Combinatoria
5. Combinatoria
6. Álgebra

2016
1. Combinatoria
2. Teoría de números
3. Geometría
4. Álgebra
5. Geometría
6. Combinatoria

2015
1. Álgebra
2. Geometría
3. Combinatoria
4. Combinatoria
5. Teoría de números
6. Álgebra

2014
1. Álgebra
2. Álgebra
3. Álgebra
4. Combinatoria/Teoría de juegos
5. Geometría
6. Teoría de números

2013
1. Geometría
2. Combinatoria
3. Combinatoria
4. Álgebra
5. Teoría de números
6. Geometría

2012
1. Combinatoria/Álgebra
2. Combinatoria
3. Teoría de números
4. Teoría de números/Álgebra
5. Geometría
6. Álgebra/Combinatoria

2011
1. Álgebra/Desigualdades
2. Combinatoria
3. Geometría
4. Teoría de números
5. Geometría
6. Combinatoria

2010
1. Geometría
2. Combinatoria
3. Álgebra
4. Geometría/Teoría de números
5. Álgebra/Teoría de números
6. Combinatoria

2009
1. Geometría
2. Combinatoria
3. Combinatoria/Teoría de Grafos
4. Álgebra
5. Geometría
6. Teoría de números

2008
1. Teoría de números
2. Geometría
3. Combinatoria
4. Combinatoria
5. Teoría de números/Combinatoria
6. Teoría de Grafos/Álgebra Lineal

2007
1. Teoría de números/Álgebra
2. Geometría
3. Combinatoria
4. Teoría de Grafos
5. Teoría de números
6. Geometría

2006
1. Teoría de números
2. Álgebra/Combinatoria
3. Teoría de números/Álgebra
4. Álgebra
5. Álgebra/Combinatoria
6. Geometría

2005
1. Teoría de números/Teoría de Grafos
2. Teoría de números
3. Geometría
4. Geometría/Álgebra
5. Combinatoria
6. Álgebra

2004
1. Geometría/Desigualdades
2. Álgebra
3. Combinatoria/Geometría
4. Combinatoria
5. Desigualdades
6. Geometría

2003
1. Teoría de números
2. Geometría/Álgebra
3. Álgebra
4. Geometría
5. Desigualdades
6. Combinatoria

Fuentes.

https://www.maa.org/math-competitions/usamo-archive

https://web.evanchen.cc/problems.html

https://www.russianschool.com/blog/competitions/usamo-problems-and-solutions

https://artofproblemsolving.com/wiki/index.php/USAMO_Problems_and_Solutions

Todo este material ha sido agrupado en un único archivo "pdf" mediante la aplicación online https://www.ilovepdf.com/

1987 USAMO Problems

Problems from the **1987 USAMO.**

Contents

Problem 1

Find all solutions to $(m^2+n)(m+n^2)=(m-n)^3$, where m and n are non-zero integers.

Solution

Problem 2

The feet of the angle bisectors of ΔABC form a right-angled triangle. If the right-angle is at X, where AX is the bisector of $\angle A$, find all possible values for $\angle A$.

Solution

Problem 3

X is the smallest set of polynomials $p(x)$ such that:

1. $p(x)=x$ belongs to X.
2. If $r(x)$ belongs to X, then $x\cdot r(x)$ and $(x+(1-x)\cdot r(x))$ both belong to X.

Show that if $r(x)$ and $s(x)$ are distinct elements of X, then $r(x)\neq s(x)$ for any $0<x<1$.

Solution

Problem 4

M is the midpoint of XY. The points P and Q lie on a line through Y on opposite sides of Y, such that $|XQ|=2|MP|$ and $\frac{|XY|}{2}<|MP|<\frac{3|XY|}{2}$. For what value of $\frac{|PY|}{|QY|}$ is $|PQ|$ a minimum?

Solution

Problem 5

$a_1,a_2,\cdots,a_n$ is a sequence of 0's and 1's. T is the number of triples (a_i,a_j,a_k) with $i<j<k$ which are not equal to (0, 1, 0) or (1, 0, 1). For $1\leq i\leq n$, $f(i)$ is the number of $j<i$ with $a_j=a_i$ plus the number of $j>i$ with $a_j\neq a_i$. Show that $T=\sum_{i=1}^{n}f(i)\cdot\left(\frac{f(i)-1}{2}\right)$. If n is odd, what is the smallest value of T?

Solution

See Also

1987 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1987))	
Preceded by **1986 USAMO**	Followed by **1988 USAMO**
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All USAMO Problems and Solutions	

1988 USAMO Problems

Problems from the **1988 USAMO.**

Contents

Problem 1

The repeating decimal $0.ab\cdots k\overline{pq\cdots u}=\frac{m}{n}$, where m and n are relatively prime integers, and there is at least one decimal before the repeating part. Show that n is divisible by 2 or 5 (or both). (For example, $0.011\overline{36}=0.01136363636\cdots=\frac{1}{88}$, and 88 is divisible by 2.)

Solution

Problem 2

The cubic polynomial x^3+ax^2+bx+c has real coefficients and three real roots $r\ge s\ge t$. Show that $k=a^2-3b\ge 0$ and that $\sqrt{k}\le r-t$.

Solution

Problem 3

Let X be the set $\{1,2,\cdots,20\}$ and let P be the set of all 9-element subsets of X. Show that for any map $f:P\mapsto X$ we can find a 10-element subset Y of X, such that $f(Y-\{k\})\neq k$ for any k in Y.

Solution

Problem 4

ΔABC is a triangle with incenter I. Show that the circumcenters of ΔIAB, ΔIBC, and ΔICA lie on a circle whose center is the circumcenter of ΔABC.

Solution

Problem 5

Let $p(x)$ be the polynomial $(1-x)^a(1-x^2)^b(1-x^3)^c\cdots(1-x^{32})^k$, where $a,b,\cdots,k$ are integers. When expanded in powers of x, the coefficient of x^1 is -2 and the coefficients of x^2, x^3, ..., x^{32} are all zero. Find k.

Solution

See Also

1988 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1988))	
Preceded by **1987 USAMO**	Followed by **1989 USAMO**
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1989 USAMO Problems

Problems from the **1989 USAMO.**

Contents

Problem 1

For each positive integer n, let

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
$$T_n = S_1 + S_2 + S_3 + \cdots + S_n$$
$$U_n = \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}.$$

Find, with proof, integers $0 < a,\ b,\ c,\ d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

Solution

Problem 2

The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

Solution

Problem 3

Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.

Solution

Problem 4

Let ABC be an acute-angled triangle whose side lengths satisfy the inequalities $AB < AC < BC$. If point I is the center of the inscribed circle of triangle ABC and point O is the center of the circumscribed circle, prove that line IO intersects segments AB and BC.

Solution

Problem 5

Let u and v be real numbers such that

$$(u + u^2 + u^3 + \cdots + u^8) + 10u^9 = (v + v^2 + v^3 + \cdots + v^{10}) + 10v^{11} = 8.$$

Determine, with proof, which of the two numbers, u or v, is larger.

Solution

See Also

1989 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1989))	
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1990 USAMO Problems

Problems from the **1990 USAMO**.

Contents

Problem 1

A certain state issues license plates consisting of six digits (from 0 through 9). The state requires that any two plates differ in at least two places. (Thus the plates $\boxed{027592}$ and $\boxed{020592}$ cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can use.

Solution

Problem 2

A sequence of functions $\{f_n(x)\}$ is defined recursively as follows:

$$f_1(x) = \sqrt{x^2 + 48}, \quad \text{and}$$
$$f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)} \quad \text{for } n \geq 1.$$

(Recall that $\sqrt{\quad}$ is understood to represent the positive square root.) For each positive integer n, find all real solutions of the equation $f_n(x) = 2x$.

Solution

Problem 3

Suppose that necklace A has 14 beads and necklace B has 19. Prove that for any odd integer $n \geq 1$, there is a way to number each of the 33 beads with an integer from the sequence

$$\{n, n+1, n+2, \dots, n+32\}$$

so that each integer is used once, and adjacent beads correspond to relatively prime integers. (Here a "necklace" is viewed as a circle in which each bead is adjacent to two other beads.)

Solution

Problem 4

Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.)

Solution

Problem 5

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Solution

See Also

1990 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1990))	
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1991 USAMO Problems

Problems from the 1991 USAMO. There were five questions administered in one three-and-a-half-hour session.

Contents

Problem 1

In triangle ABC, angle A is twice angle B, angle C is obtuse, and the three side lengths a, b, c are integers. Determine, with proof, the minimum possible perimeter.

Solution

Problem 2

For any nonempty set S of numbers, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S. Prove that

$$\sum \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(n+1),$$

where "Σ" denotes a sum involving all nonempty subsets S of $\{1, 2, 3, \ldots, n\}$.

Solution

Problem 3

Show that, for any fixed integer $n \geq 1$, the sequence

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2,\ a_{i+1} = 2^{a_i}$. Also $a_i \pmod{n}$ means the remainder which results from dividing a_i by n.]

Solution

Problem 4

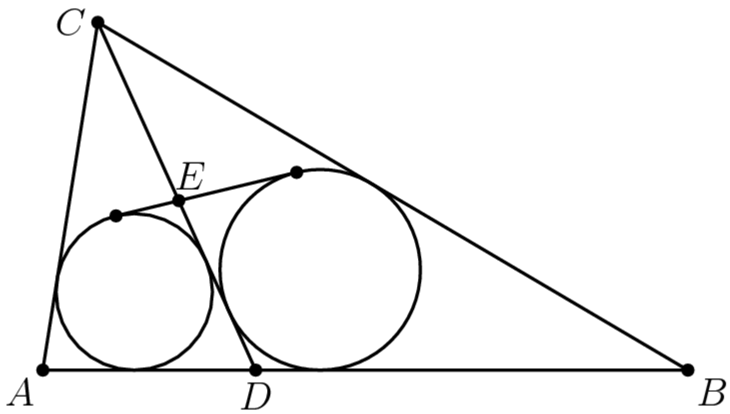
Let $a = (m^{m+1} + n^{n+1})/(m^m + n^n)$, where m and n are positive integers. Prove that $a^m + a^n \geq m^m + n^n$.

[You may wish to analyze the ratio $(a^N - N^N)/(a - N)$, for real $a \geq 0$ and integer $N \geq 1$.]

Solution

Problem 5

Let D be an arbitrary point on side AB of a given triangle ABC, and let E be the interior point where CD intersects the external common tangent to the incircles of triangles ACD and BCD. As D assumes all positions between A and B, prove that the point E traces the arc of a circle.



Solution

See Also

1991 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1991))	
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- 1991 USAMO Problems (TEX) (http://www.unl.edu/amc/a-activities/a7-problems/USAMO-IMO/q-usamo/-tex/usamo1991.tex)
- 1991 USAMO Problems on the resources page (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=1991)

21^{st} USA Mathematical Olympiad

April 30, 1992
Time Limit: $3\frac{1}{2}$ hours

1. Find, as a function of n, the sum of the digits of

$$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n} - 1\right),$$

where each factor has twice as many digits as the previous one.

2. Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

3. For a nonempty set S of integers, let $\sigma(S)$ be the sum of the elements of S. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \le 1500$, there is a subset S of A for which $\sigma(S) = n$. What is the smallest possible value of a_{10}?

4. Chords $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $AA' = BB' = CC'$.

5. Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial

$$\left(\cdots\left((z - a_1)^2 - a_2\right)^2 \cdots - a_{1991}\right)^2 - a_{1992}.$$

22^{nd} United States of America Mathematical Olympiad

April 29, 1993
Time Limit: $3\frac{1}{2}$ hours

1. For each integer $n \geq 2$, determine, with proof, which of the two positive real numbers a and b satisfying
$$a^n = a + 1, \qquad b^{2n} = b + 3a$$
is larger.

2. Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

3. Consider functions $f : [0, 1] \to \mathbf{R}$ which satisfy
   (i) $f(x) \geq 0$ for all x in $[0, 1]$,
   (ii) $f(1) = 1$,
   (iii) $f(x) + f(y) \leq f(x + y)$ whenever x, y, and $x + y$ are all in $[0, 1]$.

   Find, with proof, the smallest constant c such that
$$f(x) \leq cx$$
for every function f satisfying (i)-(iii) and every x in $[0, 1]$.

4. Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \ldots$. (Such a sequence is said to be *log concave*.) Show that for each $n > 1$,
$$\frac{a_0 + \cdots + a_n}{n+1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \frac{a_0 + \cdots + a_{n-1}}{n} \cdot \frac{a_1 + \cdots + a_n}{n}.$$

23^{rd} United States of America Mathematical Olympiad

April 28, 1994
Time Limit: $3\frac{1}{2}$ hours

1. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers, no two consecutive, and let $s_m = k_1 + k_2 + \cdots + k_m$ for $m = 1, 2, 3, \ldots$. Prove that, for each positive integer n, the interval $[s_n, s_{n+1})$ contains at least one perfect square.

2. The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, ..., red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, ..., red, yellow, blue?

3. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, and CF are concurrent. Let P be the intersection of AD and CE. Prove that $CP/PE = (AC/CE)^2$.

4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers satisfying $\sum_{j=1}^{n} a_j \geq \sqrt{n}$ for all $n \geq 1$. Prove that, for all $n \geq 1$,
$$\sum_{j=1}^{n} a_j^2 > \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

5. Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!\,(n-k)!}$. Prove that
$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$
for all integers $m \geq \sigma(S)$.

24^{th} United States of America Mathematical Olympiad

April 27, 1995
Time Limit: $3\frac{1}{2}$ hours

1. Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1, \ldots, a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading the result in base p.

2. A calculator is broken so that the only keys that still work are the $\sin$, $\cos$, $\tan$, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

3. Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e. these three lines intersect at a point.

4. Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

   (i) $m-n$ divides $q_m - q_n$ for $m > n \geq 0$,

   (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

   Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

5. Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.

25^{th} United States of America Mathematical Olympiad

Part I 9 a.m. - 12 noon

May 2, 1996

1. Prove that the average of the numbers $n \sin n^\circ$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1^\circ$.

2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

3. Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than 2/3 the area of triangle ABC.

25^{th} United States of America Mathematical Olympiad

Part II 1 p.m. - 4 p.m.

May 2, 1996

4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10°$, $\angle PBA = 20°$, $\angle PCA = 30°$, and $\angle PAC = 40°$. Prove that triangle ABC is isosceles.

6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

26^{th} United States of America Mathematical Olympiad

Part I 9 a.m. - 12 noon

May 1, 1997

1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}$, $\overleftrightarrow{FD}$, $\overleftrightarrow{DE}$, respectively, are concurrent.

3. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

26^{th} United States of America Mathematical Olympiad

Part II 1 p.m. - 4 p.m.

May 1, 1997

4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than 1/3, for all $n \geq 6$.

5. Prove that, for all positive real numbers a, b, c,

$$(a^3+b^3+abc)^{-1}+(b^3+c^3+abc)^{-1}+(c^3+a^3+abc)^{-1} \leq (abc)^{-1}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ (the greatest integer $\leq nx$) for all $1 \leq n \leq 1997$.

27^{th} United States of America Mathematical Olympiad

Part I 9 a.m. -12 noon

April 28, 1998

1. Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum
$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$
ends in the digit 9.

2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ $(B \in \mathcal{C}_2)$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

3. Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that
$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$
Prove that
$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

27^{th} United States of America Mathematical Olympiad

Part II 1 p.m. - 4 p.m.

April 28, 1998

4. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

5. Prove that for each $n \geq 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

6. Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2 \ldots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle. (Here $A_{n+j} = A_j$.)

28^{th} United States of America Mathematical Olympiad

Part I 9 a.m. – 12 noon

April 27, 1999

1. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

   (a) every square that does not contain a checker shares a side with one that does;

   (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

   Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

3. Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\{ra/p\} + \{rb/p\} + \{rc/p\} + \{rd/p\} = 2$$

   for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. (Note: $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

28^{th} United States of America Mathematical Olympiad

Part II 1 p.m. – 4 p.m.

April 27, 1999

4. Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

29^{th} United States of America Mathematical Olympiad

Part I 9 a.m. -12 noon

May 2, 2000

1. Call a real-valued function f *very convex* if
$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$
holds for all real numbers x and y. Prove that no very convex function exists.

2. Let S be the set of all triangles ABC for which
$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$
where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

3. A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of R, W, and B, the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

29^{th} United States of America Mathematical Olympiad

Part II   1 p.m. - 4 p.m.

May 2, 2000

4. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

5. Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles $\omega_2, \omega_3, \ldots, \omega_7$ such that for $k = 2, 3, \ldots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

6. Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_ia_j, b_ib_j\} \leq \sum_{i,j=1}^{n} \min\{a_ib_j, a_jb_i\}.$$

USAMO 2000 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2000 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Call a real-valued function f *very convex* if

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holds for all real numbers x and y. Prove that no very convex function exists.

2. Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right) - \frac{3}{\min\{AP,BQ,CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA respectively. Prove that all triangles in S are isosceles and similar to one another.

3. A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand.

Find, as a function of R, W, and B, the minimal total penalty a player can amass and the number of ways in which this minimum can be achieved.

4. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

5. Let $A_1A_2A_3$ be a triangle, and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exists circles ω_2, ω_3, ..., ω_7 such that for $k = 2, 3, \dots, 7$, circle ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1} (indices mod 3). Prove that $\omega_7 = \omega_1$.

6. Let $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_ia_j, b_ib_j\} \le \sum_{i,j=1}^{n} \min\{a_ib_j, a_jb_i\}.$$

§1 USAMO 2000/1

Call a real-valued function f *very convex* if

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

For $C \ge 0$, we say a function f is *C-convex*

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + C\left|x-y\right|.$$

Suppose f is C-convex. Let $a < b < c < d < e$ be any arithmetic progression, such that $t = |e-a|$. Observe that

$$\begin{aligned} f(a)+f(c) &\ge 2f(b) + C \cdot \frac{1}{2}t \\ f(c)+f(e) &\ge 2f(d) + C \cdot \frac{1}{2}t \\ f(b)+f(d) &\ge 2f(c) + C \cdot \frac{1}{2}t \end{aligned}$$

Adding the first two to twice the third gives

$$f(a)+f(e) \ge 2f(c) + 2C \cdot t.$$

So we conclude C-convex function is also $2C$-convex. This is clearly not okay for $C > 0$.

§2 USAMO 2000/2

Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right)-\frac{3}{\min\{AP,BQ,CR\}}=\frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA respectively. Prove that all triangles in S are isosceles and similar to one another.

We will prove the inequality

$$\frac{2}{AP}+\frac{5}{BQ}+\frac{5}{CR}\geq\frac{6}{r}$$

with equality when $AP:BQ:CR=1:4:4$. This implies the problem statement.

Letting $x=AP$, $y=BQ$, $z=CR$, the inequality becomes

$$\frac{2}{x}+\frac{5}{y}+\frac{5}{z}\geq 6\sqrt{\frac{x+y+z}{xyz}}.$$

Squaring both sides and collecting terms gives

$$\frac{4}{x^2}+\frac{25}{y^2}+\frac{25}{z^2}+\frac{14}{yz}\geq\frac{16}{xy}+\frac{16}{xz}.$$

If we replace $x=1/a$, $y=4/b$, $z=4/c$, then it remains to prove the inequality

$$64a^2+25(b+c)^2\geq 64a(b+c)+36bc$$

where equality holds when $a=b=c$. This follows by two applications of AM-GM:

$$16\left(4a^2+(b+c)^2\right)\geq 64a(b+c)$$
$$9(b+c)^2\geq 36bc.$$

Again one can tell this is an inequality by counting degrees of freedom.

§3 USAMO 2000/3

A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand.

Find, as a function of R, W, and B, the minimal total penalty a player can amass and the number of ways in which this minimum can be achieved.

The minimum penalty is

$$f(B, W, R) = \min(BW, 2WR, 3RB)$$

or equivalently, the natural guess of "discard all cards of one color first" is actually optimal (though not necessarily unique).

This can be proven directly by induction. Indeed the base case $BWR = 0$ (in which case zero penalty is clearly achievable). The inductive step follows from

$$f(B, W, R) = \min \begin{cases} f(B-1, W, R) + W \\ f(B, W-1, R) + 2R \\ f(B, W, R-1) + 3B. \end{cases}$$

It remains to characterize the strategies. This is a routine calculation, so we just state the result.

- If any of the three quantities BW, $2WR$, $3RB$ is strictly smaller than the other three, there is one optimal strategy.
- If $BW = 2WR < 3RB$, there are $W + 1$ optimal strategies, namely discarding from 0 to W white cards, then discarding all blue cards. (Each white card discarded still preserves $BW = 2WR$.)
- If $2WR = 3RB < BW$, there are $R + 1$ optimal strategies, namely discarding from 0 to R red cards, and then discarding discarding all white cards.
- If $3WR = RB < 2WR$, there are $B + 1$ optimal strategies, namely discarding from 0 to B blue cards, and then discarding discarding all red cards.
- Now suppose $BW = 2WR = 3RB$. Discarding a card of one color ends up in exactly one of the previous three cases. This gives an answer of $R + W + B$ strategies.

§4 USAMO 2000/4

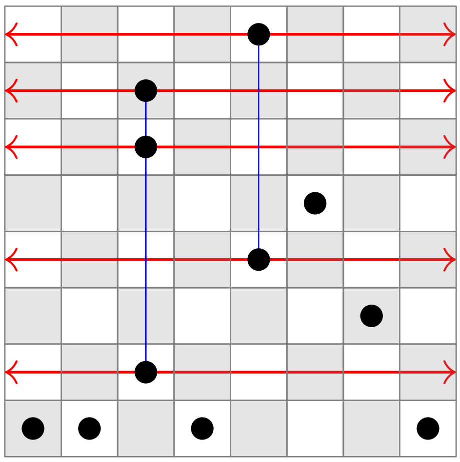
Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

The answer is $n = 1999$.

For a construction with $n = 1998$, take a punctured L as illustrated below (with 1000 replaced by 4):

$$\begin{bmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ & 1 & 1 & 1 \end{bmatrix}.$$

We now show that if there is no right triangle, there are at most 1998 tokens (colored squares). In every column with more than two tokens, we have token emit a bidirectional horizontal death ray (laser) covering its entire row: the hypothesis is that the death ray won't hit any other tokens.

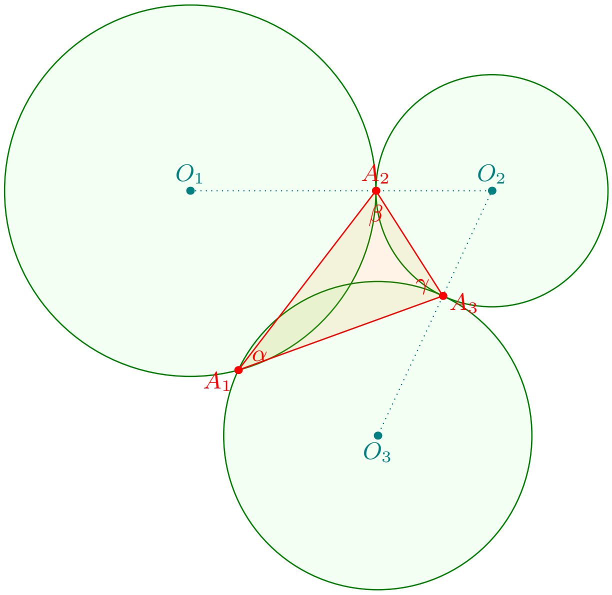


Assume there are n tokens and that $n > 1000$. Then obviously some column has more than two tokens, so at most 999 tokens don't emit a death ray (namely, any token in its own column). Thus there are at least $n - 999$ death rays. On the other hand, we can have at most 999 death rays total (since it would not be okay for the whole board to have death rays, as some row should have more than two tokens). Therefore, $n \le 999 + 999 = 1998$ as desired.

§5 USAMO 2000/5

Let $A_1A_2A_3$ be a triangle, and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exists circles ω_2, ω_3, ..., ω_7 such that for $k = 2, 3, \dots, 7$, circle ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1} (indices mod 3). Prove that $\omega_7 = \omega_1$.

The idea is to keep track of the subtended arc $\widehat{A_iA_{i+1}}$ of ω_i for each i. To this end, let $\beta = \measuredangle A_1A_2A_3$, $\gamma = \measuredangle A_2A_3A_1$ and $\alpha = \measuredangle A_1A_2A_3$.



Initially, we set $\theta = \measuredangle O_1A_2A_1$. Then we compute

$$\begin{aligned}
\measuredangle O_1A_2A_1 &= \theta \\
\measuredangle O_2A_3A_2 &= -\beta - \theta \\
\measuredangle O_3A_1A_3 &= \beta - \gamma + \theta \\
\measuredangle O_4A_2A_1 &= (\gamma - \beta - \alpha) - \theta
\end{aligned}$$

and repeating the same calculation another round gives

$$\measuredangle O_7A_2A_1 = k - (k - \theta) = \theta$$

with $k = \gamma - \beta - \alpha$. This implies $O_7 = O_1$, so $\omega_7 = \omega_1$.

§6 USAMO 2000/6, proposed by Gheorghita Zbaganu

Let $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \le \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

We present two solutions.

First solution by creating a single min (Vincent Huang and Ravi Boppana) Let $b_i = r_i a_i$ for each i, and rewrite the inequality as

$$\sum_{i,j} a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right] \ge 0.$$

We now do the key manipulation to convert the double min into a separate single min. Let $\varepsilon_i = +1$ if $r_i \ge 1$, and $\varepsilon_i = -1$ otherwise, and let $s_i = |r_i - 1|$. Then we pass to absolute values:

$$\begin{aligned} 2\min(r_i, r_j) - 2\min(1, r_i r_j) &= |r_i r_j - 1| - |r_i - r_j| - (r_i - 1)(r_j - 1) \\ &= |r_i r_j - 1| - |r_i - r_j| - \varepsilon_i \varepsilon_j s_i s_j \\ &= \varepsilon_i \varepsilon_j \min\left(|1 - r_i r_j \pm (r_i - r_j)|\right) - \varepsilon_i \varepsilon_j s_i s_j \\ &= \varepsilon_i \varepsilon_j \min\left(s_i(r_j + 1), s_j(r_i + 1)\right) - \varepsilon_i \varepsilon_j s_i s_j \\ &= (\varepsilon_i s_i)(\varepsilon_j s_j) \min\left(\frac{r_j + 1}{s_j} - 1, \frac{r_i + 1}{s_i} - 1\right). \end{aligned}$$

So let us denote $x_i = a_i \varepsilon_i s_i \in \mathbb{R}$, and $t_i = \frac{r_i + 1}{s_i} - 1 \in \mathbb{R}_{\ge 0}$. Thus it suffices to prove that:

Claim — We have
$$\sum_{i,j} x_i x_j \min(t_i, t_j) \ge 0$$
for arbitrary $x_i \in \mathbb{R}$, $t_i \in \mathbb{R}_{\ge 0}$.

Proof. One can just check this "by hand" by assuming $t_1 \le t_2 \le \dots \le t_n$; then the left-hand side becomes

$$\sum_i t_i x_i^2 + 2\sum_{i<j} 2 t_i x_i x_j = \sum_i (t_i - t_{i-1})(x_i + x_{i+1} + \dots + x_n)^2 \ge 0.$$

There is also a nice proof using the integral identity

$$\min(t_i, t_j) = \int_0^\infty \mathbf{1}(u \le t_i)\mathbf{1}(u \le t_j)\, du$$

where the $\mathbf{1}$ are indicator functions. Indeed,

$$\begin{aligned}\sum_{i,j} x_i x_j \min(t_i, t_j) &= \sum_{i,j} x_i x_j \int_0^\infty \mathbf{1}(u \le t_i)\mathbf{1}(u \le t_j)\, du \\ &= \int_0^\infty \sum_i x_i \mathbf{1}(u \le t_i) \sum_j x_j \mathbf{1}(u \le t_j)\, du \\ &= \int_0^\infty \left(\sum_i x_i \mathbf{1}(u \le t_i) \right)^2 du \\ &\ge 0.\end{aligned}$$

□

Second solution by smoothing (Alex Zhai) The case $n = 1$ is immediate, so we'll proceed by induction on $n \ge 2$.

Again, let $b_i = r_i a_i$ for each i, and write the inequality as

$$L_n(a_1, \dots, a_n, r_1, \dots, r_n) \overset{\text{def}}{=} \sum_{i,j} a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right] \ge 0.$$

First note that if $r_1 = r_2$ then

$$L_n(a_1, a_2, a_3, \dots, r_1, r_1, r_3 \dots) = L_{n-1}(a_1 + a_2, a_3, \dots, r_1, r_3, \dots)$$

and so our goal is to smooth to a situation where two of the r_i's are equal, so that we may apply induction.

On the other hand, L_n is a *piecewise linear* function in $r_1 \ge 0$. Let us smooth r_1 then. Note that if the minimum is attained at $r_1 = 0$, we can ignore a_1 and reduce to the $(n-1)$-variable case. On the other hand, the minimum must be achieved at a cusp which opens upward, which can only happen if $r_i r_j = 1$ for some j. (The $r_i = r_j$ cusps open downward, sadly.)

In this way, whenever some r_i is not equal to the reciprocal of any other $r_\bullet$, we can smooth it. This terminates; so we may smooth until we reach a situation for which

$$\{r_1, \dots, r_n\} = \{1/r_1, \dots, 1/r_n\}.$$

Now, assume WLOG that $r_1 = \max_i r_i$ and $r_2 = \min_i r_i$, hence $r_1 r_2 = 1$ and $r_1 \ge 1 \ge r_2$. We isolate the contributions from a_1, a_2, r_1 and r_2.

$$\begin{aligned}L_n(\dots) = a_1^2 &\left[r_1 - 1\right] + a_2^2 \left[r_2 - r_2^2\right] + 2a_1 a_2 \left[r_2 - 1\right] \\ &+ 2a_1 \left[(a_3 r_3 + \dots + a_n r_n) - (a_3 + \dots + a_n)\right] \\ &+ 2a_2 r_2 \left[(a_3 + \dots + a_n) - (a_3 r_3 + \dots + a_n r_n)\right] \\ &+ \sum_{i=3}^n \sum_{j=3}^n a_i a_j \left[\min(r_i, r_j) - \min(1, r_i r_j)\right].\end{aligned}$$

The idea now is to smooth via

$$(a_1, a_2, r_1, r_2) \longrightarrow \left(a_1, \frac{1}{t} a_2, \frac{1}{t} r_1, t r_2\right)$$

where $t \ge 1$ is such that $\frac{1}{t} r_1 \ge \max(1, r_3, \dots, r_n)$ holds. (This choice is such that a_1 and $a_2 r_2$ are unchanged, because we don't know the sign of $\sum_{i \ge 3}(1 - r_i) a_i$ and so the post-smoothing value is still at least the max.) Then,

$$\begin{aligned} &L_n(a_1, a_2, \dots, r_1, r_2, \dots) - L_n\left(a_1, \frac{1}{t}a_2, \dots, \frac{1}{t}r_1, tr_2\right) \\ &= a_1^2\left(r_1 - \frac{1}{t}r_1\right) + a_2^2\left(r_2 - \frac{1}{t}r_2\right) + 2a_1a_2\left(\frac{1}{t} - 1\right) \\ &= \left(1 - \frac{1}{t}\right)\left(r_1a_1^2 + r_2a_2^2 - 2a_1a_2\right) \ge 0 \end{aligned}$$

the last line by AM-GM. Now pick $t = \frac{r_1}{\max(1, r_3, \dots, r_n)}$, and at last we can induct down.

30^{th} United States of America Mathematical Olympiad

Part I 9 a.m. - 12 p.m.

May 1, 2001

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

3. Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

30^{th} United States of America Mathematical Olympiad

Part II 1 p.m. - 4 p.m.

May 1, 2001

4. Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

5. Let S be a set of integers (not necessarily positive) such that

   (a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

   (b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

   Prove that S is the set of all integers.

6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

30^{th} United States of America Mathematical Olympiad

Part I 9 a.m. – 12 noon

May 1, 2001

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Find the smallest integer n for which this is possible.

Solution: The smallest such n is 23.

We first show that $n = 22$ cannot be achieved. We present two arguments.

- *First argument* Let $m_{i,j}$ be the number of balls which are the same color as the j^{th} ball in box i (including that ball). For a fixed box i, $1 \leq i \leq 8$, consider the sums

$$S_i = \sum_{j=1}^{6} m_{i,j} \quad \text{and} \quad s_i = \sum_{j=1}^{6} \frac{1}{m_{i,j}}.$$

  For each fixed i, since no pair of colors is repeated, each of the reamining seven boxes can contributed at most one ball to S_i. Thus $S_i \leq 13$. It follows by the convexity of $f(x) = 1/x$ that s_i is minimized when one of the $m_{i,j}$ is equal to 3 and the other five equal to 2. Hence $s_i \geq 17/6$. Note that

$$n = \sum_{i=1}^{8} \sum_{j=1}^{6} \frac{1}{m_{i,j}} \geq 8 \cdot \frac{17}{6} = \frac{68}{3}.$$

  Hence there must be 23 colors.

- *Second argument* Assume that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of 6+5+5+5+2+1=24, a contradiction.

  Next, assume that no color occurs four times; this forces at least four colors to occur three times. In particular, there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of 6+5+5+3+3+1=23, again a contradiction.

We now give a construction for $n = 23$, guided by the second argument. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

1	3	4	5	6	7
1	8	9	10	11	12
1	13	14	15	16	17
2	3	8	13	18	19
2	4	9	14	20	21
2	5	10	15	22	23
6	11	16	18	20	22
7	12	17	19	21	23

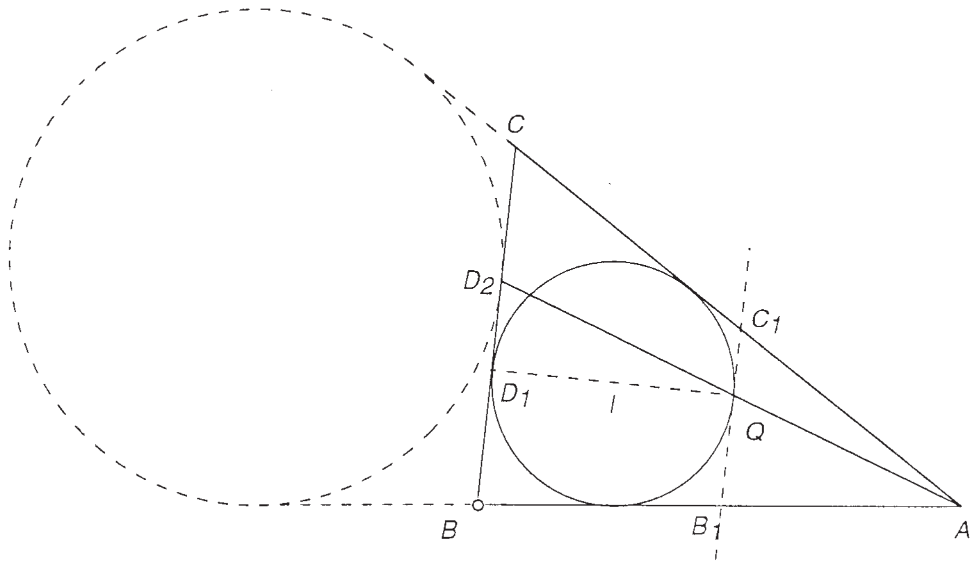
Note: Thanks to David Savitt for his help in assembling this solution; he also showed that for 10 boxes of eight balls, the minimum number of colors is 39. The general case of $n + 2$ boxes of n balls, or even more generally of $n + k$ boxes of n balls for other small values of k, may be of interest.

2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

Solution:

The key observation is the following lemma.

Lemma *Segement D_1Q is a diameter of circle ω.*



Proof: Let I be the center of circle ω, i.e., I is the incenter of triangle ABC. Extand segement D_1I through I to intersect circle ω again at Q', and extand segment AQ' through Q' to intersectt segment BC at D'. We show that $D_2 = D'$, which in turn implies that $Q = Q'$, that is, D_1Q is a diameter of ω.

Let ℓ be the line tangent to circle ω at Q', and let ℓ intersect the segments AB and AC at B' and C', respectively. Then ω is an **excircle** of triangle $AB'C'$. Let $\mathbf{H}_1$ denote the dialation with its center at A and ratio AD'/AQ'. Since $\ell \perp D_1Q'$ and $BC \perp D_1Q$, $\ell \perp BC$. Hence $AB/AB' = AC/AC' = AD'/AQ'$. Thus $\mathbf{H}_1(Q') = D'$, $\mathbf{H}_1(B') = B$, and $\mathbf{H}_1(C') = C$. It also follows that an excircle Ω of triangle ABC is tangent to the side BC at D'.

It is well known that

$$CD_1 = \frac{1}{2}(BC + CA - AB). \tag{1}$$

We compute BD'. Let X and Y denote the points of tangency of circle Ω with rays AB and AC, respectively. Then by equal tangents, $AX = AY$, $BD' = BX$, and $D'C = YC$. Hence

$$AX = AY = \frac{1}{2}(AX + AY) = \frac{1}{2}(AB + BX + YC + CA) = \frac{1}{2}(AB + BC + CA).$$

It follows that

$$BD' = BX = AX - AB = \frac{1}{2}(BC = CA - AB). \tag{2}$$

Combining (1) and (2) yields $BD' = CD_1$. Thus

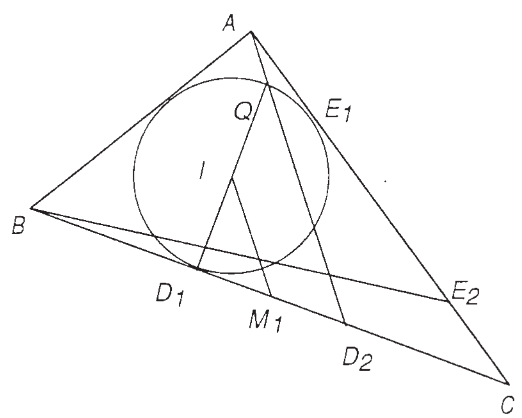
$$BD_2 = BD_1 - D_2D_1 = D_2C - D_2D_1 = CD_1 = BD',$$

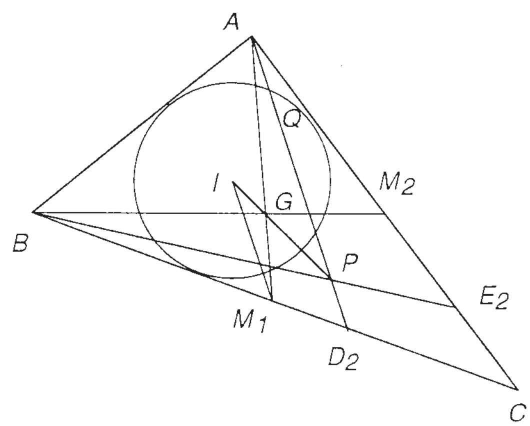
that is, $D' = D_2$, as desired. ∎

Now we prove our main result. Let M_1 and M_2 be the respective midpoints of segments BC and CA. Then M_1 is also the midpoint of segment D_1D_2, from which it follows that IM_1 is the midline of triangle D_1QD_2. Hence

$$QD_2 = 2IM_1 \tag{3}$$

and $AD_2 \parallel M_1I$. Similarly, we can prove that $BE_2 \parallel M_2I$.





Let G be the centroid of triangle ABC. Thus segments AM_1 and BM_2 intersect at G. Define transformation $\mathbf{H}_2$ as the **dialation** with its center at G and ratio $-1/2$. Then $\mathbf{H}_2(A) = M_1$ and $\mathbf{H}_2(B) = M_2$. Under the dilation, parallel lines go to parallel lines and the intersection of two lines goes to the intersection of their images. Since $AD_2 \parallel M_1I$ and $BE_2 \parallel M_2I$, $\mathbf{H}$ maps lines AD_2 and BE_2 to lines M_1I and M_2I, respectively. It also follows that $\mathbf{H}_2(I) = P$ and

$$\frac{IM_1}{AP} = \frac{GM_1}{AG} = \frac{1}{2}$$

or

$$AP = 2IM_1. \tag{4}$$

Combining (3) and (4) yields

$$AQ = AP - QP = 2IM_1 - QP = QD_2 - QP = PD_2,$$

as desired.

Note: We used three different diagrams of triangle ABC. Each diagram was desgined to assist the reader in understanding a particular part of the proof. We used directed lengths of segements in our calculations to avoid possible complications caused by the different shapes of triangle ABC.

3. Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \le ab + bc + ca - abc \le 2.$$

First Solution: From the condition, at least one of a, b, and c does not exceed 1, say $a \le 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \ge 0.$$

Now we prove the upper bound. Let us note that at least two of the three numbers a, b, and c are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are b and c. Then we have

$$(1 - b)(1 - c) \ge 0. \tag{1}$$

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \ge 2bc$ imply that

$$a^2 + 2bc + abc \le 4, \quad \text{or} \quad bc(2 + a) \le 4 - a^2.$$

Dividing both sides of the last inequality by $2 + a$ yields

$$bc \le 2 - a. \tag{2}$$

Combining (1) and (2) gives

$$ab + bc + ac - abc \le ab + 2 - a + ac(1 - b) = 2 - a(1 + bc - b - c) = 2 - a(1 - b)(1 - c) \le 2,$$

as desired.

The last equality holds if and only if $b = c$ and $a(1 - b)(1 - c) = 0$. Hence equality for the upper bound holds if and only if (a, b, c) is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$, and $(\sqrt{2}, \sqrt{2}, 0)$. Equality for the lower bound holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Second Solution: The proof for the lower bound is the same as in the first solution. Now we prove the upper bound. It is clear that $a, b, c \le 2$. If $abc = 0$, then the result is trivial. Suppose that $a, b, c > 0$. Solving for a yields

$$a = \frac{-bc + \sqrt{b^2c^2 - 4(b^2 + c^2 - 4)}}{2} = \frac{-bc + \sqrt{(4 - b^2)(4 - c^2)}}{2}.$$

This asks for the trigonometric substitution $b = 2\sin u$ and $c = 2\sin v$, where $0^\circ < u, v < 90^\circ$. Then

$$a = 2(-\sin u \sin v + \cos u \cos v) = 2\cos(u + v),$$

and if we set $u = B/2$ and $v = C/2$, then $a = 2\sin(A/2)$, $b = 2\sin(B/2)$, and $c = 2\sin(C/2)$, where A, B, and C are the angles of a triangle. We have

$$\begin{aligned}
ab &= 4\sin\frac{A}{2}\sin\frac{B}{2} = 2\sqrt{\sin A\tan\frac{A}{2}\sin B\tan\frac{B}{2}} = 2\sqrt{\sin A\tan\frac{B}{2}\sin B\tan\frac{A}{2}}\\
&\le \sin A\tan\frac{B}{2} + \sin B\tan\frac{A}{2} \qquad \text{(by the \textbf{AM-GM inequality})}\\
&= \sin A\cot\frac{A+C}{2} + \sin B\cot\frac{B+C}{2}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
bc &\le \sin B\cot\frac{B+A}{2} + \sin C\cot\frac{C+A}{2},\\
ca &\le \sin A\cot\frac{A+B}{2} + \sin C\cot\frac{C+B}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
ab + bc + ca &\le (\sin A + \sin B)\cot\frac{A+B}{2} + (\sin B + \sin C)\cot\frac{B+C}{2} + (\sin C + \sin A)\cot\frac{C+A}{2}\\
&= 2\left(\cos\frac{A-B}{2}\cos\frac{A+B}{2} + \cos\frac{B-C}{2}\cos\frac{B+C}{2} + \cos\frac{C-A}{2}\cos\frac{C+A}{2}\right)\\
&= 2(\cos A + \cos B + \cos C) = 6 - 4\left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right)\\
&= 6 - (a^2 + b^2 + c^2) = 2 + abc,
\end{aligned}$$

as desired.

30^{th} United States of America Mathematical Olympiad

Part II 1 p.m. – 4 p.m.

May 1, 2001

4. Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

First Solution: Let A be the origin. For a point Q, denote by q the vector $\overrightarrow{AQ}$, and denote by $|q|$ the length of q. The given condictions may be written as

$$|p-b|^2+|p-c|^2<|p|^2,$$

or

$$p\cdot p+b\cdot b+c\cdot c-2p\cdot b-2p\cdot c<0.$$

Adding $2b\cdot c$ on both sides of the last inequality gives

$$|p-b-c|^2<2b\cdot c.$$

Since the left-hand side of the last inequality is nonnegative, the right-hand side is positive. Hence

$$\cos\angle BAC=\frac{b\cdot c}{|b||c|}>0,$$

that is, $\angle BAC$ is acute.

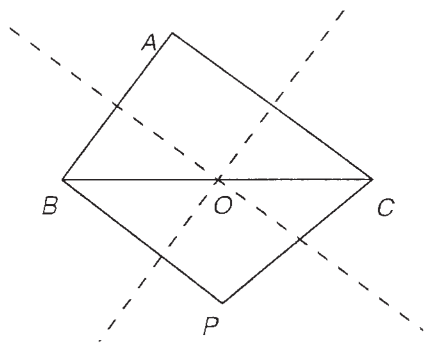
Second Solution: For the sake of contradiction, let's assume to the contrary that $\angle BAC$ is not acute. Let $AB=c$, $BC=a$, and $CA=b$. Then $a^2\geq b^2+c^2$. We claim that the quadrilateral $ABPC$ is convex. Now applying the generalized Ptolemy's Theorem to the convex quadrilateral $ABPC$ yields

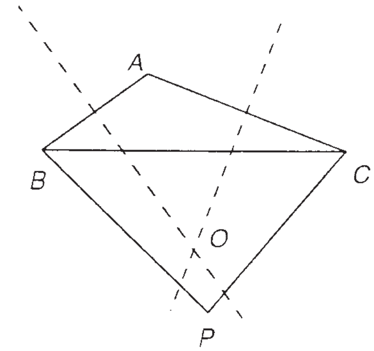
$$a\cdot PA\leq b\cdot PB+c\cdot PC\leq\sqrt{b^2+c^2}\sqrt{PB^2+PC^2}\leq a\sqrt{PB^2+PC^2},$$

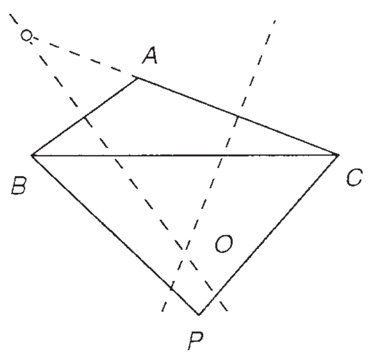
where the second inequality is by Cauchy-Schwarz. This implies $PA^2\leq PB^2+PC^2$, in contradiction with the facts that PA, PB, and PC are the sides of an obtuse triangle and $PA>\max\{PB,PC\}$.

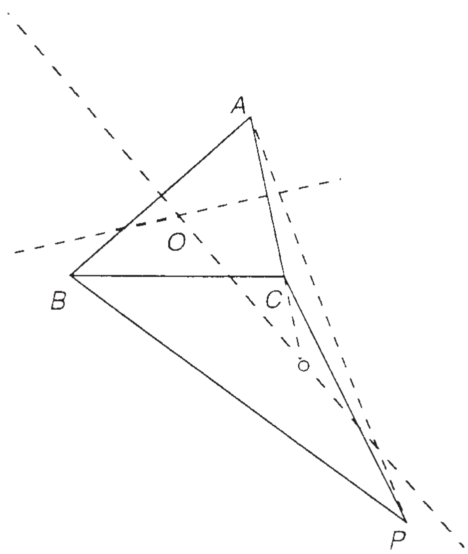
We present two arguments to prove our claim.

- *First argument* Without loss of generality, we may assume that A, B, and C are in counterclockwise order. Let line ℓ_1 and ℓ_2 be the perpendicular bisectors of segments AB and AC, respectively. Then ℓ_1 and ℓ_2 meet at O, the circumcenter of triangle ABC. Lines ℓ_1 and ℓ_2 cut the plane into four regions and A is in the interior of one of these regions. Since $PA>PB$ and $PA>PC$, P must be in the interior of the region that opposes A. Since $\angle BAC$ is not acute, ray AC does not meet ℓ_1 and ray AB does not meet ℓ_2. Hence B and C must lie in the interiors of the regions adjacent to A. Let $\mathcal{R}_X$ denote the region containing X. Then $\mathcal{R}_A$, $\mathcal{R}_B$, $\mathcal{R}_P$, and $\mathcal{R}_C$ are the four regions in counterclockwise order. Since $\angle BAC \geq 90°$, either O is on side BC or O and A are on opposite sides of line BC. In either case P and A are on opposite sides of line BC. Also, since ray AB does not meet ℓ_2 and ray AC does not meet ℓ_1, it follows that $\mathcal{R}_P$ is entirely in the interior of $\angle BAC$. Hence B and C are on opposite sides of AP. Therefore $ABPC$ is convex.









- *Second argument* Since $PA > PB$ and $PA > PC$, A cannot be inside or on the sides of triangle PBC. Since $PA > PB$, we have $\angle ABP > \angle BAP$ and hence $\angle BAC \geq 90° > \angle BAP$. Hence C cannot be inside or on the sides of triangle BAP. Symmetrically, B cannot be inside or on the sides of triangle CAP. Finally since $\angle ABP > \angle BAP$ and $\angle ACP > \angle CAP$, we have

$$\angle ABP + \angle ACP > \angle BAC \geq 90° \geq \angle ABC + \angle ACB.$$

Therefore P cannot be in inside or on the sides of triangle ABC. Since this covers all four cases, $ABPC$ is convex.

5. Let S be a set of integers (not necessarily positive) such that

(A) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

(B) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

Prove that S is the set of all integers.

Solution: In the solution below we use the expression *S is stable under $x \mapsto f(x)$* to mean that if x belongs to S then $f(x)$ also belongs to S. If $c, d \in S$, then by (B), S is stable under $x \mapsto c^2 - x$ and $x \mapsto d^2 - x$, hence stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly S is stable under $x \mapsto x + (d^2 - c^2)$. Hence S is stable under $x \mapsto x + n$ and $x \mapsto x - n$ whenever n is an integer linear combination of numbers of the form $c^2 - d^2$ with $c, d \in S$. In particular, this holds for $n = m$, where $m = \gcd\{c^2 - d^2 : c, d \in S\}$.

Since $S \neq \emptyset$ by (A), it suffices to prove that $m = 1$. For the sake of contradiction, assume that $m \neq 1$. Let p be a prime dividing m. Then $c^2 - d^2 \equiv 0 \pmod p$ for all $c, d \in S$. In other words, for each $c, d \in S$, either $d \equiv c \pmod p$ or $d \equiv -c \pmod p$. Given $c \in S$, $c^2 - c \in S$ by (B), so $c^2 - c \equiv c \pmod p$ or $c^2 - c \equiv -c \pmod p$. Hence

$$\text{For each } c \in S, \text{ either } c \equiv 0 \pmod p \text{ or } c \equiv 2 \pmod p. \qquad (*)$$

By (A), there exist some a and b in S such that $\gcd(a, b) = 1$, that is, at least one of a or b cannot be divisible by p. Denote such an element of S by α; thus, $\alpha \not\equiv 0 \pmod p$. Similarly, by (A), $\gcd(a-2, b-2) = 1$, so p cannot divide both $a - 2$ and $b - 2$. Thus, there is an element of S, call it β, such that $\beta \not\equiv 2 \pmod p$. By $(*)$, $\alpha \equiv 2 \pmod p$ and $\beta \equiv 0 \pmod p$. By (B), $\beta^2 - \alpha \in S$. Taking $c = \beta^2 - \alpha$ in $(*)$ yields either $-2 \equiv 0 \pmod p$ or $-2 \equiv 2 \pmod p$, so $p = 2$. Now $(*)$ says that all elements of S are even, contradicting (A). Hence our assumption is false and S is the set of all integers.

6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

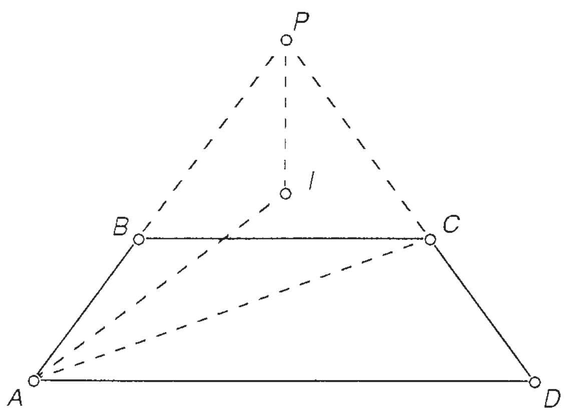
Solution: We label each upper case point with the corresponding lower case letter as its assigned number. The key step is the following lemma.

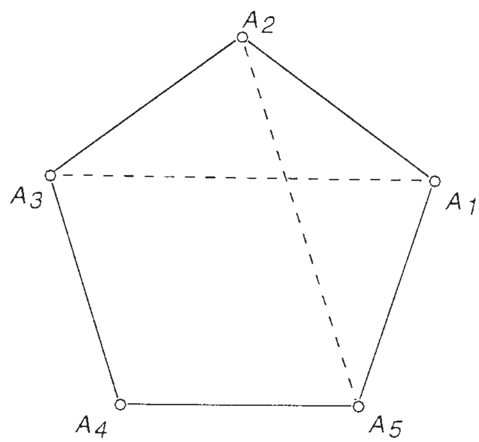
Lemma *If $ABCD$ is an isosceles trapezoid, then $a + c = b + d$.*

Proof: Assume without loss of generality that $BC \parallel AD$, and that rays AB and DC meet at P. Let I be the incenter of triangle PAC, and let line ℓ bisect $\angle APD$. Then I is on ℓ, so reflecting everything across line ℓ shows that I is also the incenter of ttriangle PDB. Therefore,

$$\frac{p+a+c}{3} = i = \frac{p+b+d}{3}.$$

Hence $a + c = b + d$, as desired. ■





For any two distinct points A_1 and A_2 in the plane, we construct a regular pentagon $A_1A_2A_3A_4A_5$. Applying the lemma to isosceles trapezoids $A_1A_3A_4A_5$ and $A_2A_3A_4A_5$ yields

$$a_1 + a_4 = a_3 + a_5 \quad \text{and} \quad a_2 + a_4 = a_3 + a_5.$$

Hence $a_1 = a_2$. Since A_1 and A_2 were arbitrary, all points in the plane are assigned the same number.

USAMO 2001 Solution Notes

Compiled by Evan Chen

April 30, 2020

This is an compilation of solutions for the 2001 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Find with proof the smallest possible n.

2. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

3. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Show that

$$0 \le ab + bc + ca - abc \le 2.$$

4. Let ABC be a triangle and P any point such that PA, PB, PC are the sides of an obtuse triangle, with PA the longest side. Prove that $\angle BAC$ is acute.

5. Let $S \subseteq \mathbb{Z}$ be such that:

(a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

(b) if x and y are element of S (possibly equal), then $x^2 - y$ also belongs to S.

Prove that $S = \mathbb{Z}$.

6. Each point in the plane is assigned a real number. Suppose that for any nondegenerate triangle, the number at its incenter is the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are equal to each other.

§1 USAMO 2001/1

Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Find with proof the smallest possible n.

The answer is $n = 23$. Shown below is a construction using that many colors, which we call $\{1, 2, \dots, 15, a, \dots, f, X, Y\}$.

$$\begin{bmatrix} X & X & X & 1 & 2 & 3 & 4 & 5 \\ 1 & 6 & 11 & 6 & 7 & 8 & 9 & 10 \\ 2 & 7 & 12 & 11 & 12 & 13 & 14 & 15 \\ 3 & 8 & 13 & Y & Y & Y & a & b \\ 4 & 9 & 14 & a & c & e & c & d \\ 5 & 10 & 15 & b & d & f & e & f \end{bmatrix}$$

We present now two proofs that $n = 23$ is best possible. I think the first is more motivated — it will actually show us how we could come up with the example above.

First solution (hands-on) We say a color x is *overrated* if it is used at least three times. First we make the following smoothing argument.

Claim — Suppose some box contains a ball of overrated color x plus a ball of color y used only once. Then we can change one ball of color x to color y while preserving all the conditions.

Proof. Obvious. (Though the color x could cease to be overrated after this operation.) □

By applying this operation as many times as possible, we arrive at a situation in which whenever we have a box with an overrated color, the other colors in the box are used twice or more.

Assume now $n \le 23$ and the assumption; we will show the equality case must of the form we gave. Since there are a total of 48 balls, at least two colors are overrated. Let X be an overrated color and take three boxes where it appears. Then there are 15 more distinct colors, say $\{1, \dots, 15\}$ lying in those boxes. Each of them must appear at least once more, so we arrive at the situation

$$\begin{bmatrix} X & X & X & 1 & 2 & 3 & 4 & 5 \\ 1 & 6 & 11 & 6 & 7 & 8 & 9 & 10 \\ 2 & 7 & 12 & 11 & 12 & 13 & 14 & 15 \\ 3 & 8 & 13 & & & & & \\ 4 & 9 & 14 & & & & & \\ 5 & 10 & 15 & & & & & \end{bmatrix}$$

up to harmless permutation of the color names. Now, note that none of these 15 colors can reappear. So it remains to fill up the last five boxes.

Now, there is at least one more overrated color, distinct from any we have seen; call it Y. In the three boxes Y appears in, there must be six new colors, and this gives the lower bound $n \ge 1 + 15 + 1 + 6 = 23$ which we sought, with equality occurring as we saw above.

Remark (Partial progresses)**.** The fact that $\binom{16}{2} = 120 = 8\binom{6}{2}$ (suggesting the bound $n \ge 16$) is misleading and not that helpful.

There is a simple argument showing that n should be much larger than 16. Imagine opening the boxes in any order. The first box must contain six new colors. The second box must contain five new colors, and so on; thus $n \ge 6+5+4+3+2+1 = 21$. This is sharp for seven boxes, as the example below shows.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 7 & 7 & 8 & 9 & 10 & 11 \\ 3 & 8 & 12 & 12 & 13 & 14 & 15 \\ 4 & 9 & 13 & 16 & 16 & 17 & 18 \\ 5 & 10 & 14 & 17 & 19 & 19 & 20 \\ 6 & 11 & 15 & 18 & 20 & 21 & 21 \end{bmatrix}$$

However, one cannot add an eight box, suggesting the answer should be a little larger than 21. One possible eight box is $\{1, 12, 19, a, b, c\}$ which gives $n \le 24$; but the true answer is a little trickier.

Second solution (slick) Here is a short proof from the official solutions of the bound. Consider the 8×6 grid of colors as before. For each ball b, count the number of times n_b its color is used, and write the fraction $\frac{1}{n_b}$.

On the one hand, we should have

$$n = \sum_{\text{all 48 balls } b} \frac{1}{n_b}.$$

On the other hand, for any given box B, we have $\sum_{b \in B} (n_b - 1) \le 7$, as among the other seven boxes at most one color from B appears. Therefore, $\sum_{b \in B} n_b \le 13$, and a smoothing argument this implies

$$\sum_{b \in B} \frac{1}{n_b} \ge \frac{1}{3} \cdot 1 + \frac{1}{2} \cdot 5 = \frac{17}{6}.$$

Thus, $n \ge 8 \cdot \frac{17}{6} = 22.66\dots$, so $n \ge 23$.

§2 USAMO 2001/2

Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

We have that P is the Nagel point

$$P = (s-a : s-b : s-c) .$$

Therefore,

$$\frac{PD_2}{AD_2} = \frac{s-a}{(s-a)+(s-b)+(s-c)} = \frac{s-a}{s}.$$

Meanwhile, Q is the antipode of D_1. The classical homothety at A mapping Q to D_1 (by mapping the incircle to the A-excircle) has ratio $\frac{s-a}{s}$ as well (by considering the length of the tangents from A), so we are done.

§3 USAMO 2001/3

Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Show that

$$0 \le ab + bc + ca - abc \le 2.$$

The left-hand side of the inequality is trivial; just note that $\min\{a, b, c\} \le 1$. Hence, we focus on the right side. We use Lagrange Multipliers.

Define

$$U = \left\{(a, b, c) \mid a, b, c > 0 \text{ and } a^2 + b^2 + c^2 < 1000\right\}.$$

This is an intersection of open sets, so it is open. Its closure is

$$\overline{U} = \left\{(a, b, c) \mid a, b, c \ge 0 \text{ and } a^2 + b^2 + c^2 \le 1000\right\}.$$

Hence the constraint set

$$\overline{S} = \left\{\mathbf{x} \in \overline{U} : g(\overline{x}) = 4\right\}$$

is compact, where $g(a, b, c) = a^2 + b^2 + c^2 + abc$.

Define

$$f(a, b, c) = a^2 + b^2 + c^2 + ab + bc + ca.$$

It's equivalent to show that $f \le 6$ subject to g. Over $\overline{S}$, it must achieve a global maximum. Now we consider two cases.

If $\mathbf{x}$ lies on the boundary, that means one of the components is zero (since $a^2+b^2+c^2 = 1000$ is clearly impossible). WLOG $c = 0$, then we wish to show $a^2 + b^2 + ab \le 6$ for $a^2 + b^2 = 4$, which is trivial.

Now for the interior U, we may use the method of Lagrange Multipliers. Consider a local maximum $\mathbf{x} \in U$. Compute

$$\nabla f = \langle 2a + b + c, 2b + c + a, 2c + a + b\rangle$$

and

$$\nabla g = \langle 2a + bc, 2b + ca, 2c + ab\rangle.$$

Of course, $\nabla g \ne \mathbf{0}$ everywhere, so introducing our multiplier yields

$$\langle 2a + b + c, a + 2b + c, a + b + 2c\rangle = \lambda \langle 2a + bc, 2b + ca, 2c + ab\rangle.$$

Note that $\lambda \ne 0$ since $a, b, c > 0$. Subtracting $2a + b + c = \lambda(2a + bc)$ from $a + 2b + c = \lambda(2b + ca)$ implies that

$$(a - b)([2\lambda - 1] - \lambda c) = 0.$$

We can derive similar equations for the others. Hence, we have three cases.

1. If $a = b = c$, then $a = b = c = 1$, and this satisfies $f(1, 1, 1) \le 6$.
2. If a, b, c are pairwise distinct, then we derive $a = b = c = 2 - \lambda^{-1}$, contradiction.
3. Now suppose that $a = b \ne c$.

   Meanwhile, the constraint (with $a = b$) reads

$$\begin{aligned} a^2 + b^2 + c^2 + abc = 4 &\iff c^2 + a^2c + (2a^2 - 4) = 0 \\ &\iff (c + 2)(c - (2 - a^2)) = 0 \end{aligned}$$

which since $c > 0$ gives $c = 2 - a^2$.

Noah Walsh points out that at this point, we don't need to calculate the critical point; we just directly substitute $a = b$ and $c = 2 - a^2$ into the desired inequality:

$$a^2 + 2a(2-a)^2 - a^2(2-a)^2 = 2 - (a-1)^2(a^2 - 4a + 2) \le 0.$$

So any point here satisfies the inequality anyways.

Remark. It can actually be shown that the critical point in the third case we skipped is pretty close: it is given by

$$a = b = \frac{1+\sqrt{17}}{4} \quad c = \frac{1}{8}\left(7 - \sqrt{17}\right).$$

This satisfies

$$f(a,b,c) = 3a^2 + 2ac + c^2 = \frac{1}{32}\left(121 + 17\sqrt{17}\right) \approx 5.97165$$

which is just a bit less than 6.

Remark. Equality holds for the upper bound if $(a,b,c) = (1,1,1)$ or $(a,b,c) = (\sqrt{2}, \sqrt{2}, 0)$ and permutations. The lower bound is achieved if $(a,b,c) = (2,0,0)$ and permutations.

§4 USAMO 2001/4

Let ABC be a triangle and P any point such that PA, PB, PC are the sides of an obtuse triangle, with PA the longest side. Prove that $\angle BAC$ is acute.

Using Ptolemy's inequality and Cauchy-Schwarz,

$$\begin{aligned} PA \cdot BC &\le PB \cdot AC + PC \cdot AB \\ &\le \sqrt{(PB^2 + PC^2)(AB^2 + AC^2)} \\ &< \sqrt{PA^2 \cdot (AB^2 + AC)^2} = PA \cdot \sqrt{AB^2 + AC^2} \end{aligned}$$

meaning $BC^2 < AB^2 + AC^2$, so $\angle BAC$ is acute.

§5 USAMO 2001/5

Let $S \subseteq \mathbb{Z}$ be such that:

(a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

(b) if x and y are element of S (possibly equal), then $x^2 - y$ also belongs to S.

Prove that $S = \mathbb{Z}$.

Call an integer $d > 0$ *shifty* if $S = S + d$ (meaning S is invariant under shifting by d). First, note that if $u, v \in S$, then for any $x \in S$,

$$v^2 - (u^2 - x) = (v^2 - u^2) + x \in S.$$

Since we can easily check that $|S| > 1$ and $S \neq \{n, -n\}$ we conclude exists a shifty integer.

We claim 1 is shifty, which implies the problem. Assume for contradiction not that 1 is not shifty. Then for GCD reasons the set of shifty integers must be $d\mathbb{Z}$ for some $d \geq 2$.

Claim — We have $S \subseteq \left\{x : x^2 \equiv m \pmod{d}\right\}$ for some fixed m.

Proof. Otherwise if we take any $p, q \in S$ with distinct squares modulo d, then $q^2 - p^2 \not\equiv 0 \pmod{d}$ is shifty, which is impossible. □

Now take $a, b \in S$ as in (a). In that case we need to have

$$a^2 \equiv b^2 \equiv (a^2 - a)^2 \equiv (b^2 - b)^2 \pmod{d}.$$

Passing to a prime $p \mid d$, we have the following:

- Since $a^2 \equiv (a^2 - a)^2 \pmod{p}$ or equivalently $a^3(a-2) \equiv 0 \pmod{p}$, either $a \equiv 0 \pmod{p}$ or $a \equiv 2 \pmod{p}$.
- Similarly, either $b \equiv 0 \pmod{p}$ or $b \equiv 2 \pmod{p}$.
- Since $a^2 \equiv b^2 \pmod{p}$, or $a \equiv \pm b \pmod{p}$, we find either $a \equiv b \equiv 0 \pmod{p}$ or $a \equiv b \equiv 2 \pmod{p}$ (even if $p = 2$).

This is a contradiction.

Remark. The condition (a) cannot be dropped, since otherwise we may take $S = \{2 \pmod{p}\}$ or $S = \{0 \pmod{p}\}$, say.

§6 USAMO 2001/6, proposed by Bjorn Poonen

Each point in the plane is assigned a real number. Suppose that for any nondegenerate triangle, the number at its incenter is the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are equal to each other.

First, we claim that in an isosceles trapezoid $ABCD$ we have $a + c = b + d$. Indeed, suppose WLOG that rays BA and CD meet at X. Then triangles XAC and XBD share an incircle, proving the claim.

Now, given any two points A and B, construct regular pentagon $ABCDE$. We have $a + c = b + d = c + e = d + a = e + b$, so $a = b = c = d = e$.

31^{st} United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part I 1 p.m. - 5:30 p.m.

May 3, 2002

1. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

   (a) the union of any two white subsets is white;

   (b) the union of any two black subsets is black;

   (c) there are exactly N white subsets.

2. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

   where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

31^{st} United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part II 1 p.m. - 5:30 p.m.

May 4, 2002

4. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

31^{st} United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part I 1 p.m. - 5:30 p.m.

May 3, 2002

1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:

   (a) the union of any two red subsets is white:

   (b) the union of any two blue subsets is black;

   (c) there are exactly N red subsets.

First Solution: We prove that this can be done for any n-element set, where n is an positive integer, $S_n = \{1, 2, \ldots, n\}$ and integer N with $0 \le N \le 2^n$.

We induct on n. The base case $n = 1$ is trivial. Assume that the desired coloring can be done to the subsets of set $S_n = \{1, 2, \ldots, n\}$ and integer N_n with $0 \le N \le 2^n$. We show that there is a desired coloring for set $S_{n+1} = \{1, 2, \ldots, n, n+1\}$ and integer N with $0 \le N_{n+1} \le 2^{n+1}$. We consider the following cases.

(i) $0 \le N_{n+1} \le 2^n$. Applying the induction hypothesis to S_n and $N_n = N_{n+1}$,we geet a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contains element $n+1$, we color all of them blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfying conditions (a), (b), (c).

(ii) $2^n + 1 \le N_{n+1} \le 2^{n+1}$. Applying the induction hypothesis to S_n and $N_n = 2^{n+1} - N_{n+1}$,we geet a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contains element $n+1$, we color all of them blue. Finally, we switch the color of each subset: if it is blue now, we recolor it red: it is red now, we recolor it blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfying conditions (a), (b), (c).

Thus our induction is complete.

Second Solution: If $N = 0$, we color every subset black; if $N = 2^{2002}$, we color every subset white. Now suppose neither of these holds. We may assume that $S = \{0, 1, 2, \ldots, 2001\}$. Write N in binary representation:

$$N = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k},$$

where the a_i are all distinct; then each a_i is an element of S. Color each a_i red, and color all the other elements of S blue. Now declare each nonempty subset of S to be the color of its largest element, and color the empty subset blue. If T, U are any two nonempty subsets of S, then the largest element of $T \cup U$ equals the largest element of T or the largest element of U, and if T is empty, then $T \cup U = U$. Statements (a) and (b) readily follow from this. To verify (c), notice that, for each i, there are 2^{a_i} subsets of S whose largest element is a_i (obtained by taking a_i in combination with any of the elements $0, 1, \ldots, a_i - 1$). If we sum over all i, each red subset is counted exactly once, and we get $2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} = N$ red subsets.

2. Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2 \cot \frac{B}{2}\right)^2 + \left(3 \cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

First Solution: For simplification, let $u = \cot \frac{A}{2}$, $v = \cot \frac{B}{2}$, $w = \cot \frac{C}{2}$. We start with a few basic facts.

- *Fact 1.* Let $[\mathcal{R}]$ denote the area region $\mathcal{R}$. Then

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = rs.$$

The first equality is the **Heron's formula**. The second equality follows from $[ABC] = [AIB] + [BIC] + [CIA] = rs$, where I is the incenter of triangle ABC.

- *Fact 2.* We have

$$u = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

Let ω be the **excircle** of triangle ABC opposite A, and let I_A be its center. Circle ω is tangent to side BC, rays AB and AC and X, Y, Z, respectively. By equal tangents, $AY = AZ$, $BX = BY$ and $CX = CZ$. Hence $AX = AY = s$. Then

$$[ABC] = [ABI_A] + [ACI_A] - [BCI_A] = \frac{r_a(b+c-a)}{2} = r_a(s-a).$$

where r_a is the radius of circle ω. Combining with the Heron's formula, we obtain

$$\sqrt{s(s-a)(s-b)(s-c)} = r_a(s-a),$$

or, $r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$. On the other hand, in right triangle AI_AY,

$$u = \cot \frac{A}{2} = \frac{AY}{YI} = \frac{s}{r_a}.$$

Putting the above equalities together gives

$$u = \frac{s}{r_a} = \frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

Likewise, we have

$$v = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} \quad \text{and} \quad w = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

- *Fact 3.* From fact 2, we obtain

$$\begin{aligned} u+v+w &= \frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{(s-a)(s-b)(s-c)}} \\ &= \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = uvw \end{aligned}$$

From fact 1, we obtain

$$\begin{aligned} \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} &= \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \frac{s^2}{[ABC]} = \frac{s^2}{rs} = \frac{s}{r}. \end{aligned}$$

Hence,

$$uvw = u+v+w = \frac{s}{r}. \tag{1}$$

By (1), and by noticing that $2^2+3^2+6^2=7^2$, we can rewrite the given relation as

$$(6^2+3^2+2^2)[u^2+(2v)^2+(3w)^2] = (6u+6v+6w)^2.$$

This means that we have equality in the Chauchy-Schwartz inequality. It follows that

$$\frac{u}{6} = \frac{2v}{3} = \frac{3w}{2},$$

or,

$$u = 36k, \quad v = 9k, \quad w = 4k,$$

for some positive real number k. Pluging these back into (1) gives $k = \frac{7}{36}$, and consequently, $u = 7$, $v = \frac{7}{4}$, and $w = \frac{7}{9}$. Hence by the **Double angle formulas**, $\sin A = \frac{7}{25}$, $\sin B = \frac{56}{65}$, and $\sin C = \frac{63}{65}$, or,

$$\sin A = \frac{13}{\frac{325}{7}}, \quad \sin B = \frac{40}{\frac{325}{7}}, \quad \sin C = \frac{45}{\frac{325}{7}}.$$

By the **Extanded law of sines**, triangle ABC is similar to triangle T with the side lengths 13, 40, and 45. (The circumradius of T is $\frac{325}{7}$.)

Second Solution: Let D be the point of tangency of the incircle of triangle ABC and side AB. Then $AI = r$ and $AE = s - a$, where I is the incenter of triangle ABC. Hence $u = \frac{AE}{AI} = \frac{s-a}{r}$. Likewise, $v = \frac{s-b}{r}$ and $w = \frac{s-c}{r}$. Since

$$\frac{s}{r} = \frac{(s-a)+(s-b)+(s-c)}{r} = u + v + w,$$

we can rewrite the given relation as

$$49[u^2 + 4v^2 + 9w^2] = 36(u + v + w)^2.$$

Exapnding the last equality and cancelling the like terms, we obtain

$$13u^2 + 160v^2 + 405w^2 - 72(uv + vw + wu) = 0,$$

or

$$(3u - 12v)^2 + (4v - 9w)^2 + (18w - 2u)^2 = 0.$$

Therefore $u : v : w = 1 : 4 : 9$.

By the **Addition formula**, we obtain

$$\begin{aligned}\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} &= \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A+B}{2}} + \cot\frac{C}{2} \\ &= \cot\frac{C}{2}\left(\cot\frac{A}{2}\cot\frac{B}{2} - 1\right) + \cot\frac{C}{2} \\ &= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},\end{aligned}$$

or, $u + v + w = uvw$. The rest is the same as the last part of the first solution.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Solution: Let $p(x)$ be monic real polynomial of degree n. If $n = 1$, then $p(x) = x + a$ for some real number a. It is easy to see that $p(x)$ is the average of $x + 2a$ and x, each of which has 1 real root. Now we assume that $n > 1$. Let polynomial

$$g(x) = (x-2)(x-4)\cdots(x-2(n-1)).$$

The degree of $g(x)$ is $n - 1$. Consider the polynomials

$$q(x) = x^n - kg(x) \quad \text{and} \quad r(x) = 2p(x) - q(x) = 2p(x) - x^n + kg(x).$$

We will show that for large enough k these two polynomials have n real roots. Since they are monic and their average is clearly $p(x)$, this will solve the problem.

Consider the values of polynomial $g(x)$ at n points $x = 1, 3, 5, \ldots, 2n-1$. These values alternate in sign and are at least 1 (since at most two of the factors have magnitude 1 and the others have magnitude at least 2). On the other hand, there is a constant $c > 0$ such that for $0 \leq x \leq n$, we have $|x^n| < c$ and $|2p(x) - x^n| < c$. Take $k > c$. Then we see that $q(x)$ and $r(x)$ evaluated at n points $x = 1, 3, 5, \ldots, 2n-1$ alternate in sign. Thus polynomials $p(x)$ and $r(x)$ each has at least $n-1$ real roots. How ever since they are polynomials of degree n, they must then each have n real roots, as desired.

31^{st} United States of America Mathematical Olympiad

Cambridge, Massachusetts

Part II   1 p.m. - 5:30 p.m.

May 4, 2002

4. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

Solution: Setting $x = y = 0$ in the given condition yields $f(0) = 0$. Since

$$-xf(-x) - yf(y) = f[(-x)^2 - y^2] = f(x^2 - y^2) = xf(x) - yf(y),$$

we have $f(x) = -f(x)$ for $x \neq 0$. Hence $f(x)$ is odd. From now on, we assume $x, y \geq 0$. Setting $y = 0$ in the given condition yields $f(x^2) = xf(x)$. Hence $f(x^2 - y^2) = f(x^2) - f(y^2)$, or, $f(x^2) = f(x^2 - y^2) + f(y^2)$. Since for $x \geq 0$ there is a unique $t \geq 0$ such that $t^2 = x$, it follows that

$$f(x) = f(x - y) + f(y) \tag{1}$$

Setting $x = 2t$ and $y = t$ in (1) gives

$$f(2t) = 2f(t). \tag{2}$$

Setting $x = t + 1$ and $y = t$ in the given condition yields

$$f(2t + 1) = (t + 1)f(t + 1) - tf(t). \tag{3}$$

By (2) and by setting $x = 2t + 1$ and $y = 1$ in (1), the left-hand side of (3) becomes

$$f(2t + 1) = f(2t) + f(1) = 2f(t) + f(1). \tag{4}$$

On the other hand, by setting $x = t + 1$ and $y = 1$ in (1), the right-hand side of (3) reads

$$(t + 1)f(t + 1) - tf(t) = (t + 1)[f(t) + f(1)] - tf(t) = f(t) + (t + 1)f(1). \tag{5}$$

Putting (3), (4), and (5) together leads to $2f(t) + f(1) = f(t) + (t + 1)f(1)$, or,

$$f(t) = tf(1)$$

for $t \geq 0$. Recall that $f(x)$ is odd, we conclude that $f(-t) = -f(t) = -tf(1)$ for $t \geq 0$. Hence $f(x) = kx$ for all x, where $k = f(1)$ is a constant. It is not difficult to see that all such functions indeed satisfy the conditions of the problem.

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

First Solution: We write $a \leftrightarrow b$ if the desired sequence exists. Note that for positive integer n with $n \geq 3$, $n \leftrightarrow 2n$ as the sequence

$$n_1 = n,\ n_2 = n(n-1),\ n_3 = n(n-1)(n-2),\ n_4 = n(n-2),\ n_5 = 2n$$

satisfies the conditions of the problem. For positive integer $n \geq 4$, $n' = (n-1)(n-2) \geq 3$, hence $n' \leftrightarrow 2n'$ by the above argument. It follows that $n \leftrightarrow n-1$ for $n \geq 4$ by $n' \leftrightarrow 2n'$ and by the sequences

$$n_1 = n,\ n_2 = n(n-1),\ n_3 = n(n-1)(n-2),\ n_4 = n(n-1)(n-2)(n-3),$$
$$n_5 = 2(n-1)(n-2) = 2n'$$

and $n_1' = n' = (n-1)(n-2)$, $n_2' = n-1$. Iterating this, we connect all integers larger than 2.

Second Solution: We write $a \leftrightarrow b$ if the desired sequence exists. Note that this relation is symmetric ($a \leftrightarrow b$ implies $b \leftrightarrow a$) and transitive ($a \leftrightarrow b$, $b \leftrightarrow c$ imply $a \leftrightarrow c$). Our crucial observation will be the following: If $d > 2$ and n is a multiple of d, then $n \leftrightarrow (d-1)n$. Indeed, $n + (d-1)n = dn \mid n^2 \mid (d-1)n^2 = n \cdot (d-1)n$.

Let us call a positive integer k *safe* if $n \leftrightarrow kn$ for all $n > 2$. Notice by transitivity that any product of safe numbers is safe. Now, we claim that 2 is safe. To prove this, define $f(n)$, for $n > 2$, to be the smallest divisor of n which is greater than 2. We show that $n \leftrightarrow 2n$ by strong induction on $f(n)$. In case $f(n) = 3$, we immediately have $n \leftrightarrow 2n$ by our earlier observation. Otherwise, notice that $f(n) - 1$ is a divisor of $(f(n) - 1)n$ which is greater than 2 and less than $f(n)$; thus $f((f(n) - 1)n) < f(n)$, and the induction hypothesis gives $(f(n) - 1)n \leftrightarrow 2(f(n) - 1)n$. We also have $n \leftrightarrow (f(n) - 1)n$ (by our earlier observation) and $2(f(n) - 1)n \leftrightarrow 2n$ (by the same observation, since $f(n) \mid n \mid 2n$). Thus, by transitivity, $n \leftrightarrow 2n$. This completes the induction step and proves the claim.

Next, we show that any prime p is safe, again by strong induction. The base case $p = 2$ has already been done. If p is an odd prime, then $p + 1$ is a product of primes strictly less than p, which are safe by the induction hypothesis; hence, $p + 1$ is safe. Thus, for any $n > 2$,

$$n \leftrightarrow (p+1)n \leftrightarrow p(p+1)n \leftrightarrow pn.$$

This completes the induction step. Thus, all primes are safe, and hence every integer ≥ 2 is safe. In particular, our given numbers a, b are safe, so we have $a \leftrightarrow ab \leftrightarrow b$, as needed.

6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations

separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

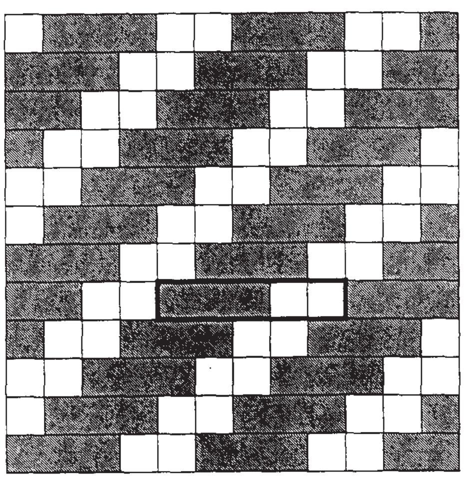
$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

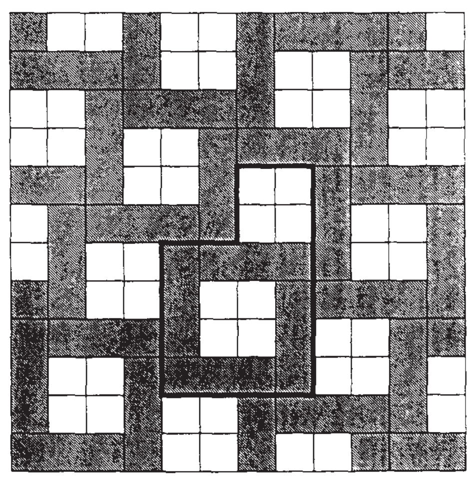
for all $n > 0$.

Solution: The upper bound requires an example of a set of $\frac{1}{5}n^2 + dn$ blocks whose removal makes it impossible to remove any further blocks. It suffices to show that we can tile the plane by tiles containing one block for every five stamps so that no more blocks can be chosen. Two such tilings are shown below with one tile outlined in heavy lines. Given an $n \times n$ section of the tiling take all blocks lying entirely within that section and add as many additional blocks as possible. If the basic tile is contained in an $m+1 \times m+1$ square, then the $n \times n$ section is covered by tiles contained in a concentric $(n+2m) \times (n+2m)$ square. Hence there are at most $\frac{1}{5}(n+2m)^2$ blocks entirely within the section. For an $n \times n$ section of the tiling, there are at most $4n$ blocks which lie partially in and partially out of that section (hence these block contain at most $8n$ stamps in the $n \times n$ square) and each of the additional blocks must contain one of these stamps. Thus there are at most $8n$ additional blocks. Thus there are at most

$$\frac{1}{5}(n+2m)^2 + 8n \leq \frac{1}{5}n^2 + \frac{4m^2+4m+40}{5}n$$

blocks total.





The lower bound requires an argument. Suppose that we have a set of $b(n)$ blocks whose removal makes removing any further blocks impossible.

1) There are $2n(n-2)$ potential blocks of three consecutive stamps in a row or column. Each of these must meet at least one of the $b(n)$ blocks removed. Conversely, each of the $b(n)$ blocks removed meets at most 14 of these potential blocks

(5 oriented the same way, including itself, and 9 oriented the orthogonal way). Therefore $14b(n) \geq 2n(n-2)$ or

$$b(n) \geq \frac{1}{7}n^2 - \frac{2}{7}n.$$

2) Call a stamp used if it belongs to one of the $b(n)$ removed blocks. Consider the $(n-2)^2$ five-stamp crosses centered at each stamp not on an edge of the sheet. Each cross must contain two used stamps. (One stamp not in the center is not enough to prevent another block from being torn out, and it is impossible to use one stamp in the center and use no other stamps in the cross.) In addition, each block not lying along an edge of the sheet lies entirely inside one cross, which thus contains three used stamps. There are at most $4n/3$ of the $b(n)$ blocks lying along the edges, hence there are at least $b(n) - 4n/3$ crosses containing three used stamps.

Now count the number of pairs of a used stamp and a cross containing that stamp, in two ways. First counting block by block, we get $3b(n)$ used stamps, and each used stamp is contained in at most five crosses (exactly five if it is not on an edge), for a total of at most $15b(n)$ pairs. Next, counting cross by cross, each of the $(n-2)^2$ crosses contains at least two used stamps and we have at least $b(n) - 4n/3$ crosses containing three used stamps, for a total of at least $2(n-2)^2 + b(n) - 4n/3$ pairs. Therefore

$$15b(n) \geq 2(n-2)^2 + b(n) - \frac{4n}{3},$$

or

$$b(n) \geq \frac{1}{7}n^2 - \frac{16}{21}n.$$

3) Call a stamp used if it belongs to one of the $b(n)$ removed blocks. Count the number of pairs consisting of a used stamp and an adjacent unused stamp, in two ways.

There are at least $(n-2)^2 - 3b(n)$ unused stamps which are not on an edge. Since no more blocks can be torn out, either the stamp to the left or right and either the stamp above or below such an unused stamp must be used. Thus we have at least $2n^2 - 8n - 6b(n)$ such pairs.

Each block removed is adjacent to at most eight other stamps. However these eight stamps contain two blocks of three consecutive stamps. Hence at most six of these eight stamps can be unused. Thus each of the $b(n)$ block removed is involved in at most six pairs. Thus there are at most $6b(n)$ pairs.

Combining these we have

$$6b(n) \geq 2n^2 - 8n - 6b(n),$$

or

$$b(n) \geq \frac{1}{6}n^2 - \frac{2}{3}n.$$

USAMO 2002 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2002 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:
   (a) the union of any two white subsets is white;
   (b) the union of any two black subsets is black;
   (c) there are exactly N white subsets.

2. Let ABC be a triangle such that
$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$
where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

4. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = xf(x) - yf(y)$$
for all pairs of real numbers x and y.

5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \le i < k$).

6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that
$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$
for all $n > 0$.

§1 USAMO 2002/1

Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

(a) the union of any two white subsets is white;

(b) the union of any two black subsets is black;

(c) there are exactly N white subsets.

We will solve the problem with 2002 replaced by an arbitrary integer $n \ge 0$. In other words, we prove:

Claim — For any nonnegative integers n and N with $0 \le N \le 2^n$, it is possible to color the 2^n subsets of $\{1, \dots, n\}$ black and white satisfying the conditions of the problem.

The proof is by induction on n. When $n = 1$ the problem is easy to do by hand, so this gives us a base case.

For the inductive step, we divide into two cases:

- If $N \le 2^{n-1}$, then we take a coloring of subsets of $\{1, \dots, n-1\}$ with N white sets; then we color the remaining 2^{n-1} sets (which contain n) black.
- If $N > 2^{n-1}$, then we take a coloring of subsets of $\{1, \dots, n-1\}$ with $N - 2^{n-1}$ white sets; then we color the remaining 2^{n-1} sets (which contain n) white.

§2 USAMO 2002/2

Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

Let $x=s-a$, $y=s-b$, $z=s-c$ in the usual fashion, then the equation reads

$$x^2+4y^2+9z^2=\left(\frac{6}{7}(x+y+z)\right)^2.$$

However, by Cauchy-Schwarz, we have

$$\left(1+\tfrac{1}{4}+\tfrac{1}{9}\right)\left(x^2+4y^2+9z^2\right)\ge(x+y+z)^2$$

with equality if and only if $1:\frac{1}{2}:\frac{1}{3}=x:2y:3z$, id est $x:y:z=1:\frac{1}{4}:\frac{1}{9}=36:9:4$. This is equivalent to $y+z:z+x:x+y=13:40:45$.

> **Remark.** You can tell this is not a geometry problem because you eliminate the cotangents right away to get an algebra problem... and then you realize the problem claims that one equation can determine three variables up to scaling, at which point you realize it has to be an inequality (otherwise degrees of freedom don't work). So of course, Cauchy-Schwarz...

§3 USAMO 2002/3

Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

First,

Lemma

If p is a monic polynomial of degree n, and $p(1)p(2) < 0$, $p(2)p(3) < 0$, $\dots$, $p(n-1)p(n) < 0$ then p has n real roots.

Proof. The intermediate value theorem already guarantees the existence of $n-1$ real roots.

The last root is obtained by considering cases on $n \pmod 2$. If n is even, then $p(1)$ and $p(n)$ have opposite sign, while we must have either

$$\lim_{x\to-\infty} p(x) = \lim_{x\to\infty} p(x) = \pm\infty$$

so we get one more root. The n odd case is similar, with $p(1)$ and $p(n)$ now having the same sign, but $\lim_{x\to-\infty} p(x) = -\lim_{x\to\infty} p(x)$ instead. □

Let $f(n)$ be the monic polynomial and let $M > 1000 \max_{t=1,\dots,n} |f(t)| + 1000$. Then we may select reals $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that for each $k = 1, \dots, n$, we have

$$\begin{aligned} a_k + b_k &= 2f(k) \\ (-1)^k a_k &> M \\ (-1)^{k+1} b_k &> M. \end{aligned}$$

We may interpolate monic polynomials g and h through the a_k and b_k (if the a_k, b_k are selected "generically" from each other). Then one can easily check $f = \frac{1}{2}(g+h)$ works.

Remark. This is like Cape Town all over again...

§4 USAMO 2002/4

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

The answer is $f(x) = cx$, $c \in \mathbb{R}$ (these obviously work).

First, by putting $x = 0$ and $y = 0$ respectively we have

$$f(x^2) = xf(x) \quad \text{and} \quad f(-y^2) = -yf(y).$$

From this we deduce that f is odd, in particular $f(0) = 0$. Then, we can rewrite the given as $f(x^2 - y^2) + f(y^2) = f(x^2)$. Combined with the fact that f is odd, we deduce that f is additive (i.e. $f(a + b) = f(a) + f(b)$).

> **Remark** (Philosophy)**.** At this point we have $f(x^2) \equiv xf(x)$ and f additive, and everything we have including the given equation is a direct corollary of these two. So it makes sense to only focus on these two conditions.

Then

$$\begin{aligned} & f((x+1)^2) = (x+1)f(x+1) \\ \implies & f(x^2) + 2f(x) + f(1) = (x+1)f(x) + (x+1)f(1) \end{aligned}$$

which readily gives $f(x) = f(1)x$.

§5 USAMO 2002/5

Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \le i < k$).

Consider a graph G on the vertex set $\{3, 4, \dots\}$ and with edges between v, w if $v + w \mid vw$; the problem is equivalent to showing that G is connected.

First, note that n is connected to $n(n-1)$, $n(n-1)(n-2)$, etc. up to $n!$. But for $n > 2$, $n!$ is connected to $(n+1)!$ too:

- $n! \to (n+1)!$ if n is even
- $n! \to 2n! \to (n+1)!$ if n is odd.

This concludes the problem.

§6 USAMO 2002/6

I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$

for all $n > 0$.

For the lower bound: there are $2n(n-2)$ places one could put a block. Note that each block eliminates at most 14 such places.

For the upper bound, the construction of $\frac{1}{5}$ is easy to build. Here is an illustration of one possible construction for $n = 9$ which generalizes readily, using only vertical blocks.

$$\begin{bmatrix}
A & & E & & I & L & & P & \\
A & & E & G & & L & & P & R \\
A & C & & G & & L & N & & R \\
 & C & & G & J & & N & & R \\
 & C & F & & J & & N & Q & \\
B & & F & & J & M & & Q & \\
B & & F & H & & M & & Q & S \\
B & D & & H & & M & O & & S \\
 & D & & H & K & & O & & S
\end{bmatrix}$$

Actually, for the lower bound, one may improve $1/7$ to $1/6$. Count the number A of pairs of adjacent squares one of which is torn out and the other which is not:

- For every deleted block, there are eight neighboring squares, at least two on each long edge which have been deleted too. Hence $N \le 6b(n)$.
- For every block still alive and not on the border, there are four neighboring squares, and clearly at least two are deleted. Hence $N \ge 2\left((n-2)^2 - 3b(n)\right)$.

Collating these solves the problem.

32^{nd} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM

April 29, 2003

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

3. Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence b such that $t(b) = b$.

32^{nd} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM

April 30, 2003

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

6. A positive integer is written at each vertex of a regular hexagon so that the sum of all numbers written is 2003^{2003}. Bert makes a sequence of moves of the following form: Bert picks a vertex and replaces the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can always make a sequence of moves ending at the position with all six numbers equal to zero.

32^{nd} United States of America Mathematical Olympiad

Proposed Solutions

May 1, 2003

Remark: The general philosophy of this marking scheme follows that of IMO 2002. This scheme encourages *complete solutions*. Partial credits will be given under more strict circumstances. Each solution by students shall be graded from one of the two approaches: (1) from 7 going down (a complete solution with possible minor errors); (2) from 0 going up (a solution missing at least one critical idea.) Most partial credits are not additive. Because there are many results need to be proved progressively in problem 3, most partial credits in this problem are accumulative. Many problems have different approaches. Graders are encouraged to choose the approach that most favorable to students. But the partial credits from different approaches are not additive.

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

   Solution: We proceed by induction. The property is clearly true for $n = 1$. Assume that $N = a_1a_2 \ldots a_n$ is divisible by 5^n and has only odd digits. Consider the numbers

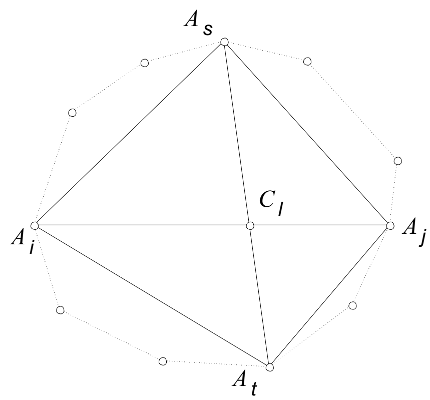
$$\begin{aligned}
N_1 &= 1a_1a_2 \ldots a_n = 1 \cdot 10^n + 5^n M = 5^n(1 \cdot 2^n + M),\\
N_2 &= 3a_1a_2 \ldots a_n = 3 \cdot 10^n + 5^n M = 5^n(3 \cdot 2^n + M),\\
N_3 &= 5a_1a_2 \ldots a_n = 5 \cdot 10^n + 5^n M = 5^n(5 \cdot 2^n + M),\\
N_4 &= 7a_1a_2 \ldots a_n = 7 \cdot 10^n + 5^n M = 5^n(7 \cdot 2^n + M),\\
N_5 &= 9a_1a_2 \ldots a_n = 9 \cdot 10^n + 5^n M = 5^n(9 \cdot 2^n + M).
\end{aligned}$$

   The numbers $1 \cdot 2^n + M, 3 \cdot 2^n + M, 5 \cdot 2^n + M, 7 \cdot 2^n + M, 9 \cdot 2^n + M$ give distinct remainders when divided by 5. Otherwise the difference of some two of them would be a multiple of 5, which is impossible, because 2^n is not a multiple of 5, nor is the difference of any two of the numbers $1, 3, 5, 7, 9$. It follows that one of the numbers N_1, N_2, N_3, N_4, N_5 is divisible by $5^n \cdot 5$, and the induction is complete.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Solution: Let $\mathcal{P} = A_1A_2\ldots A_n$, where n is an integer with $n \geq 3$. The problem is trivial for $n = 3$ because there are no diagonals and thus no dissections. We assume that $n \geq 4$. Our proof is based on the following Lemma.

Lemma 1. *Let $ABCD$ be a convex quadrilateral such that all its sides and diagonals have rational lengths. If segments AC and BD meet at P, then segments AP, BP, CP, DP all have rational lengths.*



It is clear by Lemma 1 that the desired result holds when $\mathcal{P}$ is a convex quadrilateral. Let A_iA_j $(1 \leq i < j \leq n)$ be a diagonal of $\mathcal{P}$. Assume that C_1, C_2, $\ldots$, C_m are the consecutive division points on diagonal A_iA_j (where point C_1 is the closest to vertex A_i and C_m is the closest to A_j). Then the segments $C_\ell C_{\ell+1}$, $1 \leq \ell \leq m-1$, are the sides of all polygons in the dissection. Let C_ℓ be the point where diagonal A_iA_j meets diagonal A_sA_t. Then quadrilateral $A_iA_sA_jA_t$ satisfies the conditions of Lemma 1. Consequently, segments A_iC_ℓ and $C_\ell A_j$ have rational lengths. Therefore, segments $A_iC_1, A_iC_2, \ldots, A_jC_m$ all have rational lengths. Thus, $C_\ell C_{\ell+1} = AC_{\ell+1} - AC_\ell$ is rational. Because i, j, ℓ are arbitrarily chosen, we proved that all sides of all polygons in the dissection are also rational numbers.

Now we present four proofs of Lemma 1 to finish our proof.

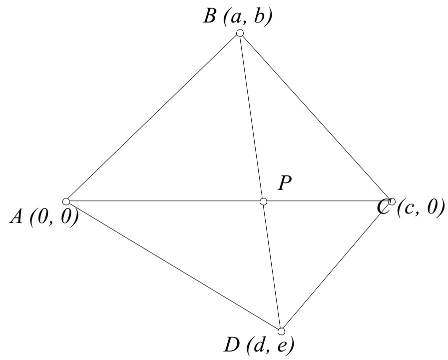
- *First approach* We show only that segment AP is rational, the others being similar. Introduce Cartesian coordinates with $A = (0,0)$ and $C = (c,0)$. Put $B = (a,b)$ and $D = (d,e)$. Then by hypothesis, the numbers

$$AB = \sqrt{a^2+b^2}, \quad AC = c, \quad AD = \sqrt{d^2+e^2},$$
$$BC = \sqrt{(a-c)^2+b^2}, \quad BD = \sqrt{(a-d)^2+(b-e)^2}, \quad CD = \sqrt{(d-c)^2+e^2},$$

are rational. In particular,

$$BC^2 - AB^2 - AC^2 = (a-c)^2 + b^2 - (a^2+b^2) - c^2 = 2ac$$

is rational. Because $c \neq 0$, a is rational. Likewise d is rational.
Now we have that $b^2 = AB^2 - a^2$, $e^2 = AD^2 - d^2$, $(b-e)^2 = BD^2 - (a-d)^2$ are rational, and so that $2be = b^2 + e^2 - (b-e)^2$ is rational. Because quadrilateral $ABCD$ is convex, b and e are nonzero and have opposite sign. Hence $\frac{b}{e} = \frac{2be}{2b^2}$ is rational.



We now calculate

$$P = \left(\frac{bd - ae}{b - e}, 0\right),$$

so

$$AP = \frac{\frac{b}{e} \cdot d - a}{\frac{b}{e} - 1}$$

is rational. ■

- *Second approach*

Note that, for an angle α, if $\cos\alpha$ is rational, then $\sin\alpha = r_\alpha\sqrt{m_\alpha}$ for some rational r and square-free positive integer m (and this expression is unique when r is written in the lowest term). We say two angles α and β with rational cosine are *equivalent* if $m_\alpha = m_\beta$, that is, if $\sin\alpha/\sin\beta$ is rational. We establish the following lemma.

Lemma 2. *Let α and β be two angles.*

(a) If α, β and $\alpha + \beta$ all have rational cosines, then all three are equivalent.

(b) If α and β have rational cosine values and are equivalent, then $\alpha + \beta$ has rational cosine value (and is equivalent to the other two).

(c) If α, β and γ are the angles of a triangle with rational sides, then all three have rational cosine values and are equivalent.

Proof: Assume that $\cos\alpha = s$ and $\cos\beta = t$.

(a) Assume that s and t are rational. By the **Addition formula**, we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad (*)$$

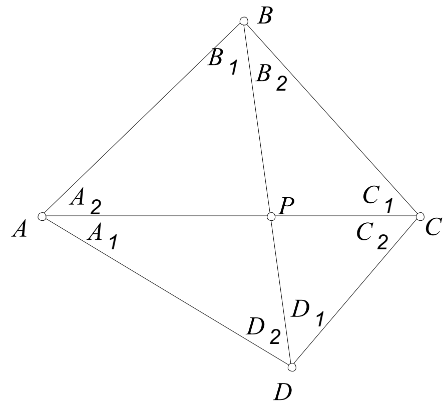
or, $\sin\alpha\sin\beta = st - \cos(\alpha + \beta)$, which is rational by the given conditions. Hence α and β are equivalent. Thus $\sin\alpha = r_a\sqrt{m}$ and $\sin\beta = r_b\sqrt{m}$ for some rational numbers r_a and r_b and some positive square free integer m. By the Addition formula, we have

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (tr_a + sr_b)\sqrt{m},$$

implying that $\alpha + \beta$ is equivalent to both α and β.

(b) By $(*)$, $\cos(\alpha + \beta)$ is rational if s, t are rational and α and β are equivalent. Then by (a), α, β, $\alpha + \beta$ are equivalent.

(c) Applying the **Law of Cosine** to triangle ABC shows that $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are all rational. Note that $\cos\gamma = \cos(180° - \alpha - \beta) = -\cos(\alpha + \beta)$. The desired conclusions follow from (a). ■



We say a triangle *rational* if all its sides are rational. By Lemma 2 (c), all the angles in a rational triangle have rational cosine values and are equivalent to each other. To prove Lemma 1, we set $\angle DAC = A_1$, $\angle CAB = A_2$, $\angle ABD = B_1$, $\angle DBC = B_2$, $\angle BCA = C_1$, $\angle ACD = C_2$, $\angle CDB = D_1$, $\angle BDA = D_2$. Because triangles ABC, ABD, ADC are rational, angles $A_2, A_1 + A_2, A_1$ all have rational cosine values. By Lemma 2 (a), A_1 and A_2 are equivalent. Similarly, we can show that B_1 and B_2, C_1 and C_2, D_1 and D_2 are equivalent. Because triangle ABC is rational, angles A_2 and C_1 are equivalent. There all angles $A_1, A_2, B_1, \ldots, D_2$ have rational cosine values and are equivalent.

Because angles A_2 and B_1 are equivalent, angle $A_2 + B_1$ has rational values and is equivalent to A_2 and B_1. Thus, $\angle APB = 180^\circ - (A_2 + B_1)$ has rational cosine value and is equivalent to A_2 and B_1. Apply the **Law of Sine** to triangle ABP gives

$$\frac{AB}{\sin \angle APB} = \frac{AP}{\sin \angle B_1} = \frac{BP}{\sin \angle A_2},$$

implying that both AP and BP have rational length. Similarly, we can show that both CP and DP has rational length, proving Lemma 1.

- *Third approach* This approach applies the techniques used in the first approach into the second approach. To prove Lemma 1, we set $\angle DAP = A_1$ and $\angle BAP = A_2$. Applying the Law of Cosine to triangle ABC, ABC, ADC shows that angles $A_1, A_2, A_1 + A_2$ all has rational cosine values. By the Addition formula, we have

  $$\sin A_1 \sin A_2 = \cos A_1 \cos A_2 - \cos(A_1 + A_2),$$

  implying that $\sin A_1 \sin A_2$ is rational.

  Thus,

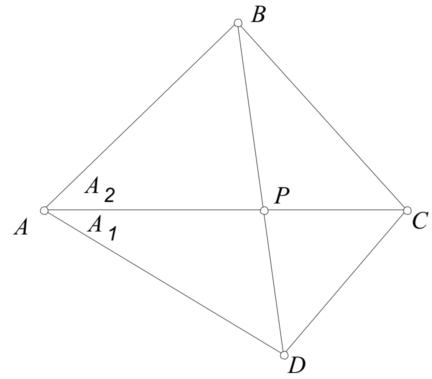
  $$\frac{\sin A_2}{\sin A_1} = \frac{\sin A_2 \sin A_1}{\sin^2 A_1} = \frac{\sin A_2 \sin A_1}{1 - \cos^2 A_1}$$

  is rational.

  Note that the ratio between areas of triangle ADP and ABP is equal to $\frac{PD}{BP}$. Therefore,

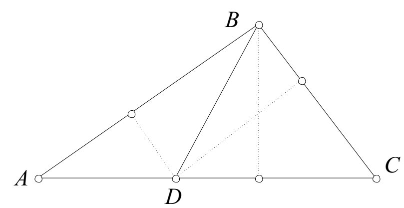
  $$\frac{BP}{PD} = \frac{[ABP]}{[ADP]} = \frac{\frac{1}{2}AB \cdot AP \cdot \sin A_2}{\frac{1}{2}AD \cdot AP \cdot \sin A_1} = \frac{AB}{AD} \cdot \frac{\sin A_2}{\sin A_1},$$

  implying that $\frac{PD}{BP}$ is rational. Because $BP + PD = BD$ is rational, both BP and PD are rational. Similarly, AP and PC are rational, proving Lemma 1.



- *Fourth approach* This approach is based on the following lemma.

  Lemma 3. *Let ABC be a triangle, D be a point on side AC, $\phi_1 = \angle DAB$, $\phi_2 = \angle DBA$, $\phi_3 = \angle DBC$, $\phi_4 = \angle DCB$, $AB = c$, $BC = a$, $AD = x$, and $DC = y$. If the numbers a, c, and $\cos\phi_i$ $(1 \le i \le 4)$ are all rational, then numbers x and y are also rational.*

  Proof: Note that $x + y = AC = c\cos\phi_1 + a\cos\phi_4$ is rational. Hence x is rational if and only if y is rational. Let $BD = z$. Projecting point D onto the lines AB and BC yields

  $$\begin{cases} x\cos\phi_1 + z\cos\phi_2 = c, \\ y\cos\phi_4 + z\cos\phi_3 = a, \end{cases}$$

  or, denoting $c_i = \cos\phi_i$ for $i = 1, 2, 3, 4$,

  $$\begin{cases} c_1x + c_2z = c, \\ c_4y + c_3z = a. \end{cases}$$

  Eliminating z, we get $(c_1c_3)x - (c_2c_4)y = c_3c - c_2a$, which is rational. Hence there exist rational numbers, r_1 and r_2, such that

  $$\begin{cases} (c_1c_3)x - (c_2c_4)y = r_1, \\ x + y = r_2. \end{cases}$$

  We consider two cases.

  - In this case, we assume that the determinant of the above system, $c_1c_3 + c_2c_4$, is not equal to 0, then this system has a unique solution (x, y) in rational numbers.
  - In this case, we assume that the determinant $c_1c_3 + c_2c_4 = 0$, or

    $$\cos\phi_1\cos\phi_3 = -\cos\phi_2\cos\phi_4.$$

    Let's denote $\theta = \angle BDC$, then $\phi_2 = \theta - \phi_1$ and $\phi_3 = 180^\circ - (\theta + \phi_4)$. Then the above equation becomes

    $$\cos\phi_1\cos(\theta + \phi_4) = \cos\phi_4\cos(\theta - \phi_1).$$

by the **Product-to-sum formulas**, we have

$$\cos(\theta + \phi_1 + \phi_4) + \cos(\theta + \phi_4 - \phi_1) = \cos(\theta + \phi_4 - \phi_1) + \cos(\theta - \phi_1 - \phi_4),$$

or

$$\cos(\theta + \phi_1 + \phi_4) = \cos(\theta - \phi_1 - \phi_4).$$

It is possible only if $[\theta + \phi_1 + \phi_4] \pm [\theta - \phi_1 - \phi_4] = 360°$, that is, either $\theta = 180°$ or $\phi_1 + \phi_4 = 180°$, which is impossible because they are angles of triangles.

Thus, the determinant $c_1c_3 + c_2c_4$ is not equal to 0 and x and y are both rational numbers. ■

Now we are ready to prove Lemma 1. Applying the Law of Cosine to triangles ABC, ACD, ABD shows that $\cos\angle BAC$, $\cos\angle CAD$, $\cos\angle ABD$, $\cos\angle ADB$ are all rational. Applying Lemma 1 to triangle ABD shows that both of the segments BP and DP have rational lengths. In exactly the same way, we can show that both of the segments AP and CP have rational lengths.

Note: It's interesting how easy it is to get a gap in the proof of the Lemma 1 by using the core idea of the proof of Lemma 3. Here is an example.

Let us project the intersection point of the diagonals, O, onto the four lines of all sides of the quadrilateral. We get the following 4×4 system of linear equations:

$$\begin{cases} \cos\phi_1\ x + \cos\phi_2 y = a, \\ \cos\phi_3 y + \cos\phi_4 z = b, \\ \cos\phi_5 z + \cos\phi_6 t = c, \\ \cos\phi_7 t + \cos\phi_8 x = d. \end{cases}$$

Using the **Kramer's Rule**, we conclude that all x, y, z, and t must be rational numbers, for all the corresponding determinants are rational. However, this logic works only if the determinant of the system is different from 0.

Unfortunately, there are many geometric configurations for which the determinant of the system vanishes (for example, this occurs for rectangles), and you cannot make a conclusion of rationality of the segments x, y, z, and t. That's why Lemma 2 plays the central role in the solution to this problem.

3. Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

Solution: Note first that the transformed sequence $t(A)$ also satisfies the inequalities $0 \leq t(a_i) \leq i$, for $i = 0, \ldots, n$. Call any integer sequence that satisfies these inequalities an *index bounded sequence.*

We prove now that that $a_i \leq t(a_i)$, for $i = 0, \ldots, n$. Indeed, this is clear if $a_i = 0$. Otherwise, let $x = a_i > 0$ and $y = t(a_i)$. None of the first x consecutive terms $a_0, a_1, \ldots, a_{x-1}$ is greater than $x - 1$ so they are all different from x and precede x (see the diagram below). Thus $y \geq x$, that is, $a_i \leq t(a_i)$, for $i = 0, \ldots, n$.

index	0	1	...	$x-1$	...	i
A	a_0	a_1	...	a_{x-1}	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	...	y

This already shows that the sequences stabilize after finitely many applications of the transformation t, because the value of the index i term in index bounded sequences cannot exceed i. Next we prove that if $a_i = t(a_i)$, for some $i = 0, \ldots, n$, then no further applications of t will ever change the index i term. We consider two cases.

- In this case, we assume that $a_i = t(a_i) = 0$. This means that no term on the left of a_i is different from 0, that is, they are all 0. Therefore the first i terms in $t(A)$ will also be 0 and this repeats (see the diagram below).

index	0	1	...	i
A	0	0	...	0
$t(A)$	0	0	...	0

- In this case, we assume that $a_i = t(a_i) = x > 0$. The first x terms are all different from x. Because $t(a_i) = x$, the terms $a_x, a_{x+1}, \ldots, a_{i-1}$ must then all be equal to x. Consequently, $t(a_j) = x$ for $j = x, \ldots, i - 1$ and further applications of t cannot change the index i term (see the diagram below).

index	0	1	...	$x-1$	x	$x+1$	...	i
A	a_0	a_1	...	a_{x-1}	x	x	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	x	x	...	x

For $0 \leq i \leq n$, the index i entry satisfies the following properties: (i) it takes integer values; (ii) it is bounded above by i; (iii) its value does not decrease under transformation t; and (iv) once it stabilizes under transformation t, it never changes again. This shows that no more than n applications of t lead to a sequence that is stable under the transformation t.

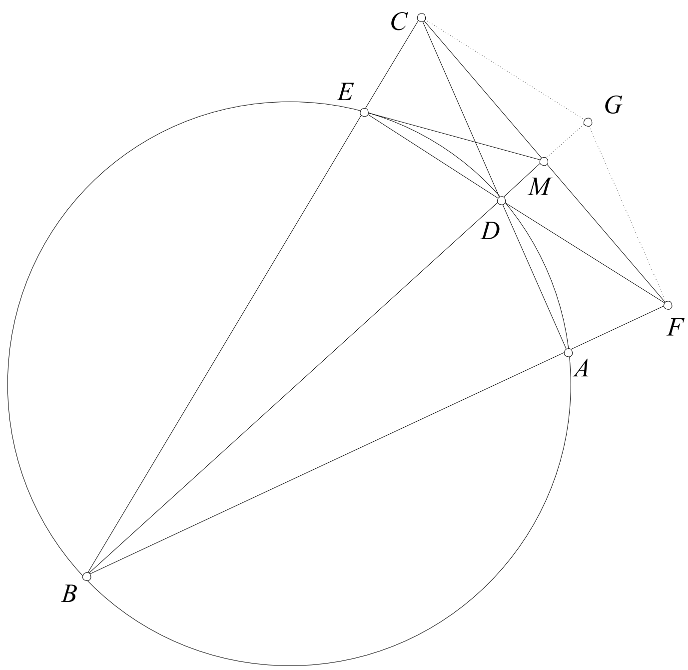
Finally, we need to show that no more than $n-1$ applications of t is needed to obtain a fixed sequence from an initial $n+1$-term index bounded sequence $A = (a_0, a_1, \ldots, a_n)$. We induct on n.

For $n = 1$, the two possible index bounded sequences $(a_0, a_1) = (0, 0)$ and $(a_0, a_1) = (0, 1)$ are already fixed by t so we need zero applications of t.

Assume that any index bounded sequences $(a_0, a_1, \ldots, a_n)$ reach a fixed sequence after no more than $n - 1$ applications of t. Consider an index bounded sequence $A = (a_0, a_1, \ldots, a_{n+1})$. It suffices to show that A will be stabilized in no more than n applications of t. We approach indirectly by assume on the contrary that $n + 1$ applications of transformations are needed. This can happen only if $a_{n+1} = 0$ and each application of t increased the index $n + 1$ term by exactly 1. Under transformation t, the resulting value of index term i will not the effected by index term j for $i < j$. Hence by the induction hypothesis, the subsequence $A' = (a_0, a_1, \ldots, a_n)$ will be stabilized in no more than $n-1$ applications of t. Because index n term is stabilized at value $x \leq n$ after no more than $\min\{x, n-1\}$ applications of t and index $n + 1$ term obtains value x after x exactly applications of t under our current assumptions. We conclude that the index $n+1$ term would become equal to the index n term after no more than $n - 1$ applications of t. However, once two consecutive terms in a sequence are equal they stay equal and stabilize together. Because the index n term needs no more than $n - 1$ transformations to be stabilized, A can be stabilized in no more than $n - 1$ applications of t, which contradicts our assumption of $n + 1$ applications needed. Thus our assumption was wrong and we need at most n applications of transformation t to stabilize an $(n + 1)$-term index bounded sequence. This completes our inductive proof.

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

First Solution: Extend segment DM through M to G such that $FG \parallel CD$.



Then $MF = MC$ if and only if quadrilateral $CDFG$ is a parallelogram, or, $FD \parallel CG$. Hence $MC = MF$ if and only if $\angle GCD = \angle FDA$, that is, $\angle FDA + \angle CGF = 180^\circ$.

Because quadrilateral $ABED$ is cyclic, $\angle FDA = \angle ABE$. It follows that $MC = MF$ if and only if

$$180^\circ = \angle FDA + \angle CGF = \angle ABE + \angle CGF,$$

that is, quadrilateral $CBFG$ is cyclic, which is equivalent to

$$\angle CBM = \angle CBG = \angle CFG = \angle DCF = \angle DCM.$$

Because $\angle DMC = \angle CMB$, $\angle CBM = \angle DCM$ if and only if triangles BCM and CDM are similar, that is

$$\frac{CM}{BM} = \frac{DM}{CM},$$

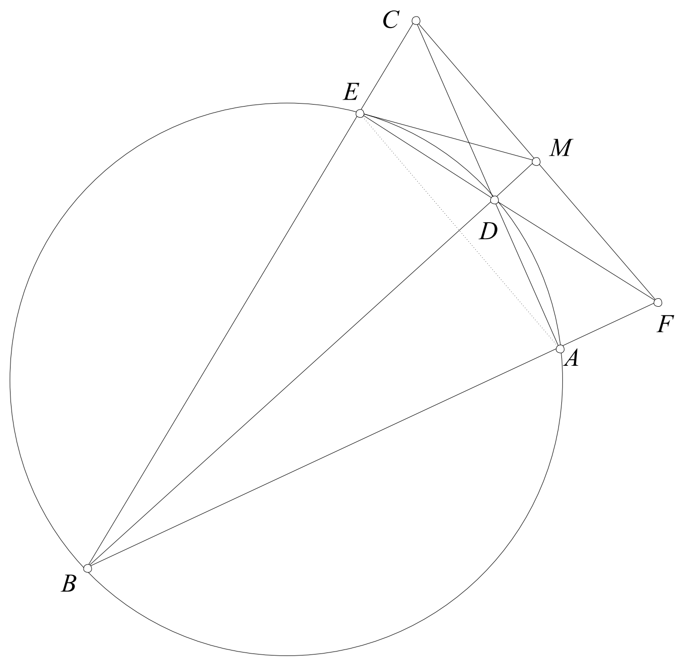
or $MB \cdot MD = MC^2$.

Second Solution:

We first assume that $MB \cdot MD = MC^2$. Because $\frac{MC}{MD} = \frac{MB}{MC}$ and $\angle CMD = \angle BMC$, triangles CMD and BMC are similar. Consequently, $\angle MCD = \angle MBC$.

Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle FCA = \angle MCD = \angle MBC = \angle DBE = \angle DAE = \angle CAE,$$



implying that $AE \parallel CF$, so $\angle AEF = \angle CFE$. Because quadrilateral $ABED$ is cyclic, $\angle ABD = \angle AED$. Hence

$$\angle FBM = \angle ABD = \angle AED = \angle AEF = \angle CFE = \angle MFD.$$

Because $\angle FBM = \angle DFM$ and $\angle FMB = \angle DMF$, triangles BFM and FDM are similar. Consequently, $\frac{FM}{DM} = \frac{BM}{FM}$, or $FM^2 = BM \cdot DM = CM^2$. Therefore $MC^2 = MB \cdot MD$ implies $MC = MF$.

Now we assume that $MC = MF$. Applying **Ceva's Theorem** to triangle BCF and **cevians** BM, CA, FE gives

$$\frac{BA}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB} = 1,$$

implying that $\frac{BA}{AF} = \frac{BE}{EC}$, so $AE \parallel CF$.

Consequently, $\angle DCM = \angle DAE$. Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle DCM = \angle DAE = \angle DBE = \angle CBM.$$

Because $\angle CBM = \angle DCM$ and $\angle CMB = \angle DMC$, triangles BCM and CDM are similar. Consequently, $\frac{CM}{DM} = \frac{BM}{CM}$, or $CM^2 = BM \cdot DM$.

Combining the above, we conclude that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

First Solution: By multiplying a, b, and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desired inequality reads

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2}\le 8.$$

Set

$$f(x)=\frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c)\le 8$. Note that

$$\begin{aligned} f(x) &= \frac{x^2+6x+9}{3(x^2-2x+3)}=\frac{1}{3}\cdot\frac{x^2+6x+9}{x^2-2x+3}\\ &= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right)=\frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right)\\ &\le \frac{1}{3}\left(1+\frac{8x+6}{2}\right)=\frac{1}{3}(4x+4). \end{aligned}$$

Hence,

$$f(a)+f(b)+f(c)\le\frac{1}{3}(4a+4+4b+4+4c+4)=8,$$

as desired.

Second Solution: Note that

$$\begin{aligned} (2x+y)^2+2(x-y)^2 &= 4x^2+4xy+y^2+2x^2-4xy+2y^2\\ &= 3(2x^2+y^2). \end{aligned}$$

Setting $x=a$ and $y=b+c$ yields

$$(2a+b+c)^2+2(a-b-c)^2=3(2a^2+(b+c)^2).$$

Thus, we have

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}=\frac{3(2a^2+(b+c)^2)-2(a-b-c)^2}{2a^2+(b+c)^2}=3-\frac{2(a-b-c)^2}{2a^2+(b+c)^2}.$$

and its analogous forms. Thus, the desired inequality is equivalent to

$$\frac{(a-b-c)^2}{2a^2+(b+c)^2}+\frac{(b-a-c)^2}{2b^2+(c+a)^2}+\frac{(c-a-b)^2}{2c^2+(a+b)^2}\ge\frac{1}{2}.$$

Because $(b+c)^2\le 2(b^2+c^2)$, we have $2a^2+(b+c)^2\le 2(a^2+b^2+c^2)$ and its analogous forms. It suffices to show that

$$\frac{(a-b-c)^2}{2(a^2+b^2+c^2)}+\frac{(b-a-c)^2}{2(a^2+b^2+c^2)}+\frac{(c-a-b)^2}{2(a^2+b^2+c^2)}\ge\frac{1}{2},$$

or,
$$(a-b-c)^2+(b-a-c)^2+(c-a-b)^2 \geq a^2+b^2+c^2. \tag{1}$$

Multiplying this out the left-hand side of the last inequality gives $3(a^2+b^2+c^2)-2(ab+bc+ca)$. Therefore the inequality (1) is equivalent to $2[a^2+b^2+c^2-(ab+bc+ca)] \geq 0$, which is evident because

$$2[a^2+b^2+c^2-(ab+bc+ca)] = (a-b)^2+(b-c)^2+(c-a)^2.$$

Equalities hold if $(b+c)^2 = 2(b^2+c^2)$ and $(c+a)^2 = 2(c^2+a^2)$, that is, $a=b=c$.

Third Solution: Given a function f of three variables, define the cyclic sum

$$\sum_{\text{cyc}} f(p,q,r) = f(p,q,r) + f(q,r,p) + f(r,p,q).$$

We first convert the inequality into

$$\frac{2a(a+2b+2c)}{2a^2+(b+c)^2} + \frac{2b(b+2c+2a)}{2b^2+(c+a)^2} + \frac{2c(c+2a+2b)}{2c^2+(a+b)^2} \leq 5.$$

Splitting the 5 among the three terms yields the equivalent form

$$\sum_{\text{cyc}} \frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq 0. \tag{2}$$

The numerator of the term shown factors as $(2a-x)(2a-5x)$, where $x=b+c$. We will show that
$$\frac{(2a-x)(2a-5x)}{3(2a^2+x^2)} \geq -\frac{4(2a-x)}{3(a+x)}. \tag{3}$$

Indeed, (3) is equivalent to

$$(2a-x)[(2a-5x)(a+x)+4(2a^2+x^2)] \geq 0,$$

which reduces to

$$(2a-x)(10a^2-3ax-x^2) = (2a-x)^2(5a+x) \geq 0,$$

evident. We proved that

$$\frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq -\frac{4(2a-b-c)}{3(a+b+c)},$$

hence (2) follows. Equality holds if and only if $2a=b+c$, $2b=c+a$, $2c=a+b$, i.e., when $a=b=c$.

Fourth Solution: Given a function f of three variables, we define the symmetric sum

$$\sum_{\text{sym}} f(x_1,\ldots,x_n) = \sum_{\sigma} f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

where σ runs over all permutations of $1, \ldots, n$ (for a total of $n!$ terms). For example, if $n = 3$, and we write x, y, z for x_1, x_2, x_3,

$$\sum_{\text{sym}} x^3 = 2x^3 + 2y^3 + 2z^3$$
$$\sum_{\text{sym}} x^2y = x^2y + y^2z + z^2x + x^2z + y^2x + z^2y$$
$$\sum_{\text{sym}} xyz = 6xyz.$$

We combine the terms in the desired inequality over a common denominator and use symmetric sum notation to simplify the algebra. The numerator of the difference between the two sides is

$$\sum_{\text{sym}} 8a^6 + 8a^5b + 2a^4b^2 + 10a^4bc + 10a^3b^3 - 52a^3b^2c + 14a^2b^2c^2.$$

Recalling **Schur's Inequality**, we have

$$\begin{aligned} & a^3 + b^3 + c^3 + 3abc - (a^2b + b^2c + c^a + ab^2 + bc^2 + ca^2) \\ = \; & a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0, \end{aligned}$$

or

$$\sum_{\text{sym}} a^3 - 2a^2b + abc \geq 0.$$

Hence,

$$0 \leq 14abc \sum_{\text{sym}} a^3 - 2a^2b + abc = 14 \sum_{\text{sym}} a^4bc - 28a^3b^2c + 14a^2b^2c^2$$

and by repeated **AM-GM Inequality**,

$$0 \leq \sum_{\text{sym}} 4a^6 - 4a^4bc$$

(because $a^46 + a^6 + a^6 + a^6 + b^6 + c^6 \geq 6a^4bc$ and its analogous forms)

and

$$0 \leq \sum_{\text{sym}} 4a^6 + 8a^5b + 2a^4b^2 + 10a^3b^3 - 24a^3b^2c.$$

Adding these three inequalities yields the desired result.

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Note: Let

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

denote a position, where A, B, C, D, E, F denote the numbers written on the vertices of the hexagon. We write

$$A \begin{matrix} B & C \\ F & E \end{matrix} D \pmod 2$$

if we consider the numbers written modulo 2.

Solution: Define the *sum* and *maximum* of a position to be the sum and maximum of the six numbers at the vertices. We will show that from any position in which the sum is odd, it is possible to reach the all-zero position.

Our strategy alternates between two steps:

(a) from a position with odd sum, move to a position with exactly one odd number;

(b) from a position with exactly one odd number, move to a position with odd sum and strictly smaller maximum, or to the all-zero position.

Note that no move will ever increase the maximum, so this strategy is guaranteed to terminate, because each step of type (b) decreases the maximum by at least one, and it can only terminate at the all-zero position. It suffices to show how each step can be carried out.

First, consider a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with odd sum. Then either $A+C+E$ or $B+D+F$ is odd; assume without loss of generality that $A+C+E$ is odd. If exactly one of A, C and E is odd, say A is odd, we can make the sequence of moves

$$1 \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to 1 \begin{matrix} \mathbf{1} & 0 \\ \mathbf{1} & 0 \end{matrix} \mathbf{0} \to \mathbf{0} \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} 0 \to 0 \begin{matrix} 1 & 0 \\ \mathbf{0} & 0 \end{matrix} 0 \pmod 2,$$

where a letter or number in boldface represents a move at that vertex, and moves that do not affect each other have been written as a single move for brevity. Hence we can reach a position with exactly one odd number. Similarly, if A, C, E are all odd, then the sequence of moves

$$1 \begin{matrix} B & 1 \\ F & 1 \end{matrix} D \to 1 \begin{matrix} \mathbf{0} & 1 \\ \mathbf{0} & 1 \end{matrix} \mathbf{0} \to 1 \begin{matrix} 0 & \mathbf{0} \\ 0 & \mathbf{0} \end{matrix} 0 \pmod 2,$$

brings us to a position with exactly one odd number. Thus we have shown how to carry out step (a).

Now assume that we have a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with A odd and all other numbers even. We want to reach a position with smaller maximum. Let M be the maximum. There are two cases, depending on the parity of M.

- In this case, M is even, so one of B, C, D, E, F is the maximum. In particular, $A < M$. We claim after making moves at B, C, D, E, and F in that order, the sum is odd and the maximum is less than M. Indeed, the following sequence

$$1\begin{matrix}0&0\\0&0\end{matrix}0 \to 1\begin{matrix}\mathbf{1}&0\\0&0\end{matrix}0 \to 1\begin{matrix}1&\mathbf{1}\\0&0\end{matrix}0 \to 1\begin{matrix}1&1\\0&0\end{matrix}\mathbf{1} \to 1\begin{matrix}1&1\\0&\mathbf{1}\end{matrix}1 \to 1\begin{matrix}1&1\\\mathbf{0}&1\end{matrix}1 \pmod 2.$$

  shows how the numbers change in parity with each move. Call this new position $A'\begin{matrix}B'&C'\\F'&E'\end{matrix}D'$. The sum is odd, since there are five odd numbers. The numbers A', B', C', D', E' are all less than M, since they are odd and M is even, and the maximum can never increase. Also, $F' = |A' - E'| \le \max\{A', E'\} < M$. So the maximum has been decreased.

- In this case, M is odd, so $M = A$ and the other numbers are all less than M.

  If $C > 0$, then we make moves at B, F, A, and F, in that order. The sequence of positions is

$$1\begin{matrix}0&0\\0&0\end{matrix}0 \to 1\begin{matrix}\mathbf{1}&0\\0&0\end{matrix}0 \to 1\begin{matrix}1&0\\\mathbf{1}&0\end{matrix}0 \to \mathbf{0}\begin{matrix}1&0\\1&0\end{matrix}0 \to 0\begin{matrix}1&0\\\mathbf{0}&0\end{matrix}0 \pmod 2.$$

  Call this new position $A'\begin{matrix}B'&C'\\F'&E'\end{matrix}D'$. The sum is odd, since there is exactly one odd number. As before, the only way the maximum could not decrease is if $B' = A$; but this is impossible, since $B' = |A - C| < A$ because $0 < C < M = A$. Hence we have reached a position with odd sum and lower maximum.

  If $E > 0$, then we apply a similar argument, interchanging B with F and C with E.

  If $C = E = 0$, then we can reach the all-zero position by the following sequence of moves:

$$A\begin{matrix}B&0\\F&0\end{matrix}D \to A\begin{matrix}\boldsymbol{A}&0\\\boldsymbol{A}&0\end{matrix}\mathbf{0} \to \mathbf{0}\begin{matrix}A&0\\A&0\end{matrix}0 \to 0\begin{matrix}\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{matrix}0.$$

  (Here 0 represents zero, not any even number.)

Hence we have shown how to carry out a step of type (b), proving the desired result. The problem statement follows since 2003 is odd.

Note: Observe that from positions of the form

$$0\begin{matrix}1&1\\1&1\end{matrix}0 \pmod 2 \qquad \text{or rotations}$$

it is impossible to reach the all-zero position, because a move at any vertex leaves the same value modulo 2. Dividing out the greatest common divisor of the six original numbers does not affect whether we can reach the all-zero position, so we may assume that the numbers in the original position are not all even. Then by a more complete analysis in step (a), one can show from any position not of the above form, it is possible to reach a position with exactly one odd number, and thus the all-zero position. This gives a complete characterization of positions from which it is possible to reach the all-zero position.

There are many ways to carry out the case analysis in this problem; the one used here is fairly economical. The important idea is the formulation of a strategy that decreases the maximum value while avoiding the “bad” positions described above.

Second Solution: We will show that if there is a pair of opposite vertices with odd sum (which of course is true if the sum of all the vertices is odd), then we can reduce to a position of all zeros.

Focus on such a pair (a, d) with smallest possible $\max(a, d)$. We will show we can always reduce this smallest maximum of a pair of opposite vertices with odd sum or reduce to the all-zero position. Because the smallest maximum takes nonnegative integer values, we must be able to achieve the all-zero position.

To see this assume without loss of generality that $a \geq d$ and consider an arc (a, x, y, d) of the position

$$a \begin{matrix} x & y \\ * & * \end{matrix} d$$

Consider updating x and y alternately, starting with x. If $\max(x, y) > a$, then in at most two updates we reduce $\max(x, y)$. Thus, we can repeat this *alternate updating* process and we must eventually reach a point when $\max(x, y) \leq a$, and hence this will be true from then on.

Under this alternate updating process, the arc of the hexagon will eventually enter an unique cycle of length four modulo 2 in at most one update. Indeed, we have

$$1 \begin{matrix} 0 & 0 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} 1 & \mathbf{1} \\ * & * \end{matrix} 0 \to 1 \begin{matrix} \mathbf{0} & 1 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} 0 & \mathbf{0} \\ * & * \end{matrix} 0 \pmod 2$$

and

$$1 \begin{matrix} 0 & 0 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} 0 & \mathbf{0} \\ * & * \end{matrix} 0 \pmod 2; \quad 1 \begin{matrix} 1 & 0 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ * & * \end{matrix} 0 \pmod 2$$

$$1 \begin{matrix} 1 & 1 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} 1 & \mathbf{1} \\ * & * \end{matrix} 0 \pmod 2; \quad 1 \begin{matrix} 0 & 1 \\ * & * \end{matrix} 0 \to 1 \begin{matrix} \mathbf{0} & 1 \\ * & * \end{matrix} 0 \pmod 2,$$

or

$$0 \begin{matrix} 0 & 1 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} \mathbf{1} & 1 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} 1 & \mathbf{0} \\ * & * \end{matrix} 1 \to 0 \begin{matrix} \mathbf{0} & 0 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} 0 & \mathbf{1} \\ * & * \end{matrix} 1 \pmod 2$$

and

$$0 \begin{matrix} 0 & 0 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} \mathbf{0} & 0 \\ * & * \end{matrix} 1 \pmod 2; \quad 0 \begin{matrix} 0 & 1 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} 0 & \mathbf{1} \\ * & * \end{matrix} 1 \pmod 2$$

$$0 \begin{matrix} 1 & 1 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} \mathbf{1} & 0 \\ * & * \end{matrix} 1 \pmod 2; \quad 0 \begin{matrix} 1 & 0 \\ * & * \end{matrix} 1 \to 0 \begin{matrix} 1 & \mathbf{0} \\ * & * \end{matrix} 1 \pmod 2.$$

Further note that each possible parity for x and y will occur equally often.

Applying this alternate updating process to both arcs (a, b, c, d) and (a, e, f, d) of

$$a \begin{matrix} b & c \\ f & e \end{matrix} d,$$

we can make the other four entries be at most a and control their parity. Thus we can create a position

$$a \begin{matrix} x_1 & x_2 \\ x_5 & x_4 \end{matrix} d$$

with $x_i + x_{i+3}$ $(i = 1, 2)$ odd and $M_i = \max(x_i, x_{i+3}) \leq a$. In fact, we can have $m = \min(M_1, M_2) < a$, as claimed, unless both arcs enter a cycle modulo 2 where the values congruent to a modulo 2 are always exactly a. More precisely, because the sum of x_i and x_{i+3} is odd, one of them is not congruent to a and so has its value strictly less than a. Thus both arcs must pass through the state (a, a, a, d) (modulo 2, this is either $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$) in a cycle of length four. It is easy to check that for this to happen, $d = 0$. Therefore, we can achieve the position

$$\begin{array}{cccc} & a & a & \\ a & & & 0. \\ & a & a & \end{array}$$

From this position, the sequence of moves

$$\begin{array}{cccc} & a & a & \\ a & & & 0 \\ & a & a & \end{array} \rightarrow \begin{array}{cccc} & \mathbf{0} & a & \\ a & & & 0 \\ & \mathbf{0} & a & \end{array} \rightarrow \begin{array}{cccc} & 0 & \mathbf{0} & \\ \mathbf{0} & & & 0 \\ & 0 & \mathbf{0} & \end{array}$$

completes the task.

USAMO 2003 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2003 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

3. Let n be a positive integer. For every sequence of integers

$$A = (a_0,\ a_1,\ a_2,\ \dots, a_n)$$

satisfying $0 \le a_i \le i$, for $i = 0, \dots, n$, we define another sequence

$$t(A) = (t(a_0),\ t(a_1),\ t(a_2),\ \dots,\ t(a_n))$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

§1 USAMO 2003/1, proposed by Titu Andreescu

Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

This is immediate by induction on n. For $n = 1$ we take 5; moving forward if M is a working n-digit number then exactly one of

$$N_1 = 10^n + M$$
$$N_3 = 3 \cdot 10^n + M$$
$$N_5 = 5 \cdot 10^n + M$$
$$N_7 = 7 \cdot 10^n + M$$
$$N_9 = 9 \cdot 10^n + M$$

is divisible by 5^{n+1}; as they are all divisible by 5^n and $N_k/5^n$ are all distinct.

§2 USAMO 2003/2

A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Suppose AB is a side of a polygon in the dissection, lying on diagonal XY, with X, A, B, Y in that order. Then

$$AB = XY - XA - YB.$$

In this way, we see that it actually just suffices to prove the result for a quadrilateral.

To do this, we apply barycentric coordinates. Consider quadrilateral $ABDC$, with $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Let $D = (x, y, z)$, with $x + y + z = 1$. By hypothesis, each of the numbers

$$\begin{aligned} -a^2yz + b^2(1-x)z + c^2(1-x)y &= AD^2 \\ a^2(1-y)z + b^2zx + c^2(1-y)x &= BD^2 \\ -a^2(1-z)y - b^2(1-z)x + c^2xy &= CD^2 \end{aligned}$$

is rational. Let $W = a^2yz + b^2zx + c^2xy$. Then,

$$\begin{aligned} b^2z + c^2y &= AD^2 + W \\ a^2z + c^2x &= BD^2 + W \\ a^2y + b^2x &= CD^2 + W. \end{aligned}$$

This implies that $AD^2+BD^2+2W-c^2 = 2S_Cz$ and cyclically (as usual $2S_C = a^2+b^2-c^2$). If any of S_A, S_B, S_C are zero, then we deduce W is rational. Otherwise, we have that

$$1 = x + y + z = \sum_{\text{cyc}} \frac{AD^2 + BD^2 + 2W - c^2}{2S_C}$$

which implies that W is rational, because it appears with coefficient $\frac{1}{S_A} + \frac{1}{S_B} + \frac{1}{S_C} \neq 0$ (since $S_{BC} + S_{CA} + S_{AB}$ is actually the area of ABC).

Hence from the rationality of W, we deduce that x is rational as long as $S_A \neq 0$, and similarly for the others. So at most one of x, y, z is irrational, but since $x + y + z = 1$ this implies they are all rational.

Finally, if $P = \overline{AD} \cap \overline{BC}$ then $AP = \frac{1}{y+z}AD$, so AP is rational too, completing the proof.

§3 USAMO 2003/3

Let n be a positive integer. For every sequence of integers

$$A = (a_0, a_1, a_2, \dots, a_n)$$

satisfying $0 \le a_i \le i$, for $i = 0, \dots, n$, we define another sequence

$$t(A) = (t(a_0), t(a_1), t(a_2), \dots, t(a_n))$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

We go by strong induction on n with the base cases $n = 1$ and $n = 2$ done by hand. Consider two cases:

- If $a_0 = 0$ and $a_1 = 1$, then $1 \le t(a_i) \le i$ for $i \ge 1$; now apply induction to

$$(t(a_1) - 1, \ t(a_2) - 1, \ \dots, \ t(a_n) - 1).$$

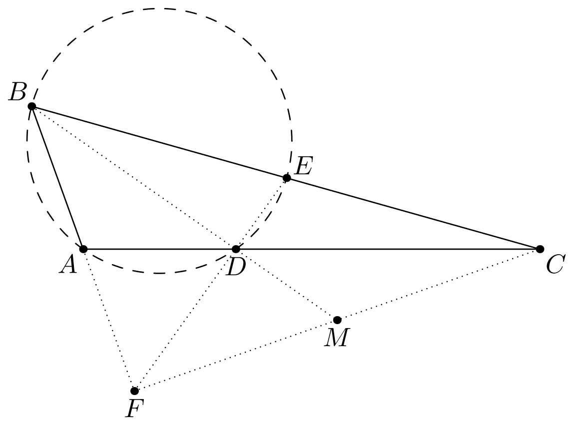
- Otherwise, assume that $a_0 = a_1 = \dots = a_{k-1} = 0$ but $a_k \ne 0$, where $k \ge 2$. Assume $k < n$ or it's obvious. Then $t(a_i) \ne 0$ for $i \ge k$, thus $t(t(a_i)) \ge k$ for $i \ge k$, and we can apply induction hypothesis to

$$(t(t(a_k)) - k, \ \dots, \ t(t(a_n)) - k).$$

§4 USAMO 2003/4, proposed by Titu Andreescu and Zuming Feng

Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Ceva theorem plus the similar triangles.



We know unconditionally that

$$\measuredangle CBD = \measuredangle EBD = \measuredangle EAD = \measuredangle EAC.$$

Moreover, by Ceva's theorem on $\triangle BCF$, we have $MF = MC \iff \overline{FC} \parallel \overline{AE}$. So we have the equivalences

$$\begin{aligned} MF = MC &\iff \overline{FC} \parallel \overline{AE} \\ &\iff \measuredangle FCA = \measuredangle EAC \\ &\iff \measuredangle MCD = \measuredangle CBD \\ &\iff MC^2 = MB \cdot MD. \end{aligned}$$

§5 USAMO 2003/5, proposed by Zuming Feng and Titu Andreescu

Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

This is a canonical example of tangent line trick. Homogenize so that $a+b+c=3$. The desired inequality reads

$$\sum_{\text{cyc}}\frac{(a+3)^2}{2a^2+(3-a)^2}\le 8.$$

This follows from

$$f(x)=\frac{(x+3)^2}{2x^2+(3-x)^2}\le\frac{1}{3}(4x+4)$$

which can be checked as $\frac{1}{3}(4x+4)(2x^2+(3-x)^2)-(x+3)^2=(x-1)^2(4x+3)\ge 0$.

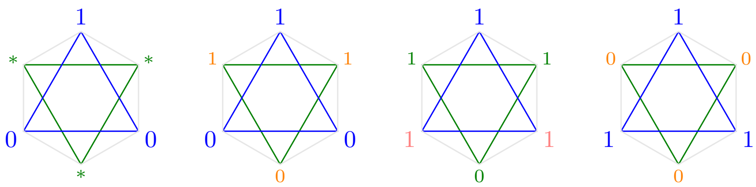
§6 USAMO 2003/6

At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

If $a \le b \le c$ are *odd* integers, the configuration which has $(a, b-a, b, c-b, c, c-a)$ around the hexagon in some order (up to cyclic permutation and reflection) is said to be *great* (picture below).

Claim — We can reach a great configuration from any configuration with odd sum.

Proof. We should be able to find an equilateral triangle whose vertices have odd sum. If all three vertices are odd, then we are already done. Otherwise, operate as in the following picture (modulo 2).

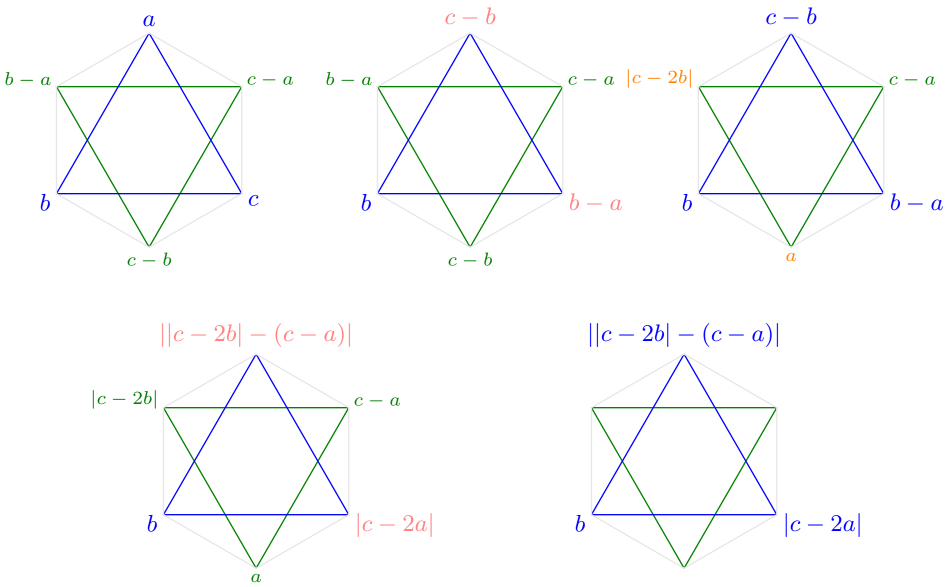


Thus we arrived at a great configuration. □

Claim — Bert's goal is possible for all great configurations.

Proof. If $a = b = c$ then we have $(t, 0, t, 0, t, 0)$ which is obviously winnable.

Otherwise, perform six moves as shown in the diagram to reach a new great configuration whose odd entries are b, $|c-2a|$, $||c-2b|-(c-a)|$ (and perform three more moves to get the even numbers). The idea is to show the largest odd entry has decreased.



This is annoying, but straightforward. Our standing assumption is $a \neq c$ (but possibly $b = c$). It's already obvious that $|c - 2a| < c$, so focus on the last term. If $c > 2b$, then $|(c-2b)-(c-a)| = |2b-a| < c$ as well for $a \neq c$. When $c \le 2b$ we instead have $|(2b-c)-(c-a)| \le \max(2b-c, c-a)$ with equality if and only if $c - a = 0$; but $2b - c \le c$ as needed. Thus, in all situations we have

$$c \neq a \implies \max\left(\left||c-2b|-(c-a)\right|, |c-2a|\right) < c.$$

Now denote the new odd entries by $a' \le b' \le c'$ (in some order). If $b < c$ then $c' < c$, while if $b = c$ then $c' = b$ but $b' < c = b$. Thus (c', b', a') precedes (c, b, a) lexicographically, and we can induct down. □

Remark. One simple idea might be to try to overwrite the maximum number at each point, decreasing the sum. However, this fails on the arrangement $(t, t, 0, t, t, 0)$.

Unfortunately, this issue is actually fatal, as the problem has a hidden parity obstruction. The configuration $(1, 1, 0, 1, 1, 0) \bmod 2$ is invariant modulo 2, and so Bert can walk into a "fatal death-trap" of this shape long before the numbers start becoming equal/zero/etc. In other words, you can mess up on the first move! This is why the initial sum is given to be odd; however, it's not possible for Bert to win so one essentially has to "tip-toe" around the 110110 trap any time one leaves the space of odd sum. That's why the great configurations defined above serve as an anchor, making sure we never veer too far from the safe 101010 configuration.

Remark. On the other hand, many other approaches are possible which anchor around a different parity configuration, like 100000 for example. The choice of 101010 by me is due to symmetry — ostensibly, if it worked, there should be fewer cases.

33^{rd} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 27, 2004

1. Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60°. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

2. Suppose $a_1, \ldots, a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties.

   (a) For $i = 1, \ldots, n$, $a_i \in S$.

   (b) For $i, j = 1, \ldots, n$ (not necessarily distinct), $a_i - a_j \in S$.

   (c) For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

   Prove that S must be equal to the set of all integers.

3. For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?

33^{rd} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 28, 2004

4. Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

5. Let a, b and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

6. A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

33^{rd} United States of America Mathematical Olympiad

1. Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60°. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

Solution: By symmetry, we only need to prove the first inequality.

Because quadrilateral $ABCD$ has an incircle, we have $AB + CD = BC + AD$, or $AB - AD = BC - CD$. It suffices to prove that

$$\frac{1}{3}(AB^2 + AB \cdot AD + AD^2) \le BC^2 + BC \cdot CD + CD^2.$$

By the given condition, $60^\circ \le \angle A, \angle C \le 120^\circ$, and so $\frac{1}{2} \ge \cos A, \cos C \ge -\frac{1}{2}$. Applying the law of cosines to triangle ABD yields

$$\begin{aligned} BD^2 &= AB^2 - 2AB \cdot AD \cos A + AD^2 \ge AB^2 - AB \cdot AD + AD^2 \\ &\ge \frac{1}{3}(AB^2 + AB \cdot AD + AD^2). \end{aligned}$$

The last inequality is equivalent to the inequality $3AB^2 - 3AB \cdot AD + 3AD^2 \ge AB^2 + AB \cdot AD + AD^2$, or $AB^2 - 2AB \cdot AD + AD^2 \ge 0$, which is evident. The last equality holds if and only if $AB = AD$.

On the other hand, applying the Law of Cosines to triangle BCD yields

$$BD^2 = BC^2 - 2BC \cdot CD \cos C + CD^2 \le BC^2 + BC \cdot CD + CD^2.$$

Combining the last two inequalities gives the desired result.

For the given inequalities to hold, we must have $AB = AD$. This condition is also sufficient, because all the entries in the equalities are 0. Thus, the given inequalities hold if and only if $ABCD$ is a kite with $AB = AD$ and $BC = CD$.

Problem originally by Titu Andreescu.

2. Suppose $a_1, \ldots, a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties.

   (a) For $i = 1, \ldots, n$, $a_i \in S$.

   (b) For $i, j = 1, \ldots, n$ (not necessarily distinct), $a_i - a_j \in S$.

   (c) For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

Prove that S must be equal to the set of all integers.

Solution: We may as well assume that none of the a_i is equal to 0. We start with the following observations.

(d) $0 = a_1 - a_1 \in S$ by (b).

(e) $-s = 0 - s \in S$ whenever $s \in S$, by (a) and (d).

(f) If $x, y \in S$ and $x - y \in S$, then $x + y \in S$ by (b) and (e).

By (f) plus strong induction on m, we have that $ms \in S$ for any $m \geq 0$ whenever $s \in S$. By (d) and (e), the same holds even if $m \leq 0$, and so we have the following.

(g) For $i = 1, \dots, n$, S contains all multiples of a_i.

We next verify that

(h) For $i, j \in \{1, \dots, n\}$ and any integers c_i, c_j, $c_i a_i + c_j a_j \in S$.

We do this by induction on $|c_i| + |c_j|$. If $|c_i| \leq 1$ and $|c_j| \leq 1$, this follows from (b), (d), (f), so we may assume that $\max\{|c_i|, |c_j|\} \geq 2$. Suppose without loss of generality (by switching i with j and/or negating both c_i and c_j) that $c_i \geq 2$; then

$$c_i a_i + c_j a_j = a_i + ((c_i - 1)a_i + c_j a_j)$$

and we have $a_i \in S$, $(c_i - 1)a_i + c_j a_j \in S$ by the induction hypothesis, and $(c_i - 2)a_i + c_j a_j \in S$ again by the induction hypothesis. So $c_i a_i + c_j a_j \in S$ by (f), and (h) is verified.

Let e_i be the largest integer such that 2^{e_i} divides a_i; without loss of generality we may assume that $e_1 \geq e_2 \geq \cdots \geq e_n$. Let d_i be the greatest common divisor of $a_1, \dots, a_i$. We prove by induction on i that S contains all multiples of d_i for $i = 1, \dots, n$; the case $i = n$ is the desired result. Our base cases are $i = 1$ and $i = 2$, which follow from (g) and (h), respectively.

Assume that S contains all multiples of d_i, for some $2 \leq i < n$. Let T be the set of integers m such that m is divisible by d_i and $m + ra_{i+1} \in S$ for all integers r. Then T contains nonzero positive and negative numbers, namely any multiple of a_i by (h). By (c), if $t \in T$ and s divisible by d_i (so in S) satisfy $t - s \in T$, then $t + s \in T$. By taking $t = s = d_i$, we deduce that $2d_i \in T$; by induction (as in the proof of (g)), we have $2md_i \in T$ for any integer m (positive, negative or zero).

From the way we ordered the a_i, we see that the highest power of 2 dividing d_i is greater than or equal to the highest power of 2 dividing a_{i+1}. In other words, a_{i+1}/d_{i+1} is odd. We can thus find integers f, g with f even such that $fd_i + ga_{i+1} = d_{i+1}$. (Choose such a pair without any restriction on f, and replace (f, g) with $(f - a_{i+1}/d_{i+1}, g + d_i/d_{i+1})$ if needed to get an even f.) Then for any integer r, we have $rfd_i \in T$ and so $rd_{i+1} \in S$. This completes the induction and the proof of the desired result.

Problem originally by Kiran Kedlaya.

3. For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?

Solution: We will show that a dissection satisfying the requirements of the problems is possible if and only if $k \neq 1$.

We first show by contradiction that such a dissection is not possible when $k = 1$. Assume that we have such a dissection. The common boundary of the two dissecting polygons must be a single broken line connecting two points on the boundary of the square (otherwise either the square is subdivided in more than two pieces or one of the polygons is inside the other). The two dissecting polygons must have the same number of vertices. They share all the vertices on the common boundary, so they have to use the same number of corners of the square as their own vertices. Therefore, the common boundary must connect two opposite sides of the square (otherwise one of the polygons will contain at least three corners of the square, while the other at most two). However, this means that each of the dissecting polygons must use an entire side of the square as one of its sides, and thus each polygon has a side of length 1. A side of longest length in one of the polygons is either a side on the common boundary or, if all those sides have length less than 1, it is a side of the square. But this is also true of the other polygon, which means that the longest side length in the two polygons is the same. This is impossible since they are similar but not congruent, so we have a contradiction.

We now construct a dissection satisfying the requirements of the problem when $k \neq 1$. Notice that we may assume that $k > 1$, because a $1 \times k$ rectangle is similar to a $1 \times \frac{1}{k}$ rectangle.

We first construct a dissection of an appropriately chosen rectangle (denoted by $ABCD$ below) into two similar noncongruent polygons. The construction depends on two parameters (n and r below). By appropriate choice of these parameters we show that the constructed rectangle can be made similar to a $1 \times k$ rectangle, for any $k > 1$. The construction follows.

Let $r > 1$ be a real number. For any positive integer n, consider the following sequence of $2n + 2$ points:

$$A_0 = (0,\ 0),\ A_1 = (1,\ 0),\ A_2 = (1,\ r),\ A_3 = (1 + r^2,\ r),$$

$$A_4 = (1 + r^2,\ r + r^3),\ A_5 = (1 + r^2 + r^4,\ r + r^3),$$

and so on, until

$$A_{2n+1} = (1 + r^2 + r^4 + \cdots + r^{2n},\ r + r^3 + r^5 + \cdots + r^{2n-1}).$$

Define a rectangle $ABCD$ by

$$A = A_0,\ B = (1 + r^2 + \cdots + r^{2n},\ 0),\ C = A_{2n+1},\ \text{ and }\ D = (0,\ r + r^3 + \ \ldots\ + r^{2n-1}).$$

The sides of the $(2n + 2)$-gon $A_1 A_2 \ldots A_{2n+1} B$ have lengths

$$r,\ r^2,\ r^3,\ \ldots\ ,\ r^{2n},\ r + r^3 + r^5 + \cdots + r^{2n-1},\ r^2 + r^4 + r^6 + \cdots + r^{2n},$$

and the sides of the $(2n+2)$-gon $A_0A_1A_2\dots A_{2n}D$ have lengths

$$1,\ r,\ r^2,\ \dots\ ,\ r^{2n-1},\ 1+r^2+r^4+\cdots+r^{2n-2},\ r+r^3+r^5+\cdots+r^{2n-1},$$

respectively. These two polygons dissect the rectangle $ABCD$ and, apart from orientation, it is clear that they are similar but noncongruent, with coefficient of similarity $r>1$. The rectangle $ABCD$ and its dissection are thus constructed.

The rectangle $ABCD$ is similar to a rectangle of size $1\times f_n(r)$, where

$$f_n(r)=\frac{1+r^2+\dots+r^{2n}}{r+r^3+\dots+r^{2n-1}}.$$

It remains to show that $f_n(r)$ can have any value $k>1$ for appropriate choices of n and r. Choose n sufficiently large so that $1+\frac{1}{n}<k$. Since

$$f_n(1)=1+\frac{1}{n}<k<k\frac{1+k^2+\dots+k^{2n}}{k^2+k^4+\dots+k^{2n}}=f_n(k)$$

and $f_n(r)$ is a continuous function for positive r, there exists an r such that $1<r<k$ and $f_n(r)=k$, so we are done.

Problem originally by Ricky Liu.

4. Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Solution: Bob can win as follows.

Claim 1. *After each of his moves, Bob can insure that in that maximum number in each row is a square in $A\cup B$, where*

$$A=\{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3)\}$$

and

$$B=\{(5,3),(4,4),(5,4),(6,4),(4,5),(5,5),(6,5),(4,6),(5,6),(6,6)\}.$$

Proof. Bob pairs each square of $A\cup B$ with a square in the same row that is not in $A\cup B$, so that each square of the grid is in exactly one pair. Whenever Alice plays in one square of a pair, Bob will play in the other square of the pair on his next turn. If Alice moves with x in $A\cup B$, Bob writes y with $y<x$ in the paired square. If Alice moves with x not in $A\cup B$, Bob writes z with $z>x$ in the paired square in $A\cup$B. So after Bob's turn, the maximum of each pair is in $A\cup B$, and thus the maximum of each row is in $A\cup B$. ∎

So when all the numbers are written, the maximum square in row 6 is in B and the maximum square in row 1 is in A. Since there is no path from B to A that stays in $A \cup B$, Bob wins.

Problem originally by Melanie Wood.

5. Let a, b and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Solution: For any positive number x, the quantities $x^2 - 1$ and $x^3 - 1$ have the same sign. Thus, we have $0 \leq (x^3 - 1)(x^2 - 1) = x^5 - x^3 - x^2 + 1$, or

$$x^5 - x^2 + 3 \geq x^3 + 2.$$

It follows that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a^3 + 2)(b^3 + 2)(c^3 + 2).$$

It suffices to show that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3. \tag{$*$}$$

We finish with two approaches.

- *First approach* Expanding both sides of inequality $(*)$ and cancelling like terms gives

  $$a^3b^3c^3+3(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)+8 \geq 3(a^2b+b^2a+b^2c+c^2b+c^2a+ac^2)+6abc. \tag{$*'$}$$

  By the AM-GM Inequality, we have $a^3 + a^3b^3 + 1 \geq 3a^2b$. Combining similar results, inequality $(*)$ reduces to

  $$a^3b^3c^3 + a^3 + b^3 + c^3 + 1 + 1 \geq 6abc,$$

  which is evident by the AM-GM Inequality.
- We rewrite the left-hand-side of inequality $(*)$ as

  $$(a^3 + 1 + 1)(1 + b^3 + 1)(1 + 1 + c^3).$$

  By Hölder's Inequality, we have

  $$(a^3 + 1 + 1)^{\frac{1}{3}}(1 + b^3 + 1)^{\frac{1}{3}}(1 + 1 + c^3)^{\frac{1}{3}} \geq (a + b + c),$$

  from which inequality $(*)$ follows.

Problem originally by Titu Andreescu.

6. A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

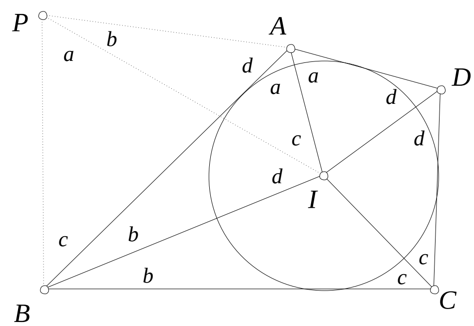
$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

Solution: Our proof is based on the following key Lemma.

Lemma *If a circle ω, centered at I, is inscribed in a quadrilateral $ABCD$, then*

$$BI^2 + \frac{AI}{DI} \cdot BI \cdot CI = AB \cdot BC. \qquad (*)$$



Proof: Since circle ω is inscribed in $ABCD$, we get $m\angle DAI = m\angle IAB = a$, $m\angle ABI = m\angle IBC = b$, $m\angle BCI = m\angle ICD = c$, $m\angle CDI = m\angle IDA = d$, and $a+b+c+d = 180°$. Construct a point P outside of the quadrilateral such that $\triangle ABP$ is similar to $\triangle DCI$. We obtain

$$\begin{aligned} m\angle PAI + m\angle PBI &= m\angle PAB + m\angle BAI + m\angle PBA + m\angle ABI \\ &= m\angle IDC + a + m\angle ICD + b \\ &= a + b + c + d = 180°, \end{aligned}$$

implying that the quadrilateral $PAIB$ is cyclic. By Ptolemy's Theorem, we have $AI \cdot BP + BI \cdot AP = AB \cdot IP$, or

$$BP \cdot \frac{AI}{IP} + BI \cdot \frac{AP}{IP} = AB. \qquad (\dagger)$$

Because $PAIB$ is cyclic, it is not difficult to see that, as indicated in the figure, $m\angle IPB = m\angle IAB = a$, $m\angle API = m\angle ABI = b$, $m\angle AIP = m\angle ABP = c$, and $m\angle PIB = m\angle PAB = d$. Note that $\triangle AIP$ and $\triangle ICB$ are similar, implying that

$$\frac{AI}{IP} = \frac{IC}{CB} \quad \text{and} \quad \frac{AP}{IP} = \frac{IB}{CB}.$$

Substituting the above equalities into the identity $(\dagger)$, we arrive at

$$BP \cdot \frac{CI}{BC} + \frac{BI^2}{BC} = AB,$$

or

$$BP \cdot CI + BI^2 = AB \cdot BC. \qquad (\dagger')$$

Note also that $\triangle BIP$ and $\triangle IDA$ are similar, implying that $\dfrac{BP}{BI} = \dfrac{IA}{ID}$, or

$$BP = \frac{AI}{ID} \cdot IB.$$

Substituting the above identity back into $(\dagger')$ gives the desired relation $(*)$, establishing the Lemma.

Now we prove our main result. By the Lemma and symmetry, we have

$$CI^2 + \frac{DI}{AI} \cdot BI \cdot CI = CD \cdot BC. \qquad (*')$$

Adding the two identities $(*)$ and $(*')$ gives

$$BI^2 + CI^2 + \left(\frac{AI}{DI} + \frac{DI}{AI}\right) BI \cdot CI = BC(AB + CD).$$

By the AM-GM Inequality, we have $\dfrac{AI}{DI} + \dfrac{DI}{AI} \geq 2$. Thus,

$$BC(AB + CD) \geq IB^2 + IC^2 + 2IB \cdot IC = (BI + CI)^2,$$

where the equality holds if and only if $AI = DI$. Likewise, we have

$$AD(AB + CD) \geq (AI + DI)^2,$$

where the equality holds if and only if $BI = CI$. Adding the last two identities gives

$$(AI + DI)^2 + (BI + CI)^2 \leq (AD + BC)(AB + CD) = (AB + CD)^2,$$

because $AD+BC = AB+CD$. (The latter equality is true because the circle ω is inscribed in the quadrilateral $ABCD$.)

By the given condition in the problem, all the equalities in the above discussion must hold, that is, $AI = DI$ and $BI = CI$. Consequently, we have $a = d$, $b = c$, and so $\angle DAB + \angle ABC = 2a + 2b = 180^\circ$, implying that $AD \parallel BC$. It is not difficult to see that $\triangle AIB$ and $\triangle DIC$ are congruent, implying that $AB = CD$. Thus, $ABCD$ is an isosceles trapezoid.

Problem originally by Zuming Feng.

USAMO 2004 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2004 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

2. Let $a_1, a_2, \dots, a_n$ be integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

(a) $a_i \in S$ for $i = 1, \dots, n$.

(b) $a_i - a_j \in S$ for $i, j = 1, \dots, n$, not necessarily distinct.

(c) If $x, y \in S$ and $x + y \in S$, then $x - y \in S$ too.

Prove that $S = \mathbb{Z}$.

3. For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

4. Alice and Bob play a game on a 6 by 6 grid. On his turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if he can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if he can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

5. Let a, b, c be positive reals. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a + b + c)^3.$$

6. A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

§1 USAMO 2004/1, proposed by Titu Andreescu

Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \le |BC^3 - CD^3| \le 3|AB^3 - AD^3|.$$

When does equality hold?

Clearly it suffices to show the left inequality. Since $AB + CD = BC + AD \implies |AB - AD| = |BC - CD|$, it suffices to prove

$$\frac{1}{3}(AB^2 + AB \cdot AD + AD^2) \le BC^2 + BC \cdot CD + CD^2.$$

This follows by noting that

$$\begin{aligned} BC^2 + BC \cdot CD + CD^2 &\ge BC^2 + CD^2 - 2(BC)(CD)\cos(\angle BCD) \\ &= BD^2 \\ &= AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD) \\ &\ge AB^2 + AD^2 - AB \cdot AD \\ &\ge \tfrac{1}{3}(AB^2 + AD^2 + AB \cdot AD) \end{aligned}$$

the last line following by AM-GM.

The equality holds iff $ABCD$ is a cite with $AB = AD$, $CB = CD$.

§2 USAMO 2004/2, proposed by Kiran Kedlaya

Let $a_1, a_2, \dots, a_n$ be integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

(a) $a_i \in S$ for $i = 1, \dots, n$.

(b) $a_i - a_j \in S$ for $i, j = 1, \dots, n$, not necessarily distinct.

(c) If $x, y \in S$ and $x + y \in S$, then $x - y \in S$ too.

Prove that $S = \mathbb{Z}$.

The idea is to show any linear combination of the a_i are in S, which implies (by Bezout) that $S = \mathbb{Z}$. This is pretty intuitive, but the details require some care (in particular there is a parity obstruction at the second lemma).

First, we make the following simple observations:

- $0 \in S$, by putting $i = j = 1$ in (b).
- $s \in S \iff -s \in S$, by putting $x = 0$ in (c).

Now, we prove that:

Lemma

For any integers c, d, and indices i, j, we have $ca_i + da_j \in S$.

Proof. We will assume $c, d > 0$ since the other cases are anologous. In that case it follows by induction on $c + d$ for example $ca_i + (d-1)a_j$, a_j, $ca_i + da_j$ in S implies $ca_i + (d+1)a_j \in S$. □

Lemma

For any nonzero integers $c_1, c_2, \dots, c_m$, and any distinct indices $\{i_1, i_2, \dots, i_m\}$, we have

$$\sum_k c_k a_{i_k} \in S.$$

Proof. By induction on m, with base case $m \le 2$ already done.

For the inductive step, we will assume that $i_1 = 1$, $i_2 = 2$, et cetera, for notational convenience. The proof is then split into two cases.

First Case: some c_i is even. WLOG $c_1 \neq 0$ is even and note that

$$\begin{aligned} x &\overset{\text{def}}{=} \frac{1}{2}c_1 a_1 + \sum_{k \ge 3} c_k a_k \in S \\ y &\overset{\text{def}}{=} -\frac{1}{2}c_1 a_1 - c_2 a_2 \in S \\ x + y &= -c_2 a_2 + \sum_{k \ge 3} c_k a_k \in S \\ \implies x - y &= \sum_{k \ge 1} c_k a_k \in S. \end{aligned}$$

Second Case: all c_i are odd. We reduce this to the first case as follows. Let $u = \frac{a_1}{\gcd(a_1,a_2)}$ and $v = \frac{a_2}{\gcd(a_1,a_2)}$. Then $\gcd(u,v) = 1$ and so WLOG u is odd. Then

$$c_1a_1 + c_2a_2 = (c_1 + v)a_1 + (c_2 - u)a_2$$

and so we can replace our given combination by $(c_1+v)a_1+(c_2-u)a_2+c_3a_3+\dots$ which now has an even coefficient for a_2. □

We then apply the lemma at $m = n$; this implies the result since Bezout's lemma implies that $\sum c_ia_i = 1$ for some choice of $c_i \in \mathbb{Z}$.

§3 USAMO 2004/3, proposed by Ricky Liu

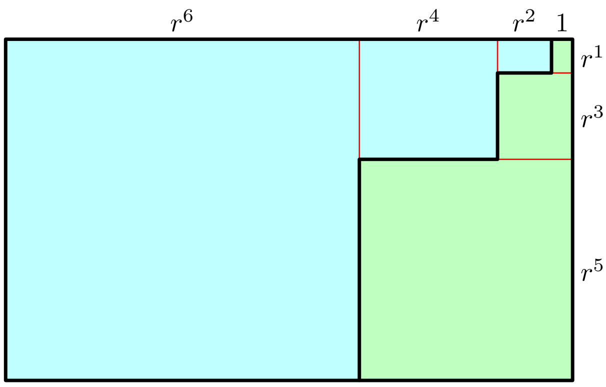
For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

Answer: the dissection is possible for every $k > 0$ except for $k = 1$.

Construction. By symmetry it suffices to give a construction for $k > 1$ (since otherwise we replace k by k^{-1}). For every integer $n \geq 2$ and real number $r > 1$, we define a shape $\mathcal{R}(n, r)$ as follows.

- We start with a rectangle of width 1 and height r. To its left, we glue a rectangle of height r and width r^2 to its left.
- Then, we glue a rectangle of width $1 + r^2$ and height r^3 below our figure, followed by a rectangle of height $r + r^3$ and width r^4 to the left of our figure.
- Next, we glue a rectangle of width $1 + r^2 + r^4$ and height r^5 below our figure, followed by a rectangle of height $r + r^3 + r^5$ and width r^6 to the left of our figure.

...and so on, up until we have put $2n$ rectangles together. The picture $\mathcal{R}(3, r)$ is shown below as an example.



Observe that by construction, the entire shape $\mathcal{R}(n, r)$ is a rectangle which consists of two similar "staircase" polygons (which are not congruent, since $r > 1$). Note that $\mathcal{R}(n, r)$ is similar to a $1 \times f_n(r)$ rectangle where $f_n(r)$ is the aspect ratio of $\mathcal{R}(n, r)$, defined by

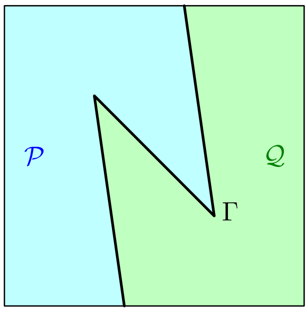
$$f_n(r) = \frac{1 + r^2 + \cdots + r^{2n}}{r + r^3 + \cdots + r^{2n-1}} = r + \frac{1}{r + r^3 + \cdots + r^{2n-1}}.$$

We claim that this is enough. Indeed for each fixed n, note that

$$\lim_{r \to 1^+} f_n(r) = 1 + \frac{1}{n} \quad \text{and} \quad \lim_{r \to \infty} f_n(r) = \infty.$$

Since f_n is continuous, it achieves all values greater than $1 + \frac{1}{n}$. Thus by taking sufficiently large n (such that $k > 1 + \frac{1}{n}$), we obtain a valid construction for any $k > 1$.

Proof of impossibility for a square. Now we show that $k = 1$ is impossible (the tricky part!). Suppose we have a squared dissected into two similar polygons $\mathcal{P} \sim \mathcal{Q}$. Let Γ be their common boundary. By counting the number of sides of $\mathcal{P}$ and $\mathcal{Q}$ we see Γ must run from one side of the square to an opposite side (possibly ending at a corner of the square). We orient the figure so Γ runs from north to south, with $\mathcal{P}$ to the west and $\mathcal{Q}$ to the east.



Let s be the longest length of a segment in Γ.

Claim — The longest side length of $\mathcal{P}$ is $\max(s, 1)$. Similarly, the longest side length of $\mathcal{Q}$ is $\max(s, 1)$ as well.

Proof. The only edges of $\mathcal{P}$ not in Γ are the west edge of our original square, which has length 1, and the north/south edges of $\mathcal{P}$ (if any), which have length at most 1. An identical argument works for $\mathcal{Q}$. □

It follows the longest sides of $\mathcal{P}$ and $\mathcal{Q}$ have the same length! Hence the two polygons are in fact congruent, ending the proof.

§4 USAMO 2004/4, proposed by Melanie Wood

Alice and Bob play a game on a 6 by 6 grid. On his turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if he can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if he can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Bob can win. Label the first two rows as follows:

$$\begin{bmatrix} a & b & c & d & e & f \\ d' & e' & f' & a' & b' & c' \end{bmatrix}$$

These twelve boxes thus come in six *pairs*, (a, a'), (b, b') and so on.

Claim — Bob can ensure that the order relation of the labels is the same between the two rows, meaning that $a < b$ if and only if $a' < b'$, and so on.

Proof. If Alice plays q in some box in the first two rows, then Bob can plays $q + \varepsilon$ in the corresponding box in the same pair, for some sufficiently small ε (in terms of the existing numbers).

When Alice writes a number in any other row, Bob writes anywhere in rows 3 to 6. □

Under this strategy the black squares in the first two rows will be a pair and therefore will not touch, so Bob wins.

§5 USAMO 2004/5, proposed by Titu Andreescu

Let a, b, c be positive reals. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a + b + c)^3 .$$

Observe that for all real numbers a, the inequality

$$a^5 - a^2 + 3 \ge a^3 + 2$$

holds. Then the problem follows by Hölder in the form

$$(a^3 + 1 + 1)(1 + b^3 + 1)(1 + 1 + c^3) \ge (a + b + c)^3.$$

§6 USAMO 2004/6, proposed by Zuming Feng

A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

Here's a completely algebraic solution. WLOG ω has radius 1, and let a, b, c, d be the lengths of the tangents from A, B, C, D to ω. It is known that

$$a + b + c + d = abc + bcd + cda + dab \qquad (\star)$$

which can be proved by, say tan-addition formula. Then, the content of the problem is to show that

$$(\sqrt{a^2+1} + \sqrt{d^2+1})^2 + (\sqrt{b^2+1} + \sqrt{c^2+1})^2 \le (a+b+c+d)^2$$

subject to $(\star)$, with equality only when $a = d = \frac{1}{b} = \frac{1}{c}$.

Let $S = ab + bc + cd + da + ac + bd$. Then the inequality is

$$\sqrt{(a^2+1)(d^2+1)} + \sqrt{(b^2+1)(c^2+1)} \le S - 2.$$

Now, by **USAMO 2014 Problem 1** and the condition $(\star)$, we have that $(a^2+1)(b^2+1)(c^2+1)(d^2+1) = (S - abcd - 1)^2$. So squaring both sides, the inequality becomes

$$(ad)^2 + (bc)^2 + a^2 + b^2 + c^2 + d^2 \le S^2 - 6S + 2abcd + 4.$$

To simplify this, we use the identities

$$\begin{aligned} S^2 &= 6abcd + \sum_{\text{sym}} a^2bc + \frac{1}{4}\sum_{\text{sym}} a^2b^2 \\ (a+b+c+d)^2 &= (abc + bcd + cda + dab)(a+b+c+d) \\ &= 4abcd + \frac{1}{2}\sum_{\text{sym}} a^2bc \end{aligned}$$

So $S^2 + 2abcd = \frac{1}{4}\sum_{\text{sym}} a^2b^2 + 2(a^2+b^2+c^2+d^2) + 4S$ and the inequality we want to prove reduces to

$$2S \le (ab)^2 + (ac)^2 + (bd)^2 + (cd)^2 + 4 + a^2 + b^2 + c^2 + d^2.$$

This follows by AM-GM since

$$\begin{aligned} (ab)^2 + 1 &\ge 2ab \\ (ac)^2 + 1 &\ge 2ac \\ (bd)^2 + 1 &\ge 2bd \\ (cd)^2 + 1 &\ge 2cd a^2 + d^2 & \ge 2ad \\ b^2 + c^2 &\ge 2bc. \end{aligned}$$

The equality case is when $ab = ac = bd = cd = 1$, $a = d$, $b = c$, as needed to imply an isosceles trapezoid.

Remark. Note that a priori one expects an inequality. Indeed,

- Quadrilaterals with incircles have four degrees of freedom.
- There is one condition imposed.
- Isosceles trapezoid with incircles have two degrees of freedom.

34^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 19, 2005

1. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

2. Prove that the system

$$x^6 + x^3 + x^3y + y = 147^{157}$$

$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

3. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

34^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 20, 2005

1. Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table? (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

2. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

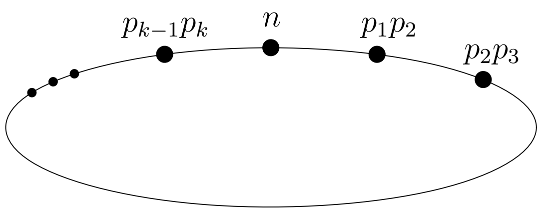
3. For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x \in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

34^{th} United States of America Mathematical Olympiad

1. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

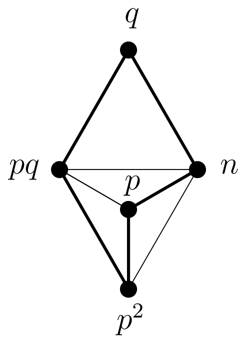
Solution. No such circular arrangement exists for $n = pq$, where p and q are distinct primes. In that case, the numbers to be arranged are p, q and pq, and in any circular arrangement, p and q will be adjacent. We claim that the desired circular arrangement exists in all other cases. If $n = p^e$ where $e \geq 2$, an arbitrary circular arrangement works. Henceforth we assume that n has prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_1 < p_2 < \cdots < p_k$ and either $k > 2$ or else $\max(e_1, e_2) > 1$. To construct the desired circular arrangement of $D_n := \{d : \ d|n \text{ and } d > 1\}$, start with the circular arrangement of n, p_1p_2, p_2p_3, $\ldots$, $p_{k-1}p_k$ as shown.



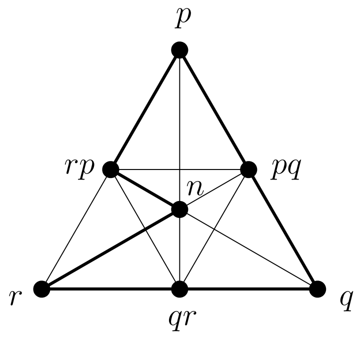
Then between n and p_1p_2, place (in arbitrary order) all other members of D_n that have p_1 as their smallest prime factor. Between p_1p_2 and p_2p_3, place all members of D_n other than p_2p_3 that have p_2 as their smallest prime factor. Continue in this way, ending by placing $p_k, p_k^2, \ldots, p_k^{e_k}$ between $p_{k-1}p_k$ and n. It is easy to see that each element of D_n is placed exactly one time, and any two adjacent elements have a common prime factor. Hence this arrangement has the desired property.

Note. In graph theory terms, this construction yields a Hamiltonian cycle[1] in the graph with vertex set D_n in which two vertices form an edge if the two corresponding numbers have a common prime factor. The graphs below illustrate the construction for the special cases $n = p^2q$ and $n = pqr$.

[1]A *cycle* of length k in a graph is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{k-1}, v_k\}, \{v_k, v_1\}$ are edges. A cycle that uses every vertex of the graph is a *Hamiltonian cycle.*



$n = p^2 q$



$n = pqr$

This problem was proposed by Zuming Feng.

2. Prove that the system

$$x^6 + x^3 + x^3y + y = 147^{157}$$

$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

First Solution. Add the two equations, then add 1 to each side to obtain

$$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1. \tag{1}$$

We prove that the two sides of this expression cannot be congruent modulo 19. We choose 19 because the least common multiple of the exponents 2 and 9 is 18, and by Fermat's Theorem, $a^{18} \equiv 1 \pmod{19}$ when a is not a multiple of 19. In particular, $(z^9)^2 \equiv 0$ or 1 (mod 19), and it follows that the possible remainders when z^9 is divided by 19 are

$$-1,\ 0,\ 1. \tag{2}$$

Next calculate n^2 modulo 19 for $n = 0, 1, \ldots, 9$ to see that the possible residues modulo 19 are

$$-8,\ -3,\ -2,\ 0,\ 1,\ 4,\ 5,\ 6,\ 7,\ 9. \tag{3}$$

Finally, apply Fermat's Theorem to see that

$$147^{157} + 157^{147} + 1 \equiv 14 \pmod{19}.$$

Because we cannot obtain 14 (or -5) by adding a number from list (2) to a number from list (3), it follows that the left side of (1) cannot be congruent to 14 modulo 19. Thus the system has no solution in integers x, y, z.

Second Solution. We will show there is no solution to the system modulo 13. Add the two equations and add 1 to obtain

$$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1.$$

By Fermat's Theorem, $a^{12} \equiv 1 \pmod{13}$ when a is not a multiple of 13. Hence we compute $147^{157} \equiv 4^1 \equiv 4 \pmod{13}$ and $157^{147} \equiv 1^3 \equiv 1 \pmod{13}$. Thus

$$(x^3 + y + 1)^2 + z^9 \equiv 6 \pmod{13}.$$

The cubes mod 13 are $0, \pm 1$, and ± 5. Writing the first equation as

$$(x^3 + 1)(x^3 + y) \equiv 4 \pmod{13},$$

we see that there is no solution in case $x^3 \equiv -1 \pmod{13}$ and for x^3 congruent to $0, 1, 5, -5$ (mod 13), correspondingly $x^3 + y$ must be congruent to $4, 2, 5, -1$. Hence

$$(x^3 + y + 1)^2 \equiv 12, 9, 10, \text{ or } 0 \pmod{13}.$$

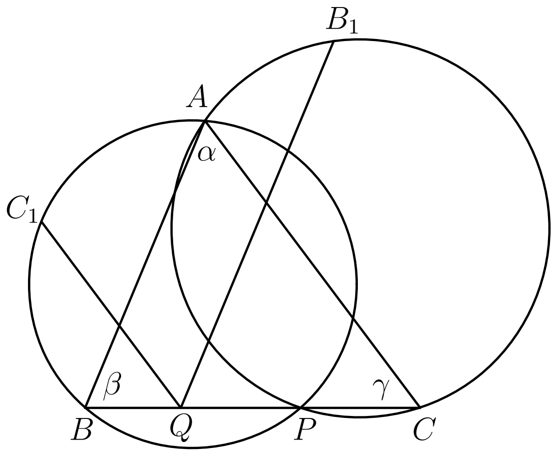
Also z^9 is a cube, hence z^9 must be 0, 1, 5, 8, or 12 (mod 13). It is easy to check that 6 (mod 13) is not obtained by adding one of 0, 9, 10, 12 to one of 0, 1, 5, 8, 12. Hence the system has no solutions in integers.

Note. This argument shows there is no solution even if z^9 is replaced by z^3.

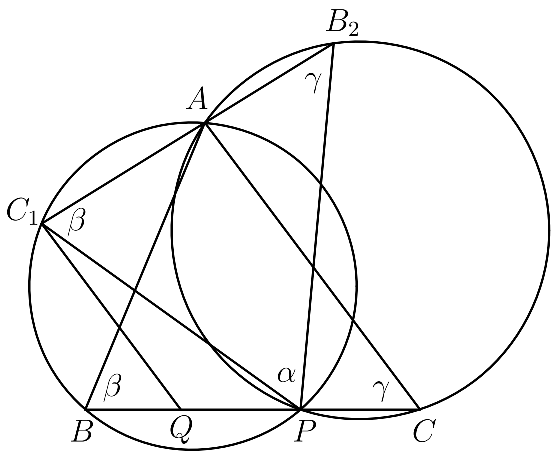
This problem was proposed by Răzvan Gelca.

3. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

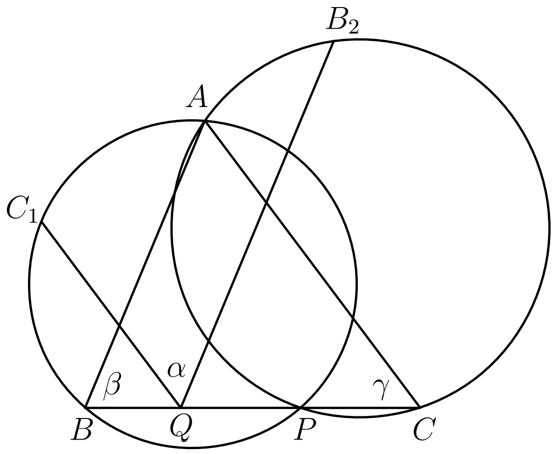
   Solution. Let α, β, γ denote the angles of ΔABC. Without loss of generality, we assume that Q is on the segment $\overline{BP}$.



We guess that B_1 is on the line through C_1 and A. To confirm that our guess is correct and prove that B_1, C_1, P, and Q lie on a circle, we start by letting B_2 be the point other than A that is on the line through C_1 and A, and on the circle through C, P, and A. Two applications of the Inscribed Angle Theorem yield $\angle PC_1A \cong \angle PBA$ and $\angle AB_2P \cong \angle ACP$, from which we conclude that $\Delta PC_1B_2 \sim \Delta ABC$.



From $QC_1 \parallel CA$ we have $m\angle PQC_1 = \pi - \gamma$ so quadrilateral PQC_1B_2 is cyclic. By the Inscribed Angle Theorem, $m\angle B_2QC_1 = \alpha$.



Finally, $m\angle PQB_2 = (\pi - \gamma) - \alpha = \beta$, from which it follows that $B_1 = B_2$ and thus P, Q, C_1, and B_1 are concyclic.

This problem was proposed by Zuming Feng.

4. Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table? (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

   Solution. Turn the table upside down so its surface lies in the xy-plane. We may assume that the corner with leg L_1 is at $(1, 0)$, and the corners with legs L_2, L_3, L_4 are at $(0, 1), (-1, 0), (0, -1)$, respectively. (We may do this because rescaling the x and y coordinates does not affect the stability of the cut table.) For $i = 1, 2, 3, 4$, let ℓ_i be the length of leg L_i after it is cut. Thus $0 \le \ell_i \le n$ for each i. The table will be stable if and only if the four points $F_1(1, 0, \ell_1)$, $F_2(0, 1, \ell_2)$, $F_3(-1, 0, \ell_3)$, and $F_4(0, -1, \ell_4)$ are coplanar. This will be the case if and only if $\overline{F_1F_3}$ intersects $\overline{F_2F_4}$, and this will happen if and only if the midpoints of the two segments coincide, that is,

$$(0, 0, (\ell_1 + \ell_3)/2) = (0, 0, (\ell_2 + \ell_4)/2). \qquad (*)$$

   Because each ℓ_i is an integer satisfying $0 \le \ell_i \le n$, the third coordinate for each of these midpoints can be any of the numbers $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, n$.

   For each nonnegative integer $k \le n$, let S_k be the number of solutions of $x + y = k$ where x, y are integers satisfying $0 \le x, y \le n$. The number of stable tables (in other words, the number of solutions of $(*)$) is $N = \sum_{k=0}^{n} S_k^2$.

Next we determine S_k. For $0 \le k \le n$, the solutions to $x + y = k$ are described by the ordered pairs $(j, k-j)$, $0 \le j \le k$. Thus $S_k = k+1$ in this case. For each $n+1 \le k \le 2n$, the solutions to $x + y = k$ are given by $(x, y) = (j, k-j)$, $k - n \le j \le n$. Thus $S_k = 2n - k + 1$ in this case. The number of stable tables is therefore

$$\begin{aligned} N &= 1^2 + 2^2 + \cdots n^2 + (n+1)^2 + n^2 + \cdots + 1^2 \\ &= 2\,\frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{1}{3}(n+1)(2n^2 + 4n + 3). \end{aligned}$$

This problem was proposed by Elgin Johnston.

5. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

   Solution. We will show that every vertex of the convex hull of the set of given $2n$ points lies on a balancing line.

   Let R be a vertex of the convex hull of the given $2n$ points and assume, without loss of generality, that R is red. Since R is a vertex of the convex hull, there exists a line ℓ through R such that all of the given points (except R) lie on the same side of ℓ. If we rotate ℓ about R in the clockwise direction, we will encounter all of the blue points in some order. Denote the blue points by $B_1,\ B_2, \ldots,\ B_n$ in the order in which they are encountered as ℓ is rotated clockwise about R. For $i = 1, \ldots, n$, let b_i and r_i be the numbers of blue points and red points, respectively, that are encountered before the point B_i as ℓ is rotated (in particular, B_i is not counted in b_i and R is never counted). Then

$$b_i = i - 1,$$

   for $i = 1, \ldots, n$, and

$$0 \le r_1 \le r_2 \le \cdots \le r_n \le n - 1.$$

   We show now that $b_i = r_i$, for some $i = 1, \ldots, n$. Define $d_i = r_i - b_i$, $i = 1, \ldots, n$. Then $d_1 = r_1 \ge 0$ and $d_n = r_n - b_n = r_n - (n-1) \le 0$. Thus the sequence $d_1, \ldots, d_n$

starts nonnegative and ends nonpositive. As i grows, r_i does not decrease, while b_i always increases by exactly 1. This means that the sequence $d_1, \dots, d_n$ can never decrease by more than 1 between consecutive terms. Indeed,

$$d_i - d_{i+1} = (r_i - r_{i+1}) + (b_{i+1} - b_i) \leq 0 + 1 = 1,$$

for $i = 1, \dots, n-1$. Since the integer-valued sequence $d_1, d_2, \dots, d_n$ starts nonnegative, ends nonpositive, and never decreases by more than 1 (so it never jumps over any integer value on the way down), it must attain the value 0 at some point, i.e., there exists some $i = 1, \dots, n$ for which $d_i = 0$. For such an i, we have $r_i = b_i$ and RB_i is a balancing line.

Since $n \geq 2$, the convex hull of the $2n$ points has at least 3 vertices, and since each of the vertices of the convex hull lies on a balancing line, there must be at least two distinct balancing lines.

Notes. The main ingredient in the solution above is a discrete version of a "tortoise-and-hare" argument. Indeed, the tortoise crawls slowly but methodically and is at distance $b_i = i - 1$ from the start at the moment i, $i = 1, \dots, n$, while the hare possibly jumps ahead at first ($r_1 \geq 0 = b_1$), but eventually becomes lazy or distracted and finishes at most as far as the tortoise ($r_n \leq n - 1 = b_n$). Since the tortoise does not skip any value and the hare never goes back towards the start, the tortoise must be even with the hare at some point.

We also note that a point not on the convex hull need not lie on any balancing line (for example, let $n = 2$ and let the convex hull be a triangle).

One can show (with much more work) that there are always at least n balancing lines; this is a theorem of J. Pach and R. Pinchasi (On the number of balanced lines, *Discrete and Computational Geometry* **25** (2001), 611–628). This is the best possible bound. Indeed, if n consecutive vertices in a regular $2n$-gon are colored blue and the other n are colored red, there are exactly n balancing lines.

This problem was proposed by Kiran Kedlaya.

6. For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x \in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

Solution: For the upper bound, let p be the smallest integer such that $10^p \geq n(n+1)/2$ and let

$$S = \{10^p - 1, 2(10^p - 1), \ldots, n(10^p - 1)\}.$$

The sum of any nonempty set of elements of S will have the form $k(10^p - 1)$ for some $1 \leq k \leq n(n+1)/2$. Write $k(10^p - 1) = [(k-1)10^p] + [(10^p - 1) - (k-1)]$. The second term gives the bottom p digits of the sum and the first term gives at most p top digits. Since the sum of a digit of the second term and the corresponding digit of $k-1$ is always 9, the sum of the digits will be $9p$. Since $10^{p-1} < n(n+1)/2$, this example shows that

$$f(n) \leq 9p < 9\log_{10}(5n(n+1)).$$

Since $n \geq 2$, $5(n+1) < n^4$, and hence

$$f(n) < 9\log_{10} n^5 = 45\log_{10} n.$$

For the lower bound, let S be a set of $n \geq 2$ positive integers such that any nonempty $X \subset S$ has $s\left(\sum_{x \in X} x\right) = f(n)$. Since $s(m)$ is always congruent to m modulo 9, $\sum_{x \in X} x \equiv f(n) \pmod 9$ for all nonempty $X \subset S$. Hence every element of S must be a multiple of 9 and $f(n) \geq 9$. Let q be the largest positive integer such that $10^q - 1 \leq n$. Lemma 1 below shows that there is a nonempty subset X of S with $\sum_{x \in X} x$ a multiple of $10^q - 1$, and hence Lemma 2 shows that $f(n) \geq 9q$.

Lemma 1. Any set of m positive integers contains a nonempty subset whose sum is a multiple of m.

Proof. Suppose a set T has no nonempty subset with sum divisible by m. Look at the possible sums mod m of nonempty subsets of T. Adding a new element a to T will give at least one new sum mod m, namely the least multiple of a which does not already occur. Therefore the set T has at least $|T|$ distinct sums mod m of nonempty subsets and $|T| < m$.

Lemma 2. Any positive multiple M of $10^q - 1$ has $s(M) \geq 9q$.

Proof. Suppose on the contrary that M is the smallest positive multiple of $10^q - 1$ with $s(M) < 9q$. Then $M \neq 10^q - 1$, hence $M > 10^q$. Suppose the most significant digit of M is the 10^m digit, $m \geq q$. Then $N = M - 10^{m-q}(10^q - 1)$ is a smaller positive multiple of $10^q - 1$ and has $s(N) \leq s(M) < 9q$, a contradiction.

Finally, since $10^{q+1} > n$, we have $q + 1 > \log_{10} n$. Since $f(n) \geq 9q$ and $f(n) \geq 9$, we have

$$f(n) \geq \frac{9q + 9}{2} > \frac{9}{2} \log_{10} n.$$

Weaker versions of Lemmas 1 and 2 are still sufficient to prove the desired type of lower bound.

This problem was proposed by Titu Andreescu and Gabriel Dospinescu.

USAMO 2005 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2005 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

2. Prove that the system of equations
$$x^6 + x^3 + x^3 y + y = 147^{157}$$
$$x^3 + x^3 y + y^2 + y + z^9 = 157^{147}$$
has no integer solutions.

3. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

4. Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i and still have a stable table?

   (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

5. Let $n > 1$ be an integer. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

6. For a positive integer m, let $s(m)$ denote the sum of the decimal digits of m. A set S positive integers is k-*stable* if $s(\sum_{x \in X} x) = k$ for any nonempty subset $X \subseteq S$.

   For each integer $n \geq 2$ let $f(n)$ be the minimal k for which there exists a k-stable set with n integers. Prove that there are constants $0 < C_1 < C_2$ with
$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

§1 USAMO 2005/1, proposed by Zuming Feng

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

The only bad ones are $n = pq$, products of two distinct primes. Clearly they can't be so arranged, so we show all others work.

- If n is a power of a prime, the result is obvious.
- If $n = p_1^{e_1} \dots p_k^{e_k}$ for some $k \geq 3$, then first situate p_1p_2, p_2p_3, $\dots$, p_kp_1 on the circle. Then we can arbitrarily place any multiples of p_i between $p_{i-1}p_i$ and p_ip_{i+1}. This finishes this case.
- Finally suppose $n = p^aq^b$. If $a > 1$, say, we can repeat the argument by first placing pq and p^2q and then placing multiples of p in one arc and multiples of q in the other arc. On the other hand the case $a = b = 1$ is seen to be impossible.

§2 USAMO 2005/2, proposed by Razvan Gelca

Prove that the system of equations

$$\begin{aligned} x^6 + x^3 + x^3y + y &= 147^{157} \\ x^3 + x^3y + y^2 + y + z^9 &= 157^{147} \end{aligned}$$

has no integer solutions.

Sum the equations and add 1 to both sides to get

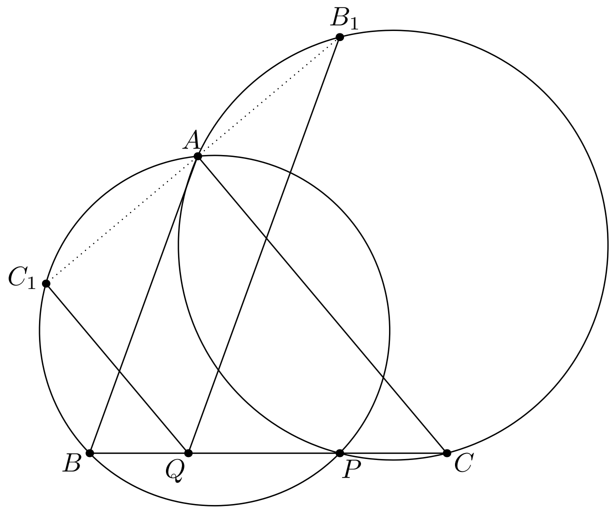
$$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1 \equiv 14 \pmod{19}$$

But $a^2 + b^9 \not\equiv 14 \pmod{19}$ for any integers a and b, since the ninth powers modulo 19 are 0, ± 1 and none of $\{13, 14, 15\}$ are squares modulo 19. Therefore, there are no integer solutions.

§3 USAMO 2005/3, proposed by Zuming Feng

Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

It is enough to prove that A, B_1, and C_1 are collinear, since then $\measuredangle C_1QP = \measuredangle ACP = \measuredangle AB_1P = \measuredangle C_1B_1P$.



First solution Let T be the second intersection of $\overline{AC_1}$ with (APC). Then readily $\triangle PC_1T \sim \triangle ABC$. Consequently, $\overline{QC_1} \parallel \overline{AC}$ implies TC_1QP cyclic. Finally, $\overline{TQ} \parallel \overline{AB}$ now follows from the cyclic condition, so $T = B_1$ as desired.

Second solution One may also use barycentric coordinates. Let $P = (0, m, n)$ and $Q = (0, r, s)$ with $m + n = r + s = 1$. Once again,

$$(APB) : -a^2yz - b^2zx - c^2xy + (x+y+z)(a^2m \cdot z) = 0.$$

Set $C_1 = (s - z, r, z)$, where $C_1Q \parallel AC$ follows by $(s - z) + r + z = 1$. We solve for this z.

$$\begin{aligned} 0 &= -a^2rz + (s-z)(-b^2z - c^2r) + a^2mz \\ &= b^2z^2 + (-sb^2 + rc^2)z - a^2rz + a^2mz - c^2rs \\ &= b^2z^2 + (-sb^2 + rc^2 + a^2(m-r))z - c^2rs \\ \implies 0 &= rb^2\left(\frac{z}{r}\right)^2 + (-sb^2 + rc^2 + a^2(m-r))\left(\frac{z}{r}\right) - c^2s. \end{aligned}$$

So the quotient of the z and y coordinates of C_1 satisfies this quadratic. Similarly, if $B_1 = (r - y, y, s)$ we obtain that

$$0 = sc^2\left(\frac{y}{s}\right)^2 + (-rc^2 + sb^2 + a^2(n-s))\left(\frac{y}{s}\right) - b^2r$$

Since these two quadratics are the same when one is written backwards (and negated), it follows that their roots are reciprocals. But the roots of the quadratics represent $\frac{z}{y}$ and $\frac{y}{z}$ for the points C_1 and B_1, respectively. This implies (with some configuration blah) that the points B_1 and C_1 are collinear with $A = (1, 0, 0)$ (in some line of the form $\frac{y}{z} = k$), as desired.

§4 USAMO 2005/4, proposed by Elgin Johnston

Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i and still have a stable table?

(The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Flip the table upside-down so that that the tabele's surface rests on the floor. Then, we see that we want the truncated legs to have endpoints A, B, C, D which are coplanar (say).

Claim — This occurs if and only if $ABCD$ is a parallelogram.

Proof. Obviously $ABCD$ being a parallelogram is necessary. Conversely, if they are coplanar, we let D' be such that $ABCD'$ is a parallelogram. Then D' also lies in the same plane as $ABCD$, but is situated directly above D (since the table was a square). This implies $D' = D$, as needed. □

In still other words, we are counting the number of solutions to

$$(n - k_1) + (n - k_3) = (n - k_2) + (n - k_4) \iff k_1 + k_3 = k_2 + k_4.$$

Define

$$a_r = \#\{(a, b) \mid a + b = r, \ 0 \le a, b \le n\}$$

so that the number of solutions to $k_1 + k_3 = k_2 + k_4 = r$ is just given by a_r^2. We now just compute

$$\begin{aligned} \sum_{r=0}^{2n} a_r^2 &= 1^2 + 2^2 + \dots + n^2 + (n+1)^2 + n^2 + \dots + 1^2 \\ &= \frac{1}{3}(n+1)(2n^2 + 4n + 3). \end{aligned}$$

§5 USAMO 2005/5, proposed by Kiran Kedlaya

Let $n > 1$ be an integer. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

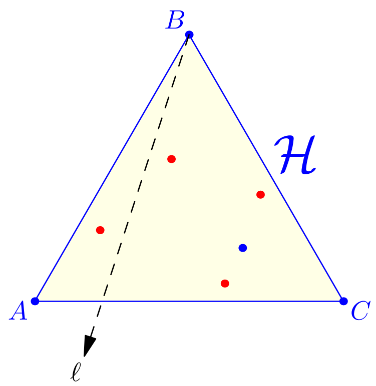
Consider the convex hull $\mathcal{H}$ of the polygon. There are two cases.

The easy case: if the convex hull $\mathcal{H}$ is not all the same color, there exist two edges of $\mathcal{H}$ (at least) which have differently colored endpoints. The extensions of those sides form balancing lines; indeed given any such line ℓ one side of ℓ has no points, the other has $n-1$ red and $n-1$ blue points.

So now assume $\mathcal{H}$ is all blue (WLOG). We will prove there are at least $|\mathcal{H}|$ balancing lines in the following way.

Claim — For any vertex B of $\mathcal{H}$ there is a balancing line through it.

Proof. Assume A, B, C are three consecutive blue vertices of $\mathcal{H}$. Imagine starting with line ℓ passing through B and A, then rotating it through B until it coincides with line BC, through the polygon.



During this process, we consider the set of points on the same side of ℓ as C, and let x be the number of such red points minus the number of such blue points. Note that:

- Every time ℓ touches a blue point, x increases by 1.
- Every time ℓ touches a red point, x decreases by 1.
- Initially, $x = +1$.
- Just before reaching the end we have $x = -1$.

So at the moment where x first equals zero, we have found our balancing line. ☐

§6 USAMO 2005/6, proposed by Titu Andreescu and Gabriel Dospinescu

For a positive integer m, let $s(m)$ denote the sum of the decimal digits of m. A set S positive integers is *k-stable* if $s(\sum_{x \in X} x) = k$ for any nonempty subset $X \subseteq S$.

For each integer $n \ge 2$ let $f(n)$ be the minimal k for which there exists a k-stable set with n integers. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \le f(n) \le C_2 \log_{10} n.$$

Lower bound: Let $n \ge 1$ and $r \ge 1$ be integers satisfying $1 + 2 + \cdots + n < 10^e$. Consider the set

$$S = \{10^e - 1,\ 2(10^e - 1),\ \dots,\ n(10^e - 1)\}.$$

For example, if $n = 6$ and $e = 3$, we have $S = \{999, 1998, 2997, 3996, 4995, 5994\}$.

The set S here is easily seen to be $9e$-good. Thus $f(n) \ge 9 \lceil \log_{10} n \rceil$, proving one direction.

Remark. I think the problem is actually more natural with a multiset S rather than a vanilla set, in which case $S = \{10^e - 1, 10^e - 1, \dots, 10^e - 1\}$ works fine, and is easier to think of.

In some sense the actual construction is obtained by starting with this one, and then pushing together the terms together in order to get the terms to be distinct, hence the $1 + 2 + \cdots + n$ appearance.

Upper bound: we are going to prove the following, which obviously sufficient.

Claim — Let r be a positive integer. In any (multi)set S of more than 12^k integers, there exists a subset whose sum of decimal digits exceeds k.

Proof. Imagine writing entries of S on a blackboard, while keeping a running sum Σ initially set to zero. For $i = 1, 2, \dots$ we will have a process such that at the end of the ith step all entries on the board are divisible by 10^i. It goes as follows:

- If the ith digit from the right of Σ is nonzero, then arbitrarily partition the numbers on the board into groups of 10, erasing any leftover numbers. Within each group of 10, we can find a nonempty subset with sum 0 mod 10^i; we then erase each group and replace it with that sum.

- If the ith digit from the right of Σ is zero, but some entry on the board is not divisible by 10^i, then we erase that entry and add it to Σ. Then we do the grouping as in the previous step.

- If the ith digit from the right of Σ is zero, and all entries on the board are divisible by 10^i, we do nothing and move on to the next step.

This process ends when no numbers remain on the blackboard. The first and second cases occur at least $k + 1$ times (the number of entries decreases by a factor of at most 12 each step), and each time Σ gets some nonzero digit, which is never changed at later steps. Therefore Σ has sum of digits at least $k + 1$ as needed. □

Remark. The official solutions contain a slicker proof: it turns out that any multiple of $10^e - 1$ has sum of decimal digits at least $9e$. However, if one does not know this lemma it seems nontrivial to imagine coming up with it.

35^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 18, 2006

1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

(For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.)

2. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \geq 0}$ is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

35^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 19, 2006

4. Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

35^{th} United States of America Mathematical Olympiad

1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

(For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.)

First Solution. First suppose that s is a divisor of $p-1$; write $d = (p-1)/s$. As x varies among $1, 2, \ldots, p-1$, $\{sx/p\}$ takes the values $1/p, 2/p, \ldots, (p-1)/p$ once each in some order. The possible values with $\{sx/p\} < s/p$ are precisely $1/p, \ldots, (s-1)/p$. From the fact that $\{sd/p\} = (p-1)/p$, we realize that the values $\{sx/p\} = (p-1)/p, (p-2)/p, \ldots, (p-s+1)/p$ occur for

$$x = d, 2d, \ldots, (s-1)d$$

(which are all between 0 and p), and so the values $\{sx/p\} = 1/p, 2/p, \ldots, (s-1)/p$ occur for

$$x = p-d, p-2d, \ldots, p-(s-1)d,$$

respectively. From this it is clear that m and n cannot exist as requested.

Conversely, suppose that s is not a divisor of $p-1$. Put $m = \lceil p/s \rceil$; then m is the smallest positive integer such that $\{ms/p\} < s/p$, and in fact $\{ms/p\} = (ms-p)/p$. However, we cannot have $\{ms/p\} = (s-1)/p$ or else we would have $(m-1)s = p-1$, contradicting our hypothesis that s does not divide $p-1$. Hence the unique $n \in \{1, \ldots, p-1\}$ for which $\{nx/p\} = (s-1)/p$ has the desired properties (since the fact that $\{nx/p\} < s/p$ forces $n \geq m$, but $m \neq n$).

Second Solution. We prove the contrapositive statement:

> Let p be a prime number and let s be an integer with $0 < s < p$. Prove that the following statements are equivalent:
>
> (a) s is a divisor of $p-1$;

(b) if integers m and n are such that $0 < m < p$, $0 < n < p$, and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p},$$

then $0 < n < m < p$.

Since p is prime and $0 < s < p$, s is relatively prime to p and

$$S = \{s, 2s, \ldots, (p-1)s, ps\}$$

is a set of complete residues classes modulo p. In particular,

(1) there is an unique integer d with $0 < d < p$ such that $sd \equiv -1 \pmod{p}$; and

(2) for every k with $0 < k < p$, there exists a unique pair of integers (m_k, a_k) with $0 < m_k < p$ such that $m_k s + a_k p = k$.

Now we consider the equations

$$m_1 s + a_1 p = 1,\ m_2 s + a_2 p = 2,\ \ldots\ , m_s s + a_s p = s.$$

Hence $\{m_k s/p\} = k/p$ for $1 \le k \le s$.

Statement (b) holds if and only $0 < m_s < m_{s-1} < \cdots < m_1 < p$. For $1 \le k \le s-1$, $m_k s - m_{k+1} s = (a_{k+1} - a_k)p - 1$, or $(m_k - m_{k+1})s \equiv -1 \pmod{p}$. Since $0 < m_{k+1} < m_k < p$, by (1), we have $m_k - m_{k+1} = d$. We conclude that (b) holds if and only if $m_s, m_{s-1}, \ldots, m_1$ form an arithmetic progression with common difference $-d$. Clearly $m_s = 1$, so $m_1 = 1 + (s-1)d = jp - d + 1$ for some j. Then $j = 1$ because m_1 and d are both positive and less than p, so $sd = p - 1$. This proves (a).

Conversely, if (a) holds, then $sd = p - 1$ and $m_k \equiv -dsm_k \equiv -dk \pmod{p}$. Hence $m_k = p - dk$ for $1 \le k \le s$. Thus $m_s, m_{s-1}, \ldots, m_1$ form an arithmetic progression with common difference $-d$. Hence (b) holds.

This problem was proposed by Kiran Kedlaya.

2. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

Solution. The minimum is $N = 2k^3 + 3k^2 + 3k$. The set

$$\{k^2 + 1, k^2 + 2, \ldots, k^2 + 2k + 1\}$$

has sum $2k^3+3k^2+3k+1 = N+1$ which exceeds N, but the sum of the k largest elements is only $(2k^3+3k^2+3k)/2 = N/2$. Thus this N is such a value.

Suppose $N < 2k^3+3k^2+3k$ and there are positive integers $a_1 < a_2 < \cdots < a_{2k+1}$ with $a_1 + a_2 + \cdots + a_{2k+1} > N$ and $a_{k+2} + \cdots + a_{2k+1} \le N/2$. Then

$$(a_{k+1}+1) + (a_{k+1}+2) + \cdots + (a_{k+1}+k) \le a_{k+2} + \cdots + a_{2k+1} \le N/2 < \frac{2k^3+3k^2+3k}{2}.$$

This rearranges to give $2ka_{k+1} \le N - k^2 - k$ and $a_{k+1} < k^2+k+1$. Hence $a_{k+1} \le k^2+k$. Combining these we get

$$2(k+1)a_{k+1} \le N + k^2 + k.$$

We also have

$$(a_{k+1}-k) + \cdots + (a_{k+1}-1) + a_{k+1} \ge a_1 + \cdots + a_{k+1} > N/2$$

or $2(k+1)a_{k+1} > N + k^2 + k$. This contradicts the previous inequality, hence no such set exists for $N < 2k^3+3k^2+3k$ and the stated value is the minimum.

This problem was proposed by Dick Gibbs.

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n\ge 0}$ is bounded above. (In particular, this requires $f(n^2) \ne 0$ for $n \ge 0$.)

Solution. The polynomial f has the required properties if and only if

$$f(x) = c(4x - a_1^2)(4x - a_2^2)\cdots(4x - a_k^2), \qquad (*)$$

where $a_1, a_2, \ldots, a_k$ are odd positive integers and c is a nonzero integer. It is straightforward to verify that polynomials given by $(*)$ have the required property. If p is a prime divisor of $f(n^2)$ but not of c, then $p|(2n - a_j)$ or $p|(2n + a_j)$ for some $j \le k$. Hence $p - 2n \le \max\{a_1, a_2, \ldots, a_k\}$. The prime divisors of c form a finite set and do affect whether or not the given sequence is bounded above. The rest of the proof is devoted to showing that any f for which $\{p(f(n^2)) - 2n\}_{n\ge 0}$ is bounded above is given by $(*)$.

Let $\mathbb{Z}[x]$ denote the set of all polynomials with integral coefficients. Given $f \in \mathbb{Z}[x]$, let $\mathcal{P}(f)$ denote the set of those primes that divide at least one of the numbers in the sequence $\{f(n)\}_{n\ge 0}$. The solution is based on the following lemma.

Lemma. *If $f \in \mathbb{Z}[x]$ is a nonconstant polynomial then $\mathcal{P}(f)$ is infinite.*

Proof. Repeated use will be made of the following basic fact: if a and b are distinct integers and $f \in \mathbb{Z}[x]$, then $a - b$ divides $f(a) - f(b)$. If $f(0) = 0$, then p divides $f(p)$ for every prime p, so $\mathcal{P}(f)$ is infinite. If $f(0) = 1$, then every prime divisor p of $f(n!)$ satisfies $p > n$. Otherwise p divides $n!$, which in turn divides $f(n!) - f(0) = f(n!) - 1$. This yields $p|1$, which is false. Hence $f(0) = 1$ implies that $\mathcal{P}(f)$ is infinite. To complete the proof, set $g(x) = f(f(0)x)/f(0)$ and observe that $g \in \mathbb{Z}[x]$ and $g(0) = 1$. The preceding argument shows that $\mathcal{P}(g)$ is infinite, and it follows that $\mathcal{P}(f)$ is infinite. $\square$

Suppose $f \in \mathbb{Z}[x]$ is nonconstant and there exists a number M such that $p(f(n^2)) - 2n \leq M$ for all $n \geq 0$. Application of the lemma to $f(x^2)$ shows that there is an infinite sequence of distinct primes $\{p_j\}$ and a corresponding infinite sequence of nonnegative integers $\{k_j\}$ such that $p_j|f(k_j^2)$ for all $j \geq 1$. Consider the sequence $\{r_j\}$ where $r_j = \min\{k_j \pmod{p_j}, p_j - k_j \pmod{p_j}\}$. Then $0 \leq r_j \leq (p_j - 1)/2$ and $p_j|f(r_j^2)$. Hence $2r_j + 1 \leq p_j \leq p(f(r_j^2)) \leq M + 2r_j$, so $1 \leq p_j - 2r_j \leq M$ for all $j \geq 1$. It follows that there is an integer a_1 such that $1 \leq a_1 \leq M$ and $a_1 = p_j - 2r_j$ for infinitely many j. Let $m = \deg f$. Then $p_j|4^m f((p_j - a_1)/2)^2)$ and $4^m f((x - a_1)/2)^2) \in \mathbb{Z}[x]$. Consequently, $p_j|f((a_1/2)^2)$ for infinitely many j, which shows that $(a_1/2)^2$ is a zero of f. Since $f(n^2) \neq 0$ for $n \geq 0$, a_1 must be odd. Then $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$. (See the note below.) Observe that $\{p(g(n^2)) - 2n\}_{n \geq 0}$ must be bounded above. If g is constant, we are done. If g is nonconstant, the argument can be repeated to show that f is given by $(*)$.

Note. The step that gives $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$ follows immediately using a lemma of Gauss. The use of such an advanced result can be avoided by first writing $f(x) = r(4x - a_1^2)g(x)$ where r is rational and $g \in \mathbb{Z}[x]$. Then continuation gives $f(x) = c(4x - a_1^2) \cdots (4x - a_k^2)$ where c is rational and the a_i are odd. Consideration of the leading coefficient shows that the denominator of c is 2^s for some $s \geq 0$ and consideration of the constant term shows that the denominator is odd. Hence c is an integer.

This problem was proposed by Titu Andreescu and Gabriel Dospinescu.

4. Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

Solution. The answer is $n = 4$ or $n \geq 6$.

I. First, we prove that each $n \in \{4, 6, 7, 8, 9, \ldots\}$ *satisfies* the condition.

(1). If $n = 2k \geq 4$ is *even*, we set $(a_1,\ a_2,\ \ldots,\ a_k) = (k,\ 2,\ 1,\ \ldots,\ 1)$:

$$a_1 + a_2 + \ \ldots\ + a_k = k + 2 + 1 \cdot (k-2) = 2k = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k = 2k = n \ .$$

(2). If $n = 2k + 3 \geq 9$ is *odd*, we set $(a_1,\ a_2,\ \ldots,\ a_k) = \left(k + \frac{3}{2},\ \frac{1}{2},\ 4,\ 1,\ \ldots,\ 1\right)$:

$$a_1 + a_2 + \ \ldots\ + a_k = k + \frac{3}{2} + \frac{1}{2} + 4 + (k-3) = 2k + 3 = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k = \left(k + \frac{3}{2}\right) \cdot \frac{1}{2} \cdot 4 = 2k + 3 = n \ .$$

(3). A very special case is $n = 7$, in which we set $(a_1,\ a_2,\ a_3) = \left(\frac{4}{3},\ \frac{7}{6},\ \frac{9}{2}\right)$. It is also easy to check that

$$a_1 + a_2 + a_3 = a_1 \cdot a_2 \cdot a_3 = 7 = n.$$

II. Second, we prove by contradiction that each $n \in \{1, 2, 3, 5\}$ *fails to satisfy* the condition.

Suppose, on the contrary, that there is a set of $k \geq 2$ positive rational numbers whose sum and product are both $n \in \{1, 2, 3, 5\}$. By the Arithmetic-Geometric Mean inequality, we have

$$n^{1/k} = \sqrt[k]{a_1 \cdot a_2 \cdot \ \ldots\ \cdot a_k} \leq \frac{a_1 + a_2 + \ \ldots\ + a_k}{k} = \frac{n}{k} \ ,$$

which gives

$$n \geq k^{\frac{k}{k-1}} = k^{1 + \frac{1}{k-1}} \ .$$

Note that $n > 5$ whenever $k = 3,\ 4$, or $k \geq 5$:

$$\begin{aligned} k = 3 \quad &\Rightarrow \quad n \geq 3\sqrt{3} = 5.196... > 5; \\ k = 4 \quad &\Rightarrow \quad n \geq 4\sqrt[3]{4} = 6.349... > 5; \\ k \geq 5 \quad &\Rightarrow \quad n \geq 5^{1 + \frac{1}{k-1}} > 5 \ . \end{aligned}$$

This proves that none of the integers 1, 2, 3, or 5 can be represented as the sum and, at the same time, as the product of three or more positive numbers $a_1,\ a_2,\ \ldots,\ a_k$, rational or irrational.

The remaining case $k = 2$ also goes to a contradiction. Indeed, $a_1 + a_2 = a_1 a_2 = n$ implies that $n = a_1^2/(a_1 - 1)$ and thus a_1 satisfies the quadratic

$$a_1^2 - na_1 + n = 0 \ .$$

Since a_1 is supposed to be *rational*, the discriminant $n^2 - 4n$ must be a perfect square (a square of a positive integer). However, it can be easily checked that this is not the case for any $n \in \{1,\ 2,\ 3,\ 5\}$. This completes the proof.

Remark. Actually, among all positive integers only $n = 4$ can be represented both as the sum and product of the same two rational numbers. Indeed, $(n-3)^2 < n^2 - 4n = (n-2)^2 - 4 < (n-2)^2$ whenever $n \geq 5$; and $n^2 - 4n < 0$ for $n = 1,\ 2,\ 3$.

This problem was proposed by Ricky Liu.

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

First Solution. For $i \geq 0$ and $k \geq 1$, let $x_{i,k}$ denote the minimum number of jumps needed to reach the integer $n_{i,k} = 2^i k$. We must prove that

$$x_{i,k} > x_{i,1} \tag{1}$$

for all $i \geq 0$ and $k \geq 2$. We prove this using the method of descent.

First note that (1) holds for $i = 0$ and all $k \geq 2$, because it takes 0 jumps to reach the starting value $n_{0,1} = 1$, and at least one jump to reach $n_{0,k} = k \geq 2$. Now assume that that (1) is not true for all choices of i and k. Let i_0 be the minimal value of i for which (1) fails for some k, let k_0 be the minimal value of $k > 1$ for which $x_{i_0,k} \leq x_{i_0,1}$. Then it must be the case that $i_0 \geq 1$ and $k_0 \geq 2$.

Let J_{i_0,k_0} be a shortest sequence of $x_{i_0,k_0}+1$ integers that the frog occupies in jumping from 1 to $2^{i_0}k_0$. The length of each jump, that is, the difference between consecutive integers in J_{i_0,k_0}, is either 1 or a positive integer power of 2. The sequence J_{i_0,k_0} cannot contain 2^{i_0} because it takes more jumps to reach $2^{i_0}k_0$ than it does to reach 2^{i_0}. Let 2^{M+1}, $M \geq 0$

be the length of the longest jump made in generating J_{i_0,k_0}. Such a jump can only be made from a number that is divisible by 2^M (and by no higher power of 2). Thus we must have $M < i_0$, since otherwise a number divisible by 2^{i_0} is visited before $2^{i_0}k_0$ is reached, contradicting the definition of k_0.

Let 2^{m+1} be the length of the jump when the frog jumps over 2^{i_0}. If this jump starts at $2^m(2t-1)$ for some positive integer t, then it will end at $2^m(2t-1)+2^{m+1} = 2^m(2t+1)$. Since it goes over 2^{i_0} we see $2^m(2t-1) < 2^{i_0} < 2^m(2t+1)$ or $(2^{i_0-m}-1)/2 < t < (2^{i_0-m}+1)/2$. Thus $t = 2^{i_0-m-1}$ and the jump over 2^{i_0} is from $2^m(2^{i_0-m}-1) = 2^{i_0} - 2^m$ to $2^m(2^{i_0-m}+1) = 2^{i_0} + 2^m$.

Considering the jumps that generate J_{i_0,k_0}, let N_1 be the number of jumps from 1 to $2^{i_0}+2^m$, and let N_2 be the number of jumps from $= 2^{i_0}+2^m$ to $2^{i_0}k$. By definition of i_0, it follows that 2^m can be reached from 1 in less than N_1 jumps. On the other hand, because $m < i_0$, the number $2^{i_0}(k_0-1)$ can be reached from 2^m in exactly N_2 jumps by using the same jump length sequence as in jumping from $2^m+2^{i_0}$ to $2^{i_0}k_0 = 2^{i_0}(k_0-1)+2^i_0$. The key point here is that the shift by 2^{i_0} does not affect any of divisibility conditions needed to make jumps of the same length. In particular, with the exception of the last entry, $2^{i_0}k_0$, all of the elements of J_{i_0,k_0} are of the form $2^p(2t+1)$ with $p < i_0$, again because of the definition of k_0. Because $2^p(2t+1)-2^{i_0} = 2^p(2t-2^{i_0-p}+1)$ and the number $2t+2^{i_0-p}+1$ is odd, a jump of size 2^{p+1} can be made from $2^p(2t+1)-2^{i_0}$ just as it can be made from $2^p(2t+1)$.

Thus the frog can reach 2^m from 1 in less than N_1 jumps, and can then reach $2^{i_0}(k_0-1)$ from 2^m in N_2 jumps. Hence the frog can reach $2^{i_0}(k_0-1)$ from 1 in less than N_1+N_2 jumps, that is, in fewer jumps than needed to get to $2^{i_0}k_0$ and hence in fewer jumps than required to get to 2^{i_0}. This contradicts the definition of k_0.

Second Solution. Suppose $x_0 = 1, x_1, \dots, x_t = 2^ik$ are the integers visited by the frog on his trip from 1 to 2^ik, $k \geq 2$. Let $s_j = x_j - x_{j-1}$ be the jump sizes. Define a reduced path y_j inductively by

$$y_j = \begin{cases} y_{j-1} + s_j & \text{if } y_{j-1} + s_j \leq 2^i, \\ y_{j-1} & \text{otherwise.} \end{cases}$$

Say a jump s_j is deleted in the second case. We will show that the distinct integers among the y_j give a shorter path from 1 to 2^i. Clearly $y_j \leq 2^i$ for all j. Suppose $2^i - 2^{r+1} < y_j \leq 2^i - 2^r$ for some $0 \leq r \leq i-1$. Then every deleted jump before y_j must have length greater than 2^r, hence must be a multiple of 2^{r+1}. Thus $y_j \equiv x_j \pmod{2^{r+1}}$. If $y_{j+1} > y_j$, then either $s_{j+1} = 1$ (in which case this is a valid jump) or $s_{j+1}/2 = 2^m$ is the exact power of 2 dividing x_j. In the second case, since $2^r \geq s_{j+1} > 2^m$, the congruence says 2^m is also the exact power of 2 dividing y_j, thus again this is a valid jump. Thus the distinct y_j form a valid path for the frog. If $j = t$ the congruence gives $y_t \equiv x_t \equiv 0 \pmod{2^{r+1}}$, but this is impossible for $2^i - 2^{r+1} < y_t \leq 2^i - 2^r$. Hence we see $y_t = 2^i$, that is, the reduced path ends at 2^i. Finally since the reduced path ends at $2^i < 2^i k$ at least one jump must have been deleted and it is strictly shorter than the original path.

This problem was proposed by Zoran Sunik.

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

First Solution. Let P be the second intersection of the circumcircles of triangles TCF and TDE. Because the quadrilateral $PEDT$ is cyclic, $\angle PET = \angle PDT$, or

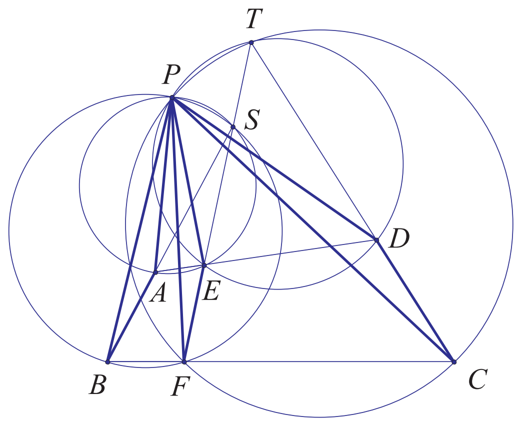
$$\angle PEF = \angle PDC. \qquad (*)$$

Because the quadrilateral $PFCT$ is cyclic,

$$\angle PFE = \angle PFT = \angle PCT = \angle PCD. \qquad (**)$$

By equations $(*)$ and $(**)$, it follows that triangle PEF is similar to triangle PDC. Hence $\angle FPE = \angle CPD$ and $PF/PE = PC/PD$. Note also that $\angle FPC = \angle FPE + \angle EPC = \angle CPD + \angle EPC = \angle EPD$. Thus, triangle EPD is similar to triangle FPC. Another way to say this is that there is a spiral similarity centered at P that sends triangle PFE to triangle PCD, which implies that there is also a spiral similarity, centered at P, that sends triangle PFC to triangle PED, and vice versa. In terms of complex numbers, this amounts to saying that

$$\frac{D-P}{E-P} = \frac{C-P}{F-P} \Longrightarrow \frac{E-P}{F-P} = \frac{D-P}{C-P}.$$



Because $AE/ED = BF/FC$, points A and B are obtained by extending corresponding segments of two similar triangles PED and PFC, namely, DE and CF, by the identical proportion. We conclude that triangle PDA is similar to triangle PCB, implying that triangle PAE is similar to triangle PBF. Therefore, as shown before, we can establish the similarity between triangles PBA and PFE, implying that

$$\angle PBS = \angle PBA = \angle PFE = \angle PFS \quad \text{and} \quad \angle PAB = \angle PEF.$$

The first equation above shows that $PBFS$ is cyclic. The second equation shows that $\angle PAS = 180^\circ - \angle BAP = 180^\circ - \angle FEP = \angle PES$; that is, $PAES$ is cyclic. We conclude that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through point P.

Note. There are two spiral similarities that send segment EF to segment CD. One of them sends E and F to D and C, respectively; the point P is the center of this spiral similarity. The other sends E and F to C and D, respectively; the center of this spiral similarity is the second intersection (other than T) of the circumcircles of triangles TFD and TEC.

Second Solution. We will give a solution using complex coordinates. The first step is the following lemma.

Lemma. *Suppose s and t are real numbers and x, y and z are complex. The circle in the complex plane passing through x, $x + ty$ and $x + (s + t)z$ also passes through the point $x + syz/(y - z)$, independent of t.*

Proof. Four points z_1, z_2, z_3 and z_4 in the complex plane lie on a circle if and only if the cross-ratio

$$cr(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is real. Since we compute

$$cr(x, x + ty, x + (s + t)z, x + syz/(y - z)) = \frac{s + t}{s}$$

the given points are on a circle. □

Lay down complex coordinates with $S = 0$ and E and F on the positive real axis. Then there are real r_1, r_2 and R with $B = r_1 A$, $F = r_2 E$ and $D = E + R(A - E)$ and hence $AE/ED = BF/FC$ gives

$$C = F + R(B - F) = r_2(1 - R)E + r_1 RA.$$

The line CD consists of all points of the form $sC + (1 - s)D$ for real s. Since T lies on this line and has zero imaginary part, we see from $\mathrm{Im}(sC + (1 - s)D) = (sr_1 R + (1 - s)R)\mathrm{Im}(A)$ that it corresponds to $s = -1/(r_1 - 1)$. Thus

$$T = \frac{r_1 D - C}{r_1 - 1} = \frac{(r_2 - r_1)(R - 1)E}{r_1 - 1}.$$

Apply the lemma with $x = E$, $y = A - E$, $z = (r_2 - r_1)E/(r_1 - 1)$, and $s = (r_2 - 1)(r_1 - r_2)$. Setting $t = 1$ gives

$$(x, x + y, x + (s + 1)z) = (E, A, S = 0)$$

and setting $t = R$ gives

$$(x, x + Ry, x + (s + R)z) = (E, D, T).$$

Therefore the circumcircles to SAE and TDE meet at

$$x + \frac{syz}{y - z} = \frac{AE(r_1 - r_2)}{(1 - r_1)E - (1 - r_2)A} = \frac{AF - BE}{A + F - B - E}.$$

This last expression is invariant under simultaneously interchanging A and B and interchanging E and F. Therefore it is also the intersection of the circumcircles of SBF and TCF.

This problem was proposed by Zuming Feng and Zhonghao Ye.

USAMO 2006 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2006 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{ \frac{sm}{p} \right\} < \left\{ \frac{sn}{p} \right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p - 1$.

2. Let $k > 0$ be a fixed integer. Compute the minimum integer N (in terms of k) for which there exists a set of $2k + 1$ distinct positive integers that has sum greater than N, but for which every subset of size k has sum at most $N/2$.

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p(f(n^2)) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

4. Find all positive integers n for which there exist $k \geq 2$ positive rational numbers $a_1, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = a_1 a_2 \ldots a_k = n$.

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n + 1$ or to $n + 2^{m_n + 1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

§1 USAMO 2006/1, proposed by Kiran Kedlaya

Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p - 1$.

It's equivalent to $ms \bmod p < ns \bmod p < s$, where $x \bmod p$ means the remainder when x is divided by p, by slight abuse of notation. We will assume $s \ge 2$ for simplicity, since the case $s = 1$ is clear.

For any $x \in \{1, 2, \dots, s-1\}$ we define $f(x)$ to be the unique number in $\{1, \dots, p-1\}$ such that $s \cdot f(x) \bmod p = x$. Then, m and n fail to exist exactly when

$$f(s-1) < f(s-2) < \cdots < f(1).$$

We give the following explicit description of f: choose $t \equiv -s^{-1} \pmod p$, $0 < t < p$. Then $f(x) = 1 + (s-x) \cdot t \bmod p$. So our displayed inequality is equivalent to

$$(1+t) \bmod p < (1+2t) \bmod p < (1+3t) \bmod p < \cdots < (1+(s-1)t) \bmod p.$$

This just means that the sequence $1 + kt$ never "wraps around" modulo p as we take $k = 1, 2, \dots, s-1$.

Since we assumed $s \ne 1$, we have $0 < 1 + t < p$. Now since $1 + kt$ never wraps around as $k = 1, 2, \dots, s-1$, and increases in increments of t, it follows that $1 + kt < p$ for all $k = 1, 2, \dots, s-1$. Finally, as $1 + st \equiv 0 \pmod p$ we get $1 + st = p$.

In summary, m, n fail to exist precisely when $1 + st = p$. That is of course equivalent to $s \mid p - 1$.

§2 USAMO 2006/2, proposed by Dick Gibbs

Let $k > 0$ be a fixed integer. Compute the minimum integer N (in terms of k) for which there exists a set of $2k+1$ distinct positive integers that has sum greater than N, but for which every subset of size k has sum at most $N/2$.

The answer is $N = k(2k^2+3k+3)$ given by

$$S = \left\{k^2+1, k^2+2, \dots, k^2+2k+1\right\}.$$

To show this is best possible, let the set be $S = \{a_0 < a_1 < \dots < a_{2k}\}$ so that the hypothesis becomes

$$\begin{aligned} N+1 &\le a_0 + a_1 + \dots + a_{2k} \\ N/2 &\ge a_{k+1} + \dots + a_{2k}. \end{aligned}$$

Subtracting twice the latter from the former gives

$$\begin{aligned} a_0 &\ge 1 + (a_{k+1} - a_1) + (a_{k+2} - a_2) + \dots + (a_{2k} - a_k) \\ &\ge 1 + \underbrace{k + k + \dots + k}_{k \text{ terms}} = 1 + k^2. \end{aligned}$$

Now, we have

$$\begin{aligned} N/2 &\ge a_{k+1} + \dots + a_{2k} \\ &\ge (a_0 + (k+1)) + (a_0 + (k+2)) + \dots + (a_0 + 2k) \\ &= k \cdot a_0 + ((k+1) + \dots + 2k) \\ &\ge k(k^2+1) + k \cdot \frac{3k+1}{2} \end{aligned}$$

so $N \ge k(2k^2+3k+3)$.

Remark. The exact value of N is therefore very superficial. From playing with these concrete examples we find out we are essentially just trying to find an increasing set S obeying

$$a_0 + a_1 + \dots + a_k > a_{k+1} + \dots + a_{2k} \qquad (\star)$$

and indeed given a sequence satisfying these properties one simply sets $N = 2(a_{k+1} + \dots + a_{2k})$. Therefore we can focus almost entirely on a_i and not N.

Remark. It is relatively straightforward to figure out what is going on based on the small cases. For example, one can work out by hand that

- $\{2,3,4\}$ is optimal for $k=1$
- $\{5,6,7,8,9\}$ is optimal for $k=2$,
- $\{10,11,12,13,14,15,16\}$ is optimal for $k=3$.

In all the examples, the a_i are an arithmetic progression of difference 1, so that $a_j - a_i \ge j - i$ is a sharp for all $i < j$, and thus this estimate may be used freely without loss of sharpness; applying it in $(\star)$ gives a lower bound on a_0 which is then good enough to get a lower bound on N matching the equality cases we found empirically.

§3 USAMO 2006/3, proposed by Titu Andreescu and Gabriel Dospinescu

For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p(f(n^2)) - 2n\}_{n \ge 0}$$

is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \ge 0$.)

If f is the (possibly empty) product of linear factors of the form $4n - a^2$, then it satisfies the condition. We will prove no other polynomials work. In what follows, assume f is irreducible and nonconstant.

It suffices to show for every positive integer c, there exists a prime p and a nonnegative integer n such that $n \le \frac{p-1}{2} - c$ and p divides $f(n^2)$.

Firstly, recall there are infinitely many odd primes p, with $p > c$, such that p divides some $f(n^2)$, by Schur's Theorem. Looking mod such a p we can find n between 0 and $\frac{p-1}{2}$ (since $n^2 \equiv (-n)^2 \pmod p$). We claim that only finitely many p from this set can fail now. For if a p fails, then its n must be between $\frac{p-1}{2} - c$ and $\frac{p-1}{2}$. That means for some $0 \le k \le c$ we have

$$0 \equiv f\left(\left(\frac{p-1}{2} - k\right)^2\right) \equiv f\left(\left(k + \frac{1}{2}\right)^2\right) \pmod p.$$

There are only finitely many p dividing

$$\prod_{k=1}^{c} f\left(\left(k + \frac{1}{2}\right)^2\right)$$

unless one of the terms in the product is zero; this means that $4n - (2k+1)^2$ divides $f(n)$. This establishes the claim and finishes the problem.

§4 USAMO 2006/4, proposed by Ricky Liu

Find all positive integers n for which there exist $k \ge 2$ positive rational numbers $a_1, \dots, a_k$ satisfying $a_1 + a_2 + \dots + a_k = a_1 a_2 \dots a_k = n$.

The answer is all n other than $1, 2, 3, 5$.

Claim — The only solution with $n \le 5$ is $n = 4$.

Proof. The case $n = 4$ works since $2 + 2 = 2 \cdot 2 = 4$. So assume $n > 4$.

We now contend that $k > 2$. Indeed, if $a_1 + a_2 = a_1 a_2 = n$ then $(a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1 a_2 = n^2 - 4n = (n-2)^2 - 4$ is a rational integer square, hence a perfect square. This happens only when $n = 4$.

Now by AM-GM,

$$\frac{n}{k} = \frac{a_1 + \dots + a_k}{k} \ge \sqrt[k]{a_1 \dots a_k} = n^{1/k}$$

and so $n \ge k^{\frac{1}{1-1/k}} = k^{\frac{k}{k-1}}$. This last quantity is always greater than 5, since

$$\begin{aligned} 3^{3/2} &= 3\sqrt{3} > 5 \\ 4^{4/3} &= 4\sqrt[3]{4} > 5 \\ k^{\frac{k}{k-1}} &> k \ge 5 \qquad \forall k \ge 5. \end{aligned}$$

This finishes the proof. □

Now, in general:

- If $n \ge 6$ is even, we may take $(a_1, \dots, a_{n/2}) = (n/2, 2, 1, \dots, 1)$.
- If $n \ge 9$ is odd, we may take $(a_1, \dots, a_{(n-3)/2}) = (n/2, 1/2, 4, 1, \dots, 1)$.
- A special case $n = 7$: one example is $(4/3, 7/6, 9/2)$, another is $(7/6, 4/3, 3/2, 3)$.

Remark. The main hurdle in the problem is the $n = 7$ case. One good reason to believe a construction exists is that it seems quite difficult to prove that $n = 7$ fails.

§5 USAMO 2006/5, proposed by Zoran Sunik

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \ge 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

We will think about the problem in terms of finite sequences of jumps $(s_1, s_2, \dots, s_\ell)$, which we draw as

$$1 = x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \dots \xrightarrow{s_\ell} x_\ell$$

where $s_k = x_k - x_{k-1}$ is the length of some hop. We say the sequence is *valid* if it has the property required by the problem: for each k, either $s_k = 1$ or $s_k = 2^{m_{x_{k-1}}+1}$.

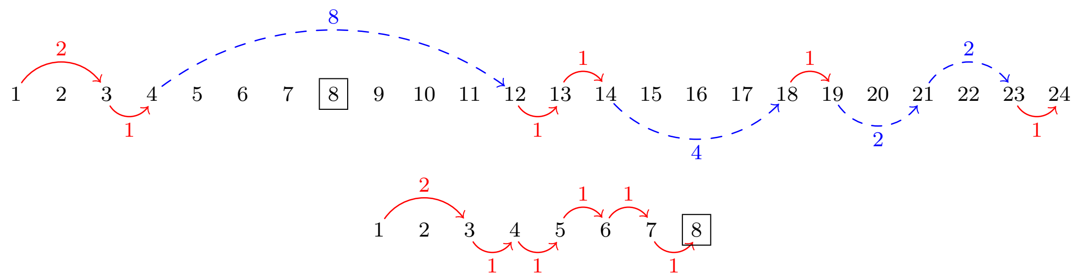
An example is shown below.

Lemma

Let $(s_1, \dots, s_\ell)$ be a sequence of jumps. Suppose we delete pick an index k and exponent $e > 0$, and delete any jumps after the kth one which are divisible by 2^e. The resulting sequence is still valid.

Proof. We only have to look after the kth jump. The launching points of the remaining jumps after the kth one are now shifted by multiples of 2^e due to the deletions; so given a jump $x \xrightarrow{s} x+s$ we end up with a jump $x' \xrightarrow{s} x'+s$ where $x - x'$ is a multiple of 2^e.

But since $s < 2^e$, we have $\nu_2(x') < e$ and hence $\nu_2(x) = \nu_2(x')$ so the jump is valid. $\square$



Now let's consider a valid path to $2^i k$ with ℓ steps, say

$$1 = x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \dots \xrightarrow{s_\ell} x_\ell = 2^i \cdot k$$

where $s_i = x_i - x_{i-1}$ is the distance jumped.

We delete jumps in the following way: starting from the largest e and going downwards until $e = 0$, we delete all the jumps of length 2^e which end at a point exceeding the target 2^i.

By the lemma, at each stage, the path remains valid. We claim more:

Claim — Let $e \ge 0$. After the jumps of length greater than 2^e are deleted, the resulting end-point is at least 2^i, and divisible by $2^{\min(i,e)}$.

Proof. By downwards induction. Consider any step where *some* jump is deleted. Then, the end-point must be strictly greater than $x = 2^i - 2^e$ (i.e. we must be within 2^e of the target 2^i).

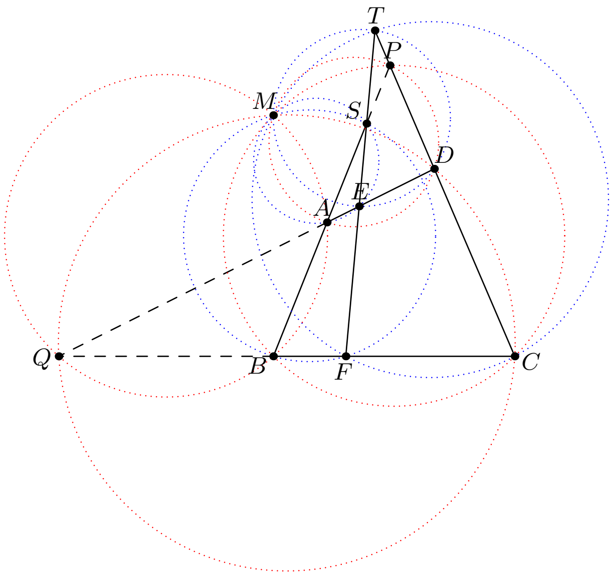
It is also divisible by $2^{\min(i,e)}$ by induction hypothesis, since we are changing the end-point by multiples of 2^e. And the smallest multiple of $2^{\min(i,e)}$ exceeding x is 2^i. $\square$

On the other hand by construction when the process ends the reduced path ends at a point at most 2^i, so it is 2^i as desired.

Therefore we have taken a path to $2^i k$ and reduced it to one to 2^i by deleting some jumps. This proves the result.

§6 USAMO 2006/6, proposed by Zuming Feng and Zhonghao Ye

Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.



Let M be the Miquel point of $ABCD$. Then M is the center of a spiral similarity taking AD to BC. The condition guarantees that it also takes E to F. Hence, we see that M is the center of a spiral similarity taking $\overline{AB}$ to $\overline{EF}$, and consequently the circumcircles of QAB, QEF, SAE, SBF concur at point M.

In other words, the Miquel point of $ABCD$ is also the Miquel point of $ABFE$. Similarly, M is also the Miquel point of $EDCF$, so all four circles concur at M.

36^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 24, 2007

1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

2. A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

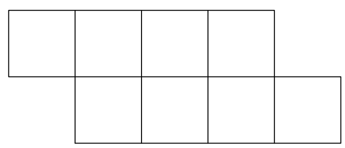
3. Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

36^{th} United States of America Mathematical Olympiad

Day II   12:30 PM – 5 PM EDT

April 25, 2007

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells.[1] The figure below shows an 8-cell animal.



A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

5. Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

6. Let ABC be an acute triangle with ω, Ω, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and Ω_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \le R^3,$$

with equality if and only if triangle ABC is equilateral.

[1] Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n-cells, one with $n + 1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

36th United States of America Mathematical Olympiad

1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

First Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We have

$$\frac{s_{k+1}}{k+1} < \frac{s_{k+1}}{k} = \frac{s_k + a_{k+1}}{k} \le \frac{s_k + k}{k} = \frac{s_k}{k} + 1.$$

On the other hand, for each k, s_k/k is a positive integer. Therefore

$$\frac{s_{k+1}}{k+1} \le \frac{s_k}{k},$$

and the sequence of quotients s_k/k is eventually constant. If $s_{k+1}/(k+1) = s_k/k$, then

$$a_{k+1} = s_{k+1} - s_k = \frac{(k+1)s_k}{k} - s_k = \frac{s_k}{k},$$

showing that the sequence a_k is eventually constant as well.

Second Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k \qquad \text{and} \qquad \frac{s_k}{k} = q_k.$$

Since $a_k \le k-1$, for $k \ge 2$, we have

$$s_k = a_1 + a_2 + a_3 + \cdots + a_k \le n + 1 + 2 + \cdots + (k-1) = n + \frac{k(k-1)}{2}.$$

Let m be a positive integer such that $n \le \frac{m(m+1)}{2}$ (such an integer clearly exists). Then

$$q_m = \frac{s_m}{m} \le \frac{n}{m} + \frac{m-1}{2} \le \frac{m+1}{2} + \frac{m-1}{2} = m.$$

We claim that

$$q_m = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \ldots \ .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \ldots$ is uniquely determined and choosing $a_{m+i} = q_m$, for $i \geq 1$, satisfies the range condition

$$0 \leq a_{m+i} = q_m \leq m \leq m + i - 1,$$

and yields

$$s_{m+i} = s_m + iq_m = mq_m + iq_m = (m+i)q_m.$$

Third Solution: For $k \geq 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We claim that for some m we have $s_m = m(m-1)$. To this end, consider the sequence which computes the differences between s_k and $k(k-1)$, i.e., whose k-th term is $s_k - k(k-1)$. Note that the first term of this sequence is positive (it is equal to n) and that its terms are strictly decreasing since

$$(s_k - k(k-1)) - (s_{k+1} - (k+1)k) = 2k - a_{k+1} \geq 2k - k = k \geq 1.$$

Further, a negative term cannot immediately follow a positive term. Suppose otherwise, namely that $s_k > k(k-1)$ and $s_{k+1} < (k+1)k$. Since s_k and s_{k+1} are divisible by k and $k+1$, respectively, we can tighten the above inequalities to $s_k \geq k^2$ and $s_{k+1} \leq (k+1)(k-1) = k^2 - 1$. But this would imply that $s_k > s_{k+1}$, a contradiction. We conclude that the sequence of differences must eventually include a term equal to zero.

Let m be a positive integer such that $s_m = m(m-1)$. We claim that

$$m - 1 = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \ldots \ .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \ldots$ is uniquely determined and choosing $a_{m+i} = m - 1$, for $i \geq 1$, satisfies the range condition

$$0 \leq a_{m+i} = m - 1 \leq m + i - 1,$$

and yields

$$s_{m+i} = s_m + i(m-1) = m(m-1) + i(m-1) = (m+i)(m-1).$$

This problem was suggested by Sam Vandervelde.

2. A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Solution: It is not possible. The proof is by contradiction. Suppose that such a covering family $\mathcal{F}$ exists. Let $D(P, \rho)$ denote the disc with center P and radius ρ. Start with an arbitrary disc $D(O, r)$ that does not overlap any member of $\mathcal{F}$. Then $D(O, r)$ covers no grid point. Take the disc $D(O, r)$ to be maximal in the sense that any further enlargement would cause it to violate the non-overlap condition. Then $D(O, r)$ is tangent to at least three discs in $\mathcal{F}$. Observe that there must be two of the three tangent discs, say $D(A, a)$ and $D(B, b)$, such that $\angle AOB \leq 120°$. By the Law of Cosines applied to triangle ABO,

$$(a + b)^2 \leq (a + r)^2 + (b + r)^2 + (a + r)(b + r),$$

which yields

$$ab \leq 3(a + b)r + 3r^2, \qquad \text{and thus} \qquad 12r^2 \geq (a - 3r)(b - 3r).$$

Note that $r < 1/\sqrt{2}$ because $D(O, r)$ covers no grid point, and $(a-3r)(b-3r) \geq (5-3r)^2$ because each disc in $\mathcal{F}$ has radius at least 5. Hence $2\sqrt{3}r \geq (5 - 3r)$, which gives $5 \leq (3+2\sqrt{3})r < (3+2\sqrt{3})/\sqrt{2}$ and thus $5\sqrt{2} < 3+2\sqrt{3}$. Squaring both sides of this inequality yields $50 < 21 + 12\sqrt{3} < 21 + 12 \cdot 2 = 45$. This contradiction completes the proof.

Remark: The above argument shows that no covering family exists where each disc has radius greater than $(3 + 2\sqrt{3})/\sqrt{2} \approx 4.571$. In the other direction, there exists a covering family in which each disc has radius $\sqrt{13}/2 \approx 1.802$. Take discs with this radius centered at points of the form $(2m + 4n + \frac{1}{2}, 3m + \frac{1}{2})$, where m and n are integers. Then any grid point is within $\sqrt{13}/2$ of one of the centers and the distance between any two centers is at least $\sqrt{13}$. The extremal radius of a covering family is unknown.

This problem was suggested by Gregory Galperin.

3. Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Solution: In order to apply induction, we generalize the result to be proved so that it reads as follows:

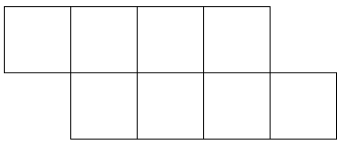
Proposition. If the n-element subsets of a set S with $(n+1)m-1$ elements are partitioned into two classes, then there are at least m pairwise disjoint sets in the same class.

Proof. Fix n and proceed by induction on m. The case of $m = 1$ is trivial. Assume $m > 1$ and that the proposition is true for $m - 1$. Let $\mathcal{P}$ be the partition of the n-element subsets into two classes. If all the n-element subsets belong to the same class, the result is obvious. Otherwise select two n-element subsets A and B from different classes so that their intersection has maximal size. It is easy to see that $|A \cap B| = n - 1$. (If $|A \cap B| = k < n-1$, then build C from B by replacing some element not in $A \cap B$ with an element of A not already in B. Then $|A \cap C| = k+1$ and $|B \cap C| = n-1$ and either A and C or B and C are in different classes.) Removing $A \cup B$ from S, there are $(n+1)(m-1)-1$ elements left. On this set the partition induced by $\mathcal{P}$ has, by the inductive hypothesis, $m-1$ pairwise disjoint sets in the same class. Adding either A or B as appropriate gives m pairwise disjoint sets in the same class. □

Remark: The value $n^2 + n - 1$ is sharp. A set S with $n^2 + n - 2$ elements can be split into a set A with $n^2 - 1$ elements and a set B of $n - 1$ elements. Let one class consist of all n-element subsets of A and the other consist of all n-element subsets that intersect B. Then neither class contains n pairwise disjoint sets.

This problem was suggested by András Gyárfás.

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells.[1] The figure below shows an 8-cell animal.



A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

[1] Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n-cells, one with $n + 1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

Solution: Let s denote the minimum number of cells in a dinosaur; the number this year is $s = 2007$.

Claim: The maximum number of cells in a primitive dinosaur is $4(s-1)+1$.

First, a primitive dinosaur can contain up to $4(s-1)+1$ cells. To see this, consider a dinosaur in the form of a cross consisting of a central cell and four arms with $s-1$ cells apiece. No connected figure with at least s cells can be removed without disconnecting the dinosaur.

The proof that no dinosaur with at least $4(s-1)+2$ cells is primitive relies on the following result.

Lemma. *Let D be a dinosaur having at least $4(s-1)+2$ cells, and let R (red) and B (black) be two complementary animals in D, i.e., $R \cap B = \varnothing$ and $R \cup B = D$. Suppose $|R| \leq s-1$. Then R can be augmented to produce animals $\tilde{R} \supset R$ and $\tilde{B} = D \setminus \tilde{R}$ such that at least one of the following holds:*

(i) $|\tilde{R}| \geq s$ *and* $|\tilde{B}| \geq s$,

(ii) $|\tilde{R}| = |R| + 1$,

(iii) $|R| < |\tilde{R}| \leq s-1$.

Proof. If there is a black cell adjacent to R that can be made red without disconnecting B, then (ii) holds. Otherwise, there is a black cell c adjacent to R whose removal disconnects B. Of the squares adjacent to c, at least one is red, and at least one is black, otherwise B would be disconnected. Then there are at most three resulting components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of B after the removal of c. Without loss of generality, $\mathcal{C}_3$ is the largest of the remaining components. (Note that $\mathcal{C}_1$ or $\mathcal{C}_2$ may be empty.) Now $\mathcal{C}_3$ has at least $\lceil (3s-2)/3 \rceil = s$ cells. Let $\tilde{B} = \mathcal{C}_3$. Then $|\tilde{R}| = |R| + |\mathcal{C}_1| + |\mathcal{C}_2| + 1$. If $|\tilde{B}| \leq 3s-2$, then $|\tilde{R}| \geq s$ and (i) holds. If $|\tilde{B}| \geq 3s-1$ then either (ii) or (iii) holds, depending on whether $|\tilde{R}| \geq s$ or not. □

Starting with $|R| = 1$, repeatedly apply the Lemma. Because in alternatives (ii) and (iii) $|R|$ increases but remains less than s, alternative (i) eventually must occur. This shows that no dinosaur with at least $4(s-1)+2$ cells is primitive.

This problem was suggested by Reid Barton.

5. Prove that for every nonnegative integer n, the number $7^{7^n}+1$ is the product of at least $2n+3$ (not necessarily distinct) primes.

Solution: The proof is by induction. The base is provided by the $n=0$ case, where $7^{7^0}+1=7^1+1=2^3$. To prove the inductive step, it suffices to show that if $x=7^{2m-1}$ for some positive integer m then $(x^7+1)/(x+1)$ is composite. As a consequence, x^7+1 has at least two more prime factors than does $x+1$. To confirm that $(x^7+1)/(x+1)$ is composite, observe that

$$\begin{aligned}\frac{x^7+1}{x+1} &= \frac{(x+1)^7-((x+1)^7-(x^7+1))}{x+1}\\ &= (x+1)^6-\frac{7x(x^5+3x^4+5x^3+5x^2+3x+1)}{x+1}\\ &= (x+1)^6-7x(x^4+2x^3+3x^2+2x+1)\\ &= (x+1)^6-7^{2m}(x^2+x+1)^2\\ &= \{(x+1)^3-7^m(x^2+x+1)\}\{(x+1)^3+7^m(x^2+x+1)\}\end{aligned}$$

Also each factor exceeds 1. It suffices to check the smaller one; $\sqrt{7x}\le x$ gives

$$\begin{aligned}(x+1)^3-7^m(x^2+x+1) &= (x+1)^3-\sqrt{7x}(x^2+x+1)\\ &\ge x^3+3x^2+3x+1-x(x^2+x+1)\\ &= 2x^2+2x+1\ge 113>1.\end{aligned}$$

Hence $(x^7+1)/(x+1)$ is composite and the proof is complete.

This problem was suggested by Titu Andreescu.

6. Let ABC be an acute triangle with ω,Ω, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and Ω_A, respectively. Define points P_B,Q_B,P_C,Q_C analogously. Prove that

$$8P_AQ_A\cdot P_BQ_B\cdot P_CQ_C\le R^3,$$

with equality if and only if triangle ABC is equilateral.

Solution: Let the incircle touch the sides AB, BC, and CA at C_1, A_1, and B_1, respectively. Set $AB = c$, $BC = a$, $CA = b$. By equal tangents, we may assume that $AB_1 = AC_1 = x$, $BC_1 = BA_1 = y$, and $CA_1 = CB_1 = z$. Then $a = y + z$, $b = z + x$, $c = x + y$. By the AM-GM inequality, we have $a \geq 2\sqrt{yz}$, $b \geq 2\sqrt{zx}$, and $c \geq 2\sqrt{xy}$. Multiplying the last three inequalities yields

$$abc \geq 8xyz, \qquad (\dagger),$$

with equality if and only if $x = y = z$; that is, triangle ABC is equilateral.

Let k denote the area of triangle ABC. By the Extended Law of Sines, $c = 2R \sin \angle C$. Hence

$$k = \frac{ab \sin \angle C}{2} = \frac{abc}{4R} \quad \text{or} \quad R = \frac{abc}{4k}. \qquad (\ddagger)$$

We are going to show that

$$P_AQ_A = \frac{xa^2}{4k}. \qquad (*)$$

In exactly the same way, we can also establish its cyclic analogous forms

$$P_BQ_B = \frac{yb^2}{4k} \quad \text{and} \quad P_CQ_C = \frac{zc^2}{4k}.$$

Multiplying the last three equations together gives

$$P_AQ_A \cdot P_BQ_B \cdot P_CQ_C = \frac{xyza^2b^2c^2}{64k^3}.$$

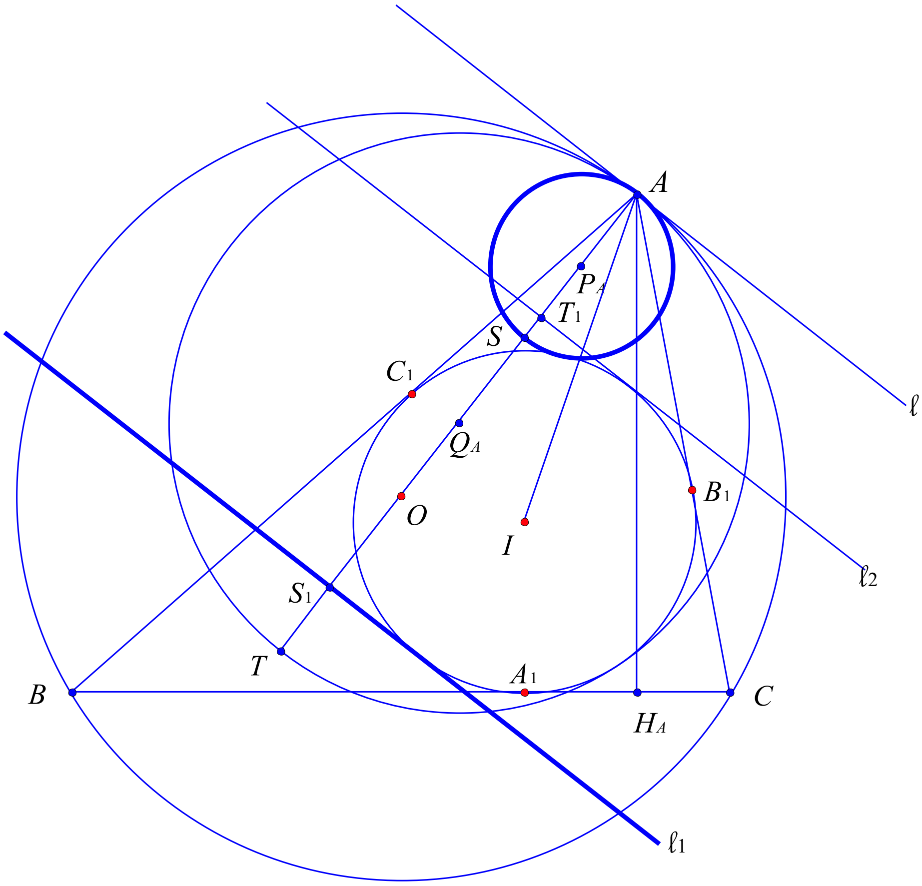
Further considering ($\dagger$) and ($\ddagger$), we have

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQC = \frac{8xyza^2b^2c^2}{64k^3} \leq \frac{a^3b^3c^3}{64k^3} = R^3,$$

with equality if and only if triangle ABC is equilateral.

Hence it suffices to show ($*$). Let r, r_A, r'_A denote the radii of $\omega, \omega_A, \Omega_A$, respectively. We consider the inversion $\mathbf{I}$ with center A and radius x. Clearly, $\mathbf{I}(B_1) = B_1$, $\mathbf{I}(C_1) = C_1$, and $\mathbf{I}(\omega) = \omega$. Let ray AO intersect ω_A and Ω_A at S and T, respectively. It is not difficult to see that $AT > AS$, because ω is tangent to ω_A and Ω_A externally and internally, respectively. Set $S_1 = \mathbf{I}(S)$ and $T_1 = \mathbf{I}(T)$. Let ℓ denote the line tangent to Ω at A. Then the image of ω_A (under the inversion) is the line (denoted by ℓ_1) passing through S_1 and parallel to ℓ, and the image of Ω_A is the line (denoted by ℓ_2) passing through T_1 and parallel to

ℓ. Furthermore, since ω is tangent to both ω_A and Ω_A, ℓ_1 and ℓ_2 are also tangent to the image of ω, which is ω itself. Thus the distance between these two lines is $2r$; that is, $S_1T_1 = 2r$. Hence we can consider the following configuration. (The darkened circle is ω_A, and its image is the darkened line ℓ_1.)



By the definition of inversion, we have $AS_1 \cdot AS = AT_1 \cdot AT = x^2$. Note that $AS = 2r_A$, $AT = 2r'_A$, and $S_1T_1 = 2r$. We have

$$r_A = \frac{x^2}{2AS_1}. \quad \text{and} \quad r'_A = \frac{x^2}{2AT_1} = \frac{x^2}{2(AS_1 - 2r)}.$$

Hence

$$P_AQ_A = AQ_A - AP_A = r'_A - r_A = \frac{x^2}{2}\left(\frac{1}{AS_1 - 2r} + \frac{1}{AS_1}\right).$$

Let H_A be the foot of the perpendicular from A to side BC. It is well known that $\angle BAS_1 = \angle BAO = 90^\circ - \angle C = \angle CAH_A$. Since ray AI bisects $\angle BAC$, it follows that rays AS_1 and AH_A are symmetric with respect to ray AI. Further note that both line ℓ_1

(passing through S_1) and line BC (passing through H_A) are tangent to ω. We conclude that $AS_1 = AH_A$. In light of this observation and using the fact $2k = AH_A \cdot BC = (AB + BC + CA)r$, we can compute P_AQ_A as follows:

$$\begin{aligned}
P_AQ_A &= \frac{x^2}{2}\left(\frac{1}{AH_A - 2r} - \frac{1}{AH_A}\right) = \frac{x^2}{4k}\left(\frac{2k}{AH_A - 2r} - \frac{2k}{AH_A}\right) \\
&= \frac{x^2}{4k}\left(\frac{1}{\frac{1}{BC} - \frac{2}{AB+BC+CA}} - BC\right) = \frac{x^2}{4k}\left(\frac{1}{\frac{1}{y+z} - \frac{1}{x+y+z}} - (y+z)\right) \\
&= \frac{x^2}{4k}\left(\frac{(y+z)(x+y+z)}{x} - (y+z)\right) \\
&= \frac{x(y+z)^2}{4k} = \frac{xa^2}{4k},
\end{aligned}$$

establishing $(*)$. Our proof is complete.

Note: Trigonometric solutions of $(*)$ are also possible.
Query: For a given triangle, how can one construct ω_A and Ω_A by ruler and compass?

This problem was suggested by Kiran Kedlaya and Sungyoon Kim.

USAMO 2007 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2007 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. (For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \dots$.) Prove that for any n the sequence a_1, a_2, ... eventually becomes constant.

2. Decide whether it possible to cover all lattice points in $\mathbb{R}^2$ by an (infinite) family of disks whose interiors are disjoint such that the radius of each disk is at least 5.

3. Let S be a set containing $n^2 + n - 1$ elements. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells (equivalently, a polyomino with n cells). A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

5. Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

6. Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

   Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

   $$8 P_A Q_A \cdot P_B Q_B \cdot P_C Q_C \le R^3$$

   with equality if and only if triangle ABC is equilateral.

§1 USAMO 2007/1, proposed by Sam Vandervelde

Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. (For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \dots$.) Prove that for any n the sequence $a_1, a_2, \dots$ eventually becomes constant.

For each k, the number

$$b_k \overset{\text{def}}{=} \frac{1}{k}(a_1 + \cdots + a_k)$$

is a nonnegative integer. Moreover, since

$$b_{k+1} = \frac{a_1 + \cdots + a_k + a_{k+1}}{k+1} < \frac{kb_k + k}{k+1} < b_k$$

the sequence b_k must eventually be constant. This can only happen once the sequence is constant.

§2 USAMO 2007/2, proposed by Gregory Galperin

Decide whether it possible to cover all lattice points in $\mathbb{R}^2$ by an (infinite) family of disks whose interiors are disjoint such that the radius of each disk is at least 5.

The answer is no.

Assume not. Take a disk $\odot O$ not touching any member of the family, and then enlarge it until it is maximal. Then, it must be tangent to at least three other disks, say $\odot A$, $\odot B$, $\odot C$. Suppose WLOG that $\angle AOB \le 120^\circ$. Denote the radii of $\odot O$, $\odot A$, $\odot B$ by r, a, b.

But the Law of Cosines gives

$$(a+b)^2 \le (a+r)^2 + (b+r)^2 + (a+r)(b+r)$$

which rewrites as

$$12r^2 \ge (a-3r)(b-3r) \ge (5-3r)^2$$

which one can check is impossible for $r \le 1/\sqrt{2}$. Thus $r > 1/\sqrt{2}$.

In particular $(\odot O)$ must contain a lattice point as it contains a unit square.

Remark. The order of the argument here matters in subtle ways. A common approach is to try and reduce to the "optimal" case where we have three mutually tangent circles, and then apply the Descarte circle theorem. There are ways in which this approach can fail if the execution is not done with care. (In particular, one cannot simply say to reduce to this case, without some justification.)

For example: it is not true that, given an infinite family of disks, we can enlarge disks until we get three mutually tangent ones. As a counterexample consider the "square grid" in which a circle is centered at $(10m, 10n)$ for each $m, n \in \mathbb{Z}$ and has radius 5. Thus it is also not possible to simply pick three nearby circles and construct a circle tangent to all three: that newly constructed circle might intersect a fourth disk not in the picture.

Thus, when constructing the small disk $\odot O$ in the above solution, it seems easiest to start with a point not covered and grow $\odot O$ until it is tangent to *some* three circles, and then argue by cosine law. Otherwise it not easy to determine which three circles to start with.

In all solutions it seems easier to prove that a disjoint circle of radius $1/\sqrt{2}$ exists, and then *finally* deduce it has a lattice point, rather than trying to work the lattice point into the existence proof.

§3 USAMO 2007/3, proposed by Andras Gyarfas

Let S be a set containing n^2+n-1 elements. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

We present two solutions which are really equivalent, but phrased differently. We refer to the two classes as "red" and "blue", respectively.

First solution (Grant Yu) We define a set of $n+1$ elements to be *useful* if it has n-element subsets in each class.

Consider a **maximal collection of disjoint useful sets** and assume there are p such sets. Then, let T be the set of elements remaining (i.e. not in one of chosen useful sets).

Claim — All subsets of T of size n are the same color.

Proof. Assume there was a red set R in T. Replace the elements of R one by one until we get to any other subset R' of T. At each step, because no sets of T form a useful set, the set remains red — so R' is red too. Since R' is arbitrary, this proves the claim. □

We have $|T| = n^2+n-1-p(n+1)$, and in particular $p < n$. WLOG all sets in T are red. We can extract another red set from each of our chosen useful sets. So we can get at least

$$p + \left\lfloor \frac{|T|}{n} \right\rfloor = p + \left\lfloor n+1-p-\frac{1+p}{n} \right\rfloor \ge p + (n-p) = n.$$

Second solution (by induction) We prove more strongly that:

Claim — Let S be a set containing $k \cdot (n+1) - 1$ elements. Then we can find k pairwise disjoint sets of the same color.

The proof is by induction on $k \ge 1$. The base case $k=1$ this is immediate; $\binom{S}{n}$ is a single set.

For the inductive step, assume for contradiction the problem fails. Let T be any subset of S of size $(k-1)(n+1)-1$. By the induction hypothesis, among the subsets of T alone, we can already find $k-1$ pairwise disjoint sets of the same color. Now $S \setminus T$ has size $k+1$, and so we would have to have that all $\binom{k+1}{k}$ subsets of $S \setminus T$ are the same color.

By varying T, the set $S \setminus T$ ranges over all of $\binom{S}{k+1}$. This causes all sets to be the same color, contradiction.

Remark. Victor Wang writes the following:

> I don't really like this problem, but I think the main motivation for generalizing the problem is that the original problem doesn't allow you to look at small cases. (Also, it's not initially clear where the n^2+n-1 comes from.) And pretty much the simplest way to get lots of similarly-flavored small cases is to start with $k=2,3$ in "find the smallest $N(n,k)$ such that when we partition the n-subsets of a $\ge N(n,k)$-set into 2 classes, we can find some k pairwise disjoint sets in the same class".

§4 USAMO 2007/4, proposed by Reid Barton

An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells (equivalently, a polyomino with n cells). A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

In fact it's true for any tree with maximum degree ≤ 4. Here is the solution of Andrew Geng.

Let T be such a tree (a spanning tree of the dinosaur graph).

Claim — There exists a vertex v such that when v is deleted, no dinosaurs result.

Proof. Assume for contradiction that all vertices are bad (leave a dinosaur when deleted). Consider two adjacent vertices v, w in T. By checking possibilities, one sees that, say, the dinosaur in $T - v$ contains w and the dinosaur of $T - w$. We can repeat in this way; since T is acyclic, this eventually becomes a contradiction. □

When this vertex is deleted, we get at most 4 components, each with ≤ 2006 vertices, giving the answer of $4 \cdot 2006 + 1 = 8025$. The construction is easy (take a "cross", for example).

§5 USAMO 2007/5, proposed by Titu Andreescu

Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

We prove this by induction on n by showing that

$$\frac{X^7 + 1}{X + 1} = X^6 - X^5 + \cdots + 1$$

is never prime for $X = 7^{7^n}$, hence we gain at least two additional prime factors whenever we increase n by one.

Indeed, the quotient may be written as

$$(X + 1)^6 - 7X \cdot (X^2 + X + 1)^2$$

which becomes a difference of squares, hence composite.

§6 USAMO 2007/6, proposed by Sung-Yoon Kim

Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \le R^3$$

with equality if and only if triangle ABC is equilateral.

It turns out we can compute P_AQ_A explicitly. Let us invert around A with radius $s-a$ (hence fixing the incircle) and then compose this with a reflection around the angle bisector of $\angle BAC$. We denote the image of the composed map via

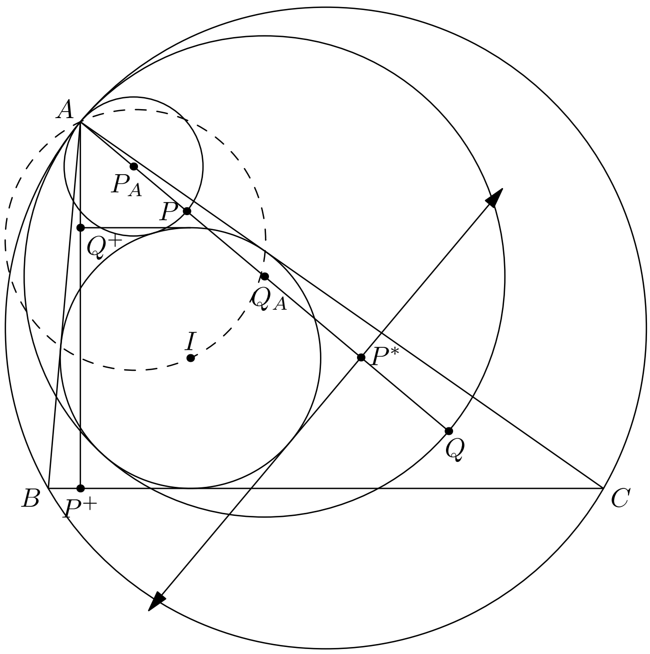
$$\bullet \mapsto \bullet^* \mapsto \bullet^+.$$

We overlay this inversion with the original diagram.

Let P_AQ_A meet ω_A again at P and S_A again at Q. Now observe that ω_A^* is a line parallel to S^*; that is, it is perpendicular to $\overline{PQ}$. Moreover, it is tangent to $\omega^* = \omega$.

Now upon the reflection, we find that $\omega^+ = \omega^* = \omega$, but line $\overline{PQ}$ gets mapped to the altitude from A to $\overline{BC}$, since $\overline{PQ}$ originally contained the circumcenter O (isogonal to the orthocenter). But this means that ω_A^* is none other than the $\overline{BC}$! Hence P^+ is actually the foot of the altitude from A onto $\overline{BC}$.

By similar work, we find that Q^+ is the point on $\overline{AP^+}$ such that $P^+Q^+ = 2r$.



Now we can compute all the lengths directly. We have that

$$AP_A = \frac{1}{2}AP = \frac{(s-a)^2}{2AP^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a}$$

and

$$AQ_A = \frac{1}{2}AQ = \frac{(s-a)^2}{2AQ^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a - 2r}$$

where $h_a = \frac{2K}{a}$ is the length of the A-altitude, with K the area of ABC as usual. Now it follows that

$$P_A Q_A = \frac{1}{2}(s-a)^2 \left(\frac{2r}{h_a(h_a - 2r)} \right).$$

This can be simplified, as

$$h_a - 2r = \frac{2K}{a} - \frac{2K}{s} = 2K \cdot \frac{s-a}{as}.$$

Hence

$$P_A Q_A = \frac{a^2 rs(s-a)}{4K^2} = \frac{a^2(s-a)}{4K}.$$

Hence, the problem is just asking us to show that

$$a^2 b^2 c^2 (s-a)(s-b)(s-c) \le 8(RK)^3.$$

Using $abc = 4RK$ and $(s-a)(s-b)(s-c) = \frac{1}{s}K^2 = rK$, we find that this becomes

$$2(s-a)(s-b)(s-c) \le RK \iff 2r \le R$$

which follows immediately from $IO^2 = R(R-2r)$. Alternatively, one may rewrite this as Schur's Inequality in the form

$$abc \ge (-a+b+c)(a-b+c)(a+b-c).$$

37^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 29, 2008

1. Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots,$ k_n, all strictly greater than 1, such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.

2. Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

3. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that
$$|x| + \left|y + \frac{1}{2}\right| < n.$$
A *path* is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates).

   Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

37^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 30, 2008

4. Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

5. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1+a_2r_2+a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

37th United States of America Mathematical Olympiad

1. Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.

First solution: We proceed by induction. The case $n = 1$ is clear, since we may pick $k_0 = 3$ and $k_1 = 7$.

Let us assume now that for a certain n there are pairwise relatively prime integers $1 < k_0 < k_1 < \cdots < k_n$ such that $k_0 k_1 \cdots k_n - 1 = a_n(a_n - 1)$, for some positive integer a_n. Then choosing $k_{n+1} = a_n^2 + a_n + 1$ yields

$$k_0 k_1 \cdots k_{n+1} = (a_n^2 - a_n + 1)(a_n^2 + a_n + 1) = a_n^4 + a_n^2 + 1,$$

so $k_0 k_1 \cdots k_{n+1} - 1$ is the product of the two consecutive integers a_n^2 and $a_n^2 + 1$. Moreover,

$$\gcd(k_0 k_1 \cdots k_n, k_{n+1}) = \gcd(a_n^2 - a_n + 1, a_n^2 + a_n + 1) = 1,$$

hence $k_0, k_1, \ldots, k_{n+1}$ are pairwise relatively prime. This completes the proof. □

Second solution: Write the relation to be proved as

$$4k_0 k_1 \cdots k_n = 4a(a+1) + 4 = (2a+1)^2 + 3.$$

There are infinitely many primes for which -3 is a quadratic residue. Let $2 < p_0 < p_1 < \ldots < p_n$ be such primes. Using the Chinese Remainder Theorem to specify a modulo p_n, we can find an integer a such that $(2a+1)^2 + 3 = 4p_0 p_1 \cdots p_n m$ for some positive integer m. Grouping the factors of m appropriately with the p_i's, we obtain $(2a+1)^2 + 3 = 4k_0 k_1 \cdots k_n$ with k_i pairwise relatively prime. We then have $k_0 k_1 \cdots k_n - 1 = a(a+1)$, as desired. □

Third solution: We are supposed to show that for every positive integer n, there is a positive integer x such that $x(x+1) + 1 = x^2 + x + 1$ has at least n distinct prime divisors. We can actually prove a more general statement.

Claim. *Let $P(x) = a_d x^d + \cdots + a_1 x + 1$ be a polynomial of degree $d \geq 1$ with integer coefficients. Then for every positive integer n, there is a positive integer x such that $P(x)$ has at least n distinct prime divisors.*

The proof follows from the following two lemmas.

Lemma 1. *The set*

$$Q = \{p \,|\, p \text{ a prime for which there is an integer } x \text{ such that } p \text{ divides } P(x)\}$$

is infinite.

Proof. The proof is analogous to Euclid's proof that there are infinitely many primes. Namely, if we assume that there are only finitely many primes $p_1, p_2, \ldots, p_k$ in Q, then for each integer m, $P(mp_1p_2\cdots p_k)$ is an integer with no prime factors, which must equal 1 or -1. However, since P has degree $d \geq 1$, it takes each of the values 1 and -1 at most d times, a contradiction. □

Lemma 2. *Let $p_1, p_2, \ldots, p_n$, $n \geq 1$ be primes in Q. Then there is a positive integer x such that $P(x)$ is divisible by $p_1p_2\cdots p_n$.*

Proof. For $i = 1, 2, \ldots, n$, since $p_i \in Q$ we can find an integer c_i such that $P(x)$ is divisible by p_i whenever $x \equiv c_i (\text{mod } p_i)$. By the Chinese Remainder Theorem, the system of n congruences $x \equiv c_i (\text{mod } p_i)$, $i = 1, 2, \ldots, n$ has positive integer solutions. For every positive integer x that solves this system, $P(x)$ is divisible by $p_1p_2\cdots p_n$. □

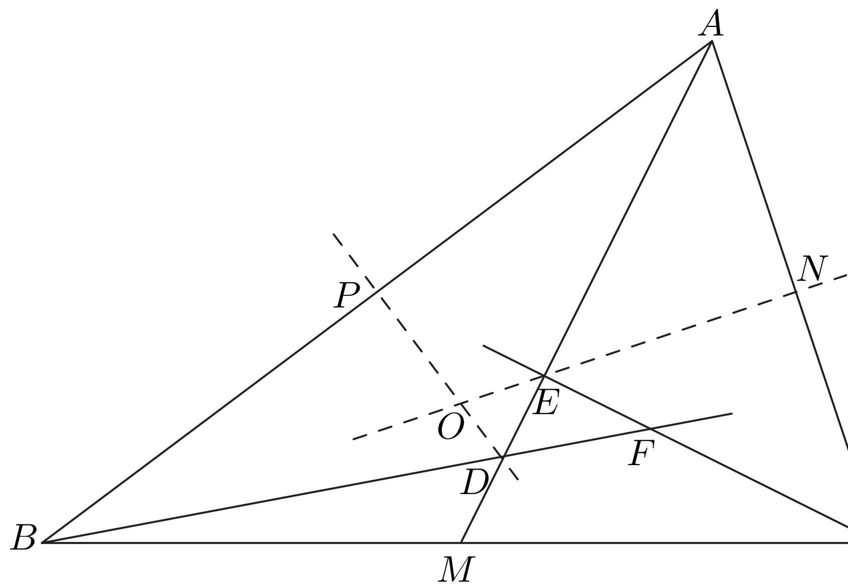
This problem was suggested by Titu Andreescu.

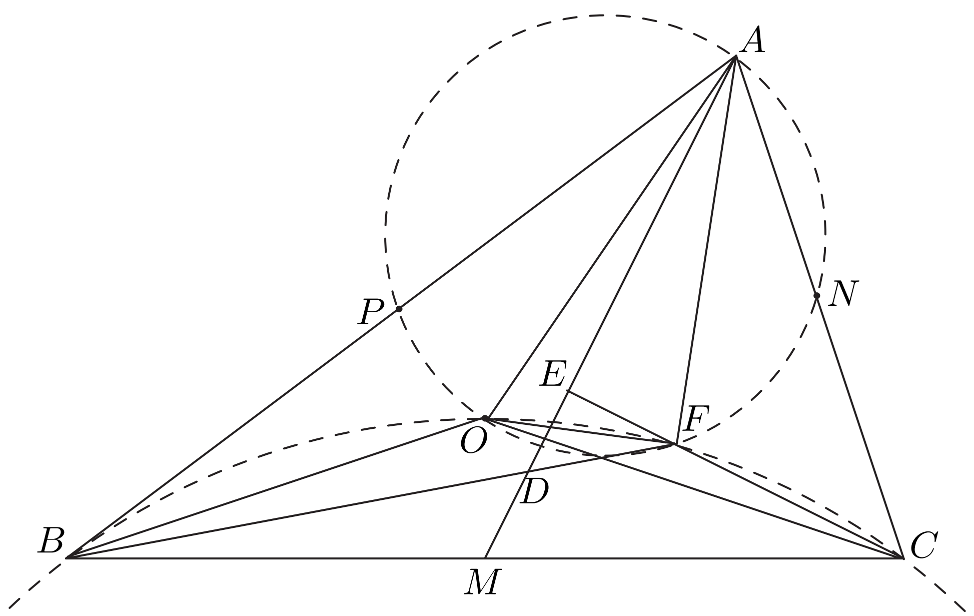
2. Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

First solution: Let O be the circumcenter of triangle ABC. We prove that

$$\angle APO = \angle ANO = \angle AFO = 90^\circ. \tag{1}$$

It will then follow that A, P, O, F, N lie on the circle with diameter $\overline{AO}$. Indeed, the fact that the first two angles in (1) are right is immediate because $\overline{OP}$ and $\overline{ON}$ are the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$, respectively. Thus we need only prove that $\angle AFO = 90^\circ$.





We may assume, without loss of generality, that $AB > AC$. This leads to configurations similar to the ones shown above. The proof can be adapted to other configurations. Because PO is the perpendicular bisector of AB, it follows that triangle ADB is an isosceles triangle with $AD = BD$. Likewise, triangle AEC is isosceles with $AE = CE$. Let $x = \angle ABD = \angle BAD$ and $y = \angle CAE = \angle ACE$, so $x + y = \angle BAC$.

Applying the Law of Sines to triangles ABM and ACM gives

$$\frac{BM}{\sin x} = \frac{AB}{\sin \angle BMA} \qquad \text{and} \qquad \frac{CM}{\sin y} = \frac{AC}{\sin \angle CMA}.$$

Taking the quotient of the two equations and noting that $\sin \angle BMA = \sin \angle CMA$ we find

$$\frac{BM}{CM}\frac{\sin y}{\sin x} = \frac{AB}{AC}\frac{\sin \angle CMA}{\sin \angle BMA} = \frac{AB}{AC}.$$

Because $BM = MC$, we have

$$\frac{\sin x}{\sin y} = \frac{AC}{AB}. \tag{2}$$

Applying the law of sines to triangles ABF and ACF we find

$$\frac{AF}{\sin x} = \frac{AB}{\sin \angle AFB} \qquad \text{and} \qquad \frac{AF}{\sin y} = \frac{AC}{\sin \angle AFC}.$$

Taking the quotient of the two equations yields

$$\frac{\sin x}{\sin y} = \frac{AC}{AB}\frac{\sin \angle AFB}{\sin \angle AFC}, \qquad \text{so by (2),} \qquad \sin \angle AFB = \sin \angle AFC. \tag{3}$$

Because $\angle ADF$ is an exterior angle to triangle ADB, we have $\angle EDF = 2x$. Similarly, $\angle DEF = 2y$. Hence

$$\angle EFD = 180^\circ - 2x - 2y = 180^\circ - 2\angle BAC.$$

Thus $\angle BFC = 2\angle BAC = \angle BOC$, so $BOFC$ is cyclic. In addition,

$$\angle AFB + \angle AFC = 360^\circ - 2\angle BAC > 180^\circ,$$

and hence, from (3), $\angle AFB = \angle AFC = 180^\circ - \angle BAC$. Because $BOFC$ is cyclic and $\triangle BOC$ is isosceles with vertex angle $\angle BOC = 2\angle BAC$, we have $\angle OFB = \angle OCB = 90^\circ - \angle BAC$. Therefore,

$$\angle AFO = \angle AFB - \angle OFB = (180^\circ - \angle BAC) - (90^\circ - \angle BAC) = 90^\circ.$$

This completes the proof. $\square$

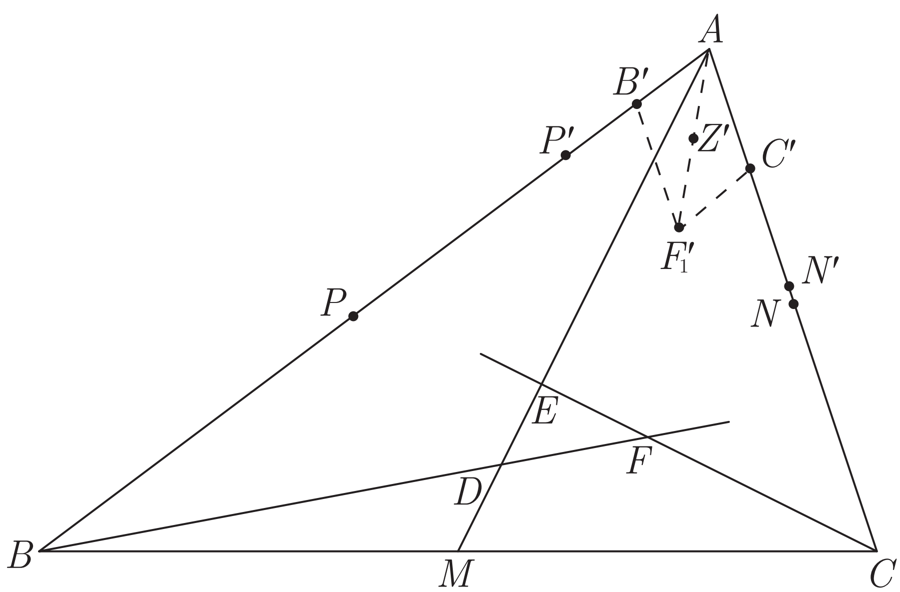
Second solution: Invert the figure about a circle centered at A, and let X' denote the image of the point X under this inversion. Find point F_1' so that $AB'F_1'C'$ is a parallelogram and let Z' denote the center of this parallelogram. Note that $\triangle BAC \sim$

$\triangle C'AB'$ and $\triangle BAD \sim \triangle D'AB'$. Because M is the midpoint of BC and Z' is the midpoint of $B'C'$, we also have $\triangle BAM \sim \triangle C'AZ'$. Thus

$$\angle AF_1'B' = \angle F_1'AC' = \angle Z'AC' = \angle MAB = \angle DAB = \angle DBA = \angle AD'B'.$$

Hence quadrilateral $AB'D'F_1'$ is cyclic and, by a similar argument, quadrilateral $AC'E'F_1'$ is also cyclic. Because the images under the inversion of lines BDF and CFE are circles that intersect in A and F', it follows that $F_1' = F'$.

Next note that B', Z', and C' are collinear and are the images of P', F', and N', respectively, under a homothety centered at A and with ratio $1/2$. It follows that P', F' and N' are collinear, and then that the points A, P, F and N lie on a circle. $\square$



This problem was suggested by Zuming Feng. The second solution was contributed by Gabriel Carroll.

3. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that
$$|x| + \left|y + \frac{1}{2}\right| < n.$$
A *path* is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates).

Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Solution: Color the points in S_n as follows (see Figure 1):

- if $y \geq 0$, color (x, y) white if $x + y - n$ is even and black if $x + y - n$ is odd;
- if $y < 0$, color (x, y) white if $x + y - n$ is odd and black if $x + y - n$ is even.

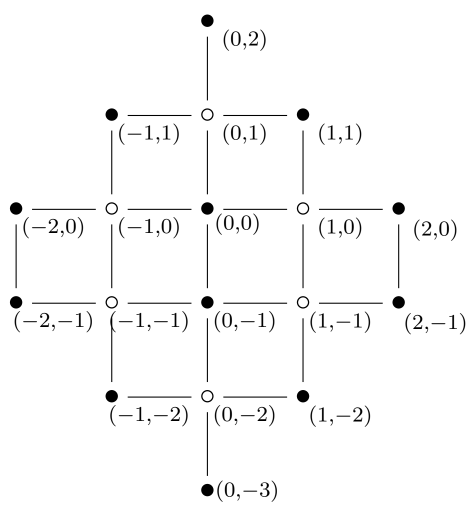


Figure 1: Coloring of S_3

Consider a path $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n. A pair of successive points (x_{i-1}, y_{i-1}) and (x_i, y_i) in the path is called a pair of successive black points if both points in the pair are colored black.

Suppose now that the points of S_n are partitioned into m paths and the total number of successive pairs of black points in all paths is k. By breaking the paths at each pair of successive black points, we obtain $k+m$ paths in each of which the number of black points exceeds the number of white points by at most one. Therefore, the total number of black points in S_n cannot exceed the number of white points by more than $k+m$. On the other hand, the total number of black points in S_n exceeds the total number of white points by exactly $2n$ (there is exactly one more black point in each row of S_n). Therefore,

$$2n \leq k + m.$$

There are exactly n adjacent black points in S_n (call two points in S_n *adjacent* if their distance is 1), namely the pairs

$$(x, 0) \text{ and } (x, -1),$$

for $x = -n+1, -n+3, \ldots, n-3, n-1$. Therefore $k \le n$ (the number of successive pairs of black points in the paths in the partition of S_n cannot exceed the total number of adjacent pairs of black points in S_n) and we have $2n \le k + m \le n + m$, yielding

$$n \le m.$$

□

This problem was suggested by Gabriel Carroll.

4. Let $\mathcal{P}$ be a convex polygon with n sides, $n \ge 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

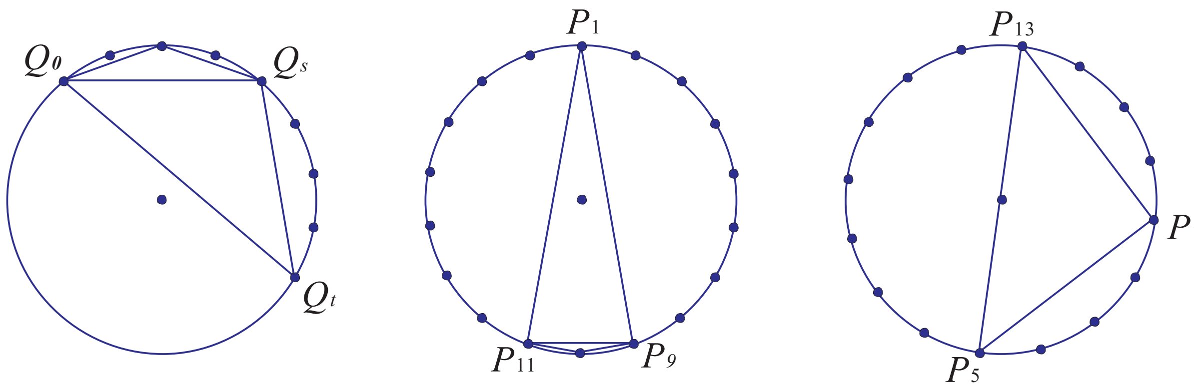
Solution: The answer is $n = 2^{m+1} + 2^k$, where m and k are nonnegative integers. In other words, n is either a power of 2 (when $m + 1 = k$) or the sum of two nonequal powers of 2 (with $1 = 2^0$ being considered as a power of 2).

We start with the following observation.

Lemma. *Let $\mathcal{Q} = Q_0Q_1 \ldots Q_t$ be a convex polygon with $Q_0Q_1 = Q_1Q_2 = \cdots = Q_{t-1}Q_t$. Suppose that $\mathcal{Q}$ is cyclic and its circumcenter does not lie in its interior. If there is a triangulation of $\mathcal{Q}$ consisting only of isosceles triangles, then $t = 2^a$, where a is a positive integer.*

Proof. We call an arc *minor* if its arc measure is less than or equal to 180°. By the given conditions, points $Q_1, \ldots, Q_{t-1}$ lie on the minor arc $\widehat{Q_0Q_t}$ of the circumcircle, so none of the angles $Q_iQ_jQ_k$ $(0 \le i < j < k \le t)$ is acute. (See the left-hand side diagram shown below.) It is not difficult to see that Q_0Q_t is longer than each other side or diagonal of $\mathcal{Q}$. Thus Q_0Q_t must be the base of an isosceles triangle in the triangulation of $\mathcal{Q}$. Therefore, t must be even. We write $t = 2s$. Then $Q_0Q_sQ_t$ is an isosceles triangle in the triangulation. We can apply the same process to polygon $Q_0Q_1 \ldots Q_s$ and show that s is even. Repeating this process leads to the conclusion that $t = 2^a$ for some positive integer a.

The results of the lemma can be generalized by allowing $a = 0$ if we consider the degenerate case $\mathcal{Q} = Q_0Q_1$. □



We are ready to prove our main result. Let $\mathcal{P} = P_1P_2 \dots P_n$ denote the regular polygon. There is an isosceles triangle in the triangulation such that the center of $\mathcal{P}$ lies within the boundary of the triangle. Without loss of generality, we may assume that $P_1P_iP_j$, with $P_1P_i = P_1P_j$ (that is, $P_j = P_{n-i+2}$), is this triangle. Applying the Lemma to the polygons $P_1 \dots P_i$, $P_i \dots P_j$, and $P_j \dots P_1$, we conclude that there are $2^m - 1$, $2^k - 1$, $2^m - 1$ (where m and k are nonnegative integers) vertices in the interiors of the minor arcs $\widehat{P_1P_i}$, $\widehat{P_iP_j}$, $\widehat{P_jP_1}$, respectively. (In other words, $i = 2^m + 1$, $j = 2^k + i$.) Hence

$$n = 2^m - 1 + 2^k - 1 + 2^m - 1 + 3 = 2^{m+1} + 2^k,$$

where m and k are nonnegative integers. The above discussion can easily lead to a triangulation consisting of only isosceles triangles for $n = 2^{m+1} + 2^k$. (The middle diagram shown above illustrates the case $n = 18 = 2^{3+1} + 2^1$. The right-hand side diagram shown above illustrates the case $n = 16 = 2^{2+1} + 2^3$.) □

This problem was suggested by Gregory Galperin.

5. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1+a_2r_2+a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

Solution: If two of the a_i vanish, say a_2 and a_3, then r_1 must be zero and we are done. Assume at most one a_i vanishes. If any one a_i vanishes, say a_3, then $r_2/r_1 = -a_1/a_2$

is a nonnegative rational number. Write this number in lowest terms as p/q, and put $r = r_2/p = r_1/q$. We can then write $r_1 = qr$ and $r_2 = pr$. Performing the Euclidean algorithm on r_1 and r_2 will ultimately leave r and 0 on the blackboard. Thus we are done again.

Thus it suffices to consider the case where none of the a_i vanishes. We may also assume none of the r_i vanishes, as otherwise there is nothing to check. In this case we will show that we can perform an operation to obtain r_1', r_2', r_3' for which either one of r_1', r_2', r_3' vanishes, or there exist integers a_1', a_2', a_3', not all zero, with $a_1'r_1' + a_2'r_2' + a_3'r_3' = 0$ and

$$|a_1'| + |a_2'| + |a_3'| < |a_1| + |a_2| + |a_3|.$$

After finitely many steps we must arrive at a case where one of the a_i vanishes, in which case we finish as above.

If two of the r_i are equal, then we are immediately done by choosing them as x and y. Hence we may suppose $0 < r_1, r_2 < r_3$. Since we are free to negate all the a_i, we may assume $a_3 > 0$. Then either $a_1 < -\frac{1}{2}a_3$ or $a_2 < -\frac{1}{2}a_3$ (otherwise $a_1r_1 + a_2r_2 + a_3r_3 > (a_1 + \frac{1}{2}a_3)r_1 + (a_2 + \frac{1}{2}a_3)r_2 > 0$). Without loss of generality, we may assume $a_1 < -\frac{1}{2}a_3$. Then choosing $x = r_1$ and $y = r_3$ gives the triple $(r_1', r_2', r_3') = (r_1, r_2, r_3 - r_1)$ and $(a_1', a_2', a_3') = (a_1 + a_3, a_2, a_3)$. Since $a_1 < a_1 + a_3 < \frac{1}{2}a_3 < -a_1$, we have $|a_1'| = |a_1 + a_3| < |a_1|$ and hence this operation has the desired effect. □

This problem was suggested by Kiran Kedlaya.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

Solution: Let n be the number of participants at the conference. We proceed by induction on n.

If $n = 1$, then we have one participant who can eat in either room; that gives us total of $2 = 2^1$ options.

Let $n \geq 2$. The case in which some participant, P, has no friends is trivial. In this case, P can eat in either of the two rooms, so the total number of ways to split n participants is twice as many as the number of ways to split $(n-1)$ participants besides the participant P. By induction, the latter number is a power of two, 2^k, hence the number of ways to split n participants is $2 \times 2^k = 2^{k+1}$, also a power of two. So we assume from here on that every participant has at least one friend.

We consider two different cases separately: the case when some participant has an odd number of friends, and the case when each participant has an even number of friends.

Case 1: *Some participant, Z, has an odd number of friends.*

Remove Z from consideration and for each pair $(X,\ Y)$ of Z's friends, reverse the relationship between X and Y (from friends to strangers or vice versa).

Claim. *The number of possible seatings is unchanged after removing Z and reversing the relationship between X and Y in each pair $(X,\ Y)$ of Z's friends.*

Proof of the claim. Suppose we have an arrangement prior to Z's departure. By assumption, Z has an even number of friends in the room with him.

If this number is 0, the room composition is clearly still valid after Z leaves the room.

If this number is positive, let X be one of Z's friends in the room with him. By assumption, person X also has an even number of friends in the same room. Remove Z from the room; then X will have an odd number of friends left in the room, and there will be an odd number of Z's friends in this room besides X. Reversing the relationship between X and each of Z's friends in this room will therefore restore the parity to even.

The same reasoning applies to any of Z's friends in the other dining room. Indeed, there will be an odd number of them in that room, hence each of them will reverse relationships with an even number of individuals in that room, preserving the parity of the number of friends present.

Moreover, a legitimate seating without Z arises from exactly one arrangement including Z, because in the case under consideration, only one room contains an even number of Z's friends. □

Thus, we have to double the number of seatings for $(n-1)$ participants which is, by the induction hypothesis, a power of 2. Consequently, for n participants we will get again a power of 2 for the number of different arrangements.

Case 2: *Each participant has an even number of friends.*

In this case, each valid split of participants in two rooms gives us an even number of friends in either room.

Let $(A,\ B)$ be any pair of friends. Remove this pair from consideration and for each pair $(C,\ D)$, where C is a friend of A and D is a friend of B, change the relationship between C and D to the opposite; do the same if C is a friend of B and D is a friend of A. Note that if C and D are friends of both A and B, their relationship will be reversed twice, leaving it unchanged.

Consider now an arbitrary participant X different from A and B and choose one of the two dining rooms. [Note that in the case under consideration, the total number of participants is at least 3, so such a triplet $(A,\ B;\ X)$ can be chosen.] Let A have m friends in this room and let B have n friends in this room; both m and n are even. When the pair $(A,\ B)$ is removed, X's relationship will be reversed with either n, or m, or $m+n-2k$ (for k the number of mutual friends of A and B in the chosen room), or 0 people within the chosen room (depending on whether he/she is a friend of only A, only B, both, or neither). Since m and n are both even, the parity of the number of X's friends in that room will be therefore unchanged in any case.

Again, a legitimate seating without A and B will arise from exactly one arrangement that includes the pair $(A,\ B)$: just add each of A and B to the room with an odd number of the other's friends, and then reverse all of the relationships between a friend of A and a friend of B. In this way we create a one-to-one correspondence between all possible seatings before and after the $(A,\ B)$ removal.

Since the number of arrangements for n participants is twice as many as that for $(n-2)$ participants, and that number for $(n-2)$ participants is, by the induction hypothesis, a power of 2, we get in turn a power of 2 for the number of arrangements for n participants. The problem is completely solved. □

This problem was suggested by Sam Vandervelde.

USAMO 2008 Solution Notes

COMPILED BY EVAN CHEN

April 17, 2020

This is an compilation of solutions for the 2008 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

2. Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

3. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that
$$|x| + \left| y + \frac{1}{2} \right| < n.$$
A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \dots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1.

   Prove that the points in S_n cannot be partitioned into fewer than n paths.

4. For which integers $n \ge 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

5. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e. is of the form 2^k for some positive integer k).

§1 USAMO 2008/1, proposed by Titu Andreescu

Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

In other words, if we let

$$P(x) = x(x+1) + 1$$

then we would like there to be infinitely many primes dividing some $P(t)$ for some integer t.

In fact, this result is true in much greater generality. We first state:

Theorem 1.1 (Schur's theorem)
If $P(x) \in \mathbb{Z}[x]$ is nonconstant and $P(0) = 1$, then there are infinitely many primes which divide $P(t)$ for some integer t.

Proof. If $P(0) = 0$, this is clear. So assume $P(0) = c \neq 0$.

Let S be any finite set of prime numbers. Consider then the value

$$P(k \prod_{p \in S} p)$$

for some integer k. It is $1 \pmod p$ for each prime p, and if k is large enough it should not be equal to 1 (because P is not constant). Therefore, it has a prime divisor not in S. □

> **Remark.** In fact the result holds without the assumption $P(0) \neq 1$. The proof requires only small modifications, and a good exercise would be to write down a similar proof that works first for $P(0) = 20$, and then for any $P(0) \neq 0$. (The $P(0) = 0$ case is vacuous, since then $P(x)$ is divisible by x.)

To finish the proof, let $p_1, \dots, p_n$ be primes and x_i be integers such that

$$\begin{aligned} P(x_1) &\equiv 0 \pmod{p_1} \\ P(x_2) &\equiv 0 \pmod{p_2} \\ &\vdots \\ P(x_n) &\equiv 0 \pmod{p_n} \end{aligned}$$

as promised by Schur's theorem. Then, by Chinese remainder theorem, we can find x such that $x \equiv x_i \pmod{p_i}$ for each i, whence $P(x)$ has at least n prime factor.

§2 USAMO 2008/2, proposed by Zuming Feng

Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

We present a barycentric solution and a synthetic solution.

Barycentric solution First, we find the coordinates of D. As D lies on $\overline{AM}$, we know $D = (t : 1 : 1)$ for some t. Now by perpendicular bisector formula, we find

$$0 = b^2(t-1) + (a^2 - c^2) \implies t = \frac{c^2 + b^2 - a^2}{b^2}.$$

Thus we obtain

$$D = \left(2S_A : c^2 : c^2\right).$$

Analogously $E = (2S_A : b^2 : b^2)$, and it follows that

$$F = \left(2S_A : b^2 : c^2\right).$$

The sum of the coordinates of F is

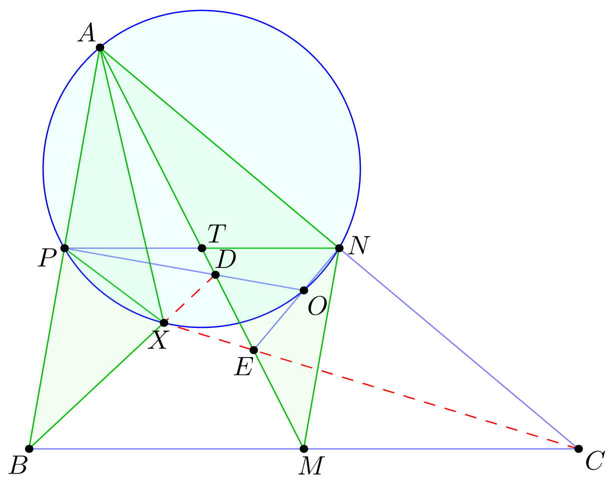
$$(b^2 + c^2 - a^2) + b^2 + c^2 = 2b^2 + 2c^2 - a^2.$$

Hence the reflection of A over F is simply

$$2F - A = \left(2(b^2 + c^2 - a^2) - (2b^2 + 2c^2 - a^2) : 2b^2 : 2c^2\right) = \left(-a^2 : 2b^2 : 2c^2\right).$$

It is evident that F' lies on $(ABC) : -a^2yz - b^2zx - c^2xy = 0$, and we are done.

Synthetic solution (harmonic) Here is a synthetic solution. Let X be the point so that $APXN$ is a cyclic harmonic quadrilateral. We contend that $X = F$. To see this it suffices to prove B, X, D collinear (and hence C, X, E collinear by symmetry).



Let T be the midpoint of $\overline{PN}$, so $\triangle APX \sim \triangle ATN$. So $\triangle ABX \sim \triangle AMN$, ergo

$$\measuredangle XBA = \measuredangle NMA = \measuredangle BAM = \measuredangle BAD = \measuredangle DBA$$

as desired.

Angle chasing solution (Mason Fang) Obviously $ANOP$ is concyclic.

Claim — Quadrilateral $BFOC$ is cyclic.

Proof. Write

$$\begin{aligned} \measuredangle BFC = \measuredangle FBC + \measuredangle BCF &= \measuredangle FBA + \measuredangle ABC + \measuredangle BCA + \measuredangle ACF \\ &= \measuredangle DBA + \measuredangle ABC + \measuredangle BCA + \measuredangle ACE \\ &= \measuredangle BAD + \measuredangle ABC + \measuredangle BCA + \measuredangle EAC \\ &= 2\angle BAC = \angle BOC. \end{aligned}$$

□

Define $Q = \overline{AA} \cap \overline{BC}$.

Claim — Point Q lies on $\overline{FO}$.

Proof. Write

$$\begin{aligned} \measuredangle BOQ = \measuredangle BOA + \measuredangle AOQ &= 2\measuredangle BCA + 90^\circ + \measuredangle AQO \\ &= 2\measuredangle BCA + 90^\circ + \measuredangle AMO \\ &= 2\measuredangle BCA + 90^\circ + \measuredangle AMC + 90^\circ \\ &= \measuredangle BCA + \measuredangle MAC = \measuredangle BCA + \measuredangle ACE \\ &= \measuredangle BCE = \measuredangle BOF. \end{aligned}$$

□

As Q is the radical center of $(ANOP)$, (ABC) and $(BFOC)$, this implies the result.

§3 USAMO 2008/3, proposed by Gabriel Carroll

Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

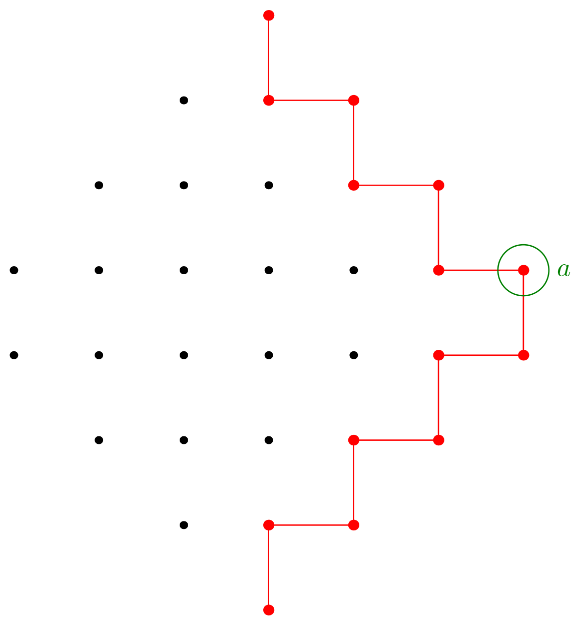
A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \dots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1.

Prove that the points in S_n cannot be partitioned into fewer than n paths.

First solution (local) We proceed by induction on n. The base case $n = 1$ is clear, so suppose $n > 1$. Let S denote the set of points

$$S = \left\{ (x, y) : x + \left|y + \frac{1}{2}\right| \geq n - 2 \right\}.$$

An example when $n = 4$ is displayed below.



For any minimal partition $\mathcal{P}$ of S_n, let P denote the path passing through the point $a = (n - 1, 0)$. Then the intersection of P with S consists of several disconnected paths; let N be the number of nodes in the component containing a, and pick $\mathcal{P}$ such that N is maximal. We claim that in this case $P = S$.

Assume not. First, note $a = (n - 1, 0)$ must be connected to $b = (n - 1, -1)$ (otherwise join them to decrease the number of paths).

Now, starting from $a = (n - 1, 0)$ walk along P away from b until one of the following three conditions is met:

- We reach a point v not in S. Let w be the point before v, and x the point in S adjacent to w. Then delete vw and add wx. This increases N while leaving the number of edges unchanged: so this case can't happen.

- We reach an endpoint v of P (which may be a), lying inside the set S, which is not the topmost point $(0, n - 1)$. Let w be the next point of S. Delete any edge touching w and add edge vw. This increases N while leaving the number of edges unchanged: so this case can't happen.

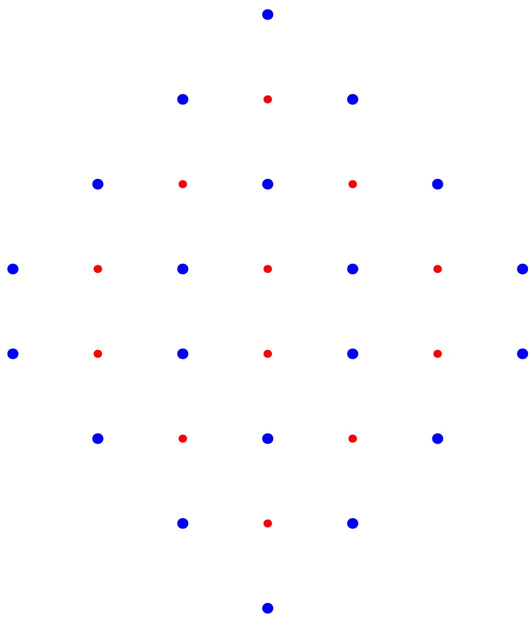
- We reach the topmost point $(0, n-1)$.

Thus we see that P must follow S until reaching the topmost point $(0, n-1)$. Similarly it must reach the bottom-most point $(0, -n)$. Hence $P = S$.

The remainder of S_n is just S_{n-1}, and hence this requires at least $n-1$ paths to cover by the inductive hypothesis. So S_n requires at least n paths, as desired.

Second solution (global) Here is a much shorter official solution, which is much trickier to find, and "global" in nature.

Color the upper half of the diagram with a blue/red checkerboard pattern such that the uppermost point $(n-1, 0)$ is blue. Reflect it over to the bottom, as shown.



Assume there are m paths. Cut in two any paths with two adjacent blue points; this occurs only along the horizontal symmetry axis. Thus:

- After cutting there are at most $m+n$ paths, since at most n cuts occur.
- On the other hand, there are $2n$ more blue points than red points. Hence after cutting there must be at least $2n$ paths (since each path alternates colors).

So $m + n \geq 2n$, hence $m \geq n$.

Remark. This problem turned out to be known already. It appears in this reference:

> Nikolai Beluhov, Nyakolko Zadachi po Shahmatna Kombinatorika, *Matematika Plyus*, 2006, issue 4, pages 61–64.

Section 1 of 2 was reprinted with revisions as Nikolai Beluhov, Dolgii Put Korolya, *Kvant*, 2010, issue 4, pages 39–41. The reprint is available at http://kvant.ras.ru/pdf/2010/2010-04.pdf.

Remark (Nikolai Beluhov)**.** As pointed out in the reference above, this problem arises naturally when we try to estimate the greatest possible length of a closed king tour on the chessboard of size $n \times n$ with n even, a question posed by Igor Akulich in Progulki Korolya, Kvant, 2000, issue 3, pages 47–48. Each one of the two references above contains a proof that the answer is $n + \sqrt{2}(n^2 - n)$.

§4 USAMO 2008/4, proposed by Gregory Galperin

For which integers $n \geq 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

The answer is n of the form $2^a(2^b+1)$ where a and b are nonnegative integers not both zero.

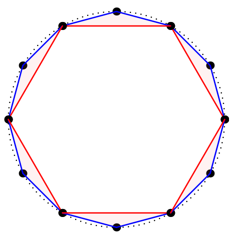
Call the polygon $A_1 \dots A_n$ with indices taken modulo n. We refer to segments A_1A_2, A_2A_3, ..., A_nA_1 as *short sides*. Each of these must be in the triangulation. Note that

- when n is even, the isosceles triangles triangle using a short side A_1A_2 are $\triangle A_nA_1A_2$ and $\triangle A_1A_2A_3$ only, which we call *small*.
- when n is odd, in addition to the small triangles, we have $\triangle A_{\frac{1}{2}(n+3)}A_1A_2$, which we call *big*.

This leads to the following two claims.

Claim — If $n > 4$ is even, then n works iff $n/2$ does.

Proof. All short sides must be part of a small triangle; after drawing these in, we obtain an $n/2$-gon.



Thus the sides of $\mathcal{P}$ must pair off, and when we finish drawing we have an $n/2$-gon. □

Since $n = 4$ works, this implies all powers of 2 work and it remains to study the case when n is odd.

Claim — If $n > 1$ is odd, then n works if and only if $n = 2^b + 1$ for some positive integer b.

Proof. We cannot have all short sides part of small triangles for parity reasons, so some side, must be part of a big triangle. Since big triangles contain the center O, there can be at most one big triangle too.

Then we get $\frac{1}{2}(n-1)$ small triangles, pairing up the remaining sides. Now repeating the argument with the triangles on each half shows that the number $n-1$ must be a power of 2, as needed. □

§5 USAMO 2008/5, proposed by Kiran Kedlaya

Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

We first show we can decrease the quantity $|a_1| + |a_2| + |a_3|$ as long as $0 \notin \{a_1, a_2, a_3\}$. Assume $a_1 > 0$ and $r_1 > r_2 > r_3$ without loss of generality and consider two cases.

- $r_2 > 0$ or $r_3 > 0$; these cases are identical. If $r_2 > 0$ then $r_3 < 0$ and get

$$0 = a_1r_1 + a_2r_2 + a_3r_3 > a_1r_3 + a_3r_3 \implies a_1 + a_3 < 0$$

  so $|a_1 + a_3| < |a_3|$, and hence we perform $(r_1, r_2, r_3) \mapsto (r_1 - r_3, r_2, r_3)$.

- Both r_2 and r_3 are less than zero. Assume for contradiction that $|a_1 + a_2| \ge -a_2$ and $|a_1 + a_3| \ge -a_3$ both hold (if either fails then we use $(r_1, r_2, r_3) \mapsto (r_1 - r_2, r_2, r_3)$ and $(r_1, r_2, r_3) \mapsto (r_1 - r_3, r_2, r_3)$, respectively). Clearly $a_1 + a_2$ and $a_1 + a_3$ are both positive in this case, so we get $a_1 + 2a_2$ and $a_1 + 2a_3 \ge 0$; adding gives $a_1 + a_2 + a_3 \ge 0$. But

$$\begin{aligned} 0 &= a_1r_1 + a_2r_2 + a_3r_3 \\ &> a_1r_2 + a_2r_2 + a_3r_2 \\ &= r_2(a_1 + a_2 + a_3) \\ \implies 0 &< a_1 + a_2 + a_3. \end{aligned}$$

Since this covers all cases, we see that we can always decrease $|a_1| + |a_2| + |a_3|$ whenever $0 \notin \{a_1, a_2, a_3\}$. Because the a_i are integers this cannot occur indefinitely, so eventually one of the a_i's is zero. At this point we can just apply the Euclidean Algorithm, so we're done.

§6 USAMO 2008/6, proposed by Sam Vandervelde

At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e. is of the form 2^k for some positive integer k).

Take the obvious graph interpretation where we are trying to 2-color a graph. Let A be the adjacency matrix of the graph over $\mathbb{F}_2$, except the diagonal of A has $\deg v \pmod 2$ instead of zero. Then let $\vec{d}$ be the main diagonal. Splittings then correspond to $A\vec{v} = \vec{d}$. It's then immediate that the number of ways is either zero or a power of two, since if it is nonempty it is a coset of $\ker A$.

Thus we only need to show that:

Claim — At least one coloring exists.

Proof. If not, consider a minimal counterexample G. Clearly there is at least one odd vertex v. Consider the graph with vertex set $G - v$, where all pairs of neighbors of v have their edges complemented. By minimality, we have a good coloring here. One can check that this extends to a good coloring on G by simply coloring v with the color matching an even number of its neighbors. This breaks minimality of G, and hence all graphs G have a coloring. □

It's also possible to use linear algebra. We prove the following lemma:

Lemma (grobber)

Let V be a finite dimensional vector space, $T\colon V \to V$ and $w \in V$. Then w is in the image of T if and only if there are no $\xi \in V^\vee$ for which $\xi(w) \neq 0$ and yet $\xi \circ T = 0$.

Proof. Clearly if $T(v) = w$, then no ξ exists. Conversely, assume w is not in the image of T. Then the image of T is linearly independent from w. Take a basis $e_1, \dots, e_m$ for the image of T, add w, and then extend it to a basis for all of V. Then have ξ kill all e_i but not w. □

Corollary

In a symmetric matrix A mod 2, there exists a vector v such that Av is a copy of the diagonal of A.

Proof. Let ξ be such that $\xi \circ T = 0$. Look at ξ as a column vector $w^\top$, and let d be the diagonal. Then

$$0 = w^\top \cdot T \cdot w = \xi(d)$$

because this extracts the sum of coefficients submatrix of T, and all the symmetric entries cancel off. Thus no ξ as in the previous lemma exists. □

This corollary gives the desired proof.

38^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 28, 2009

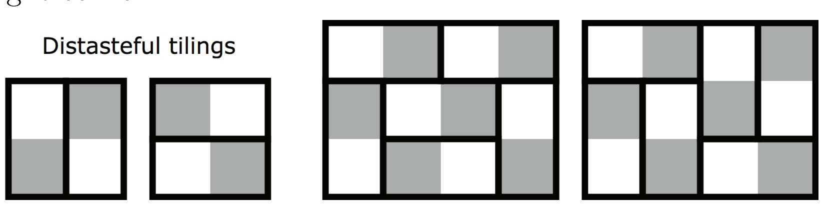
1. Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

2. Let n be a positive integer. Determine the size of the largest subset of

$$\{-n, -n+1, \ldots, n-1, n\}$$

which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c = 0$.

3. We define a *chessboard polygon* to be a polygon whose edges are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.

38^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 29, 2009

4. For $n \geq 2$ let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$.

5. Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

6. Let $s_1,\ s_2,\ s_3,\ \ldots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \cdots$. Suppose that $t_1,\ t_2,\ t_3,\ \ldots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

38th United States of America Mathematical Olympiad

1. **Solution 1.** Let ω denote the circumcircle of P, Q, R, S and let O denote the center of ω. Line XY is the radical axis of circles ω_1 and ω_2. It suffices to show that O has equal power to the two circles; that is, to show that

$$OO_1^2 - O_1S^2 = OO_2^2 - O_2Q^2 \quad \text{or} \quad OO_1^2 + O_2Q^2 = OO_2^2 + O_1S^2.$$

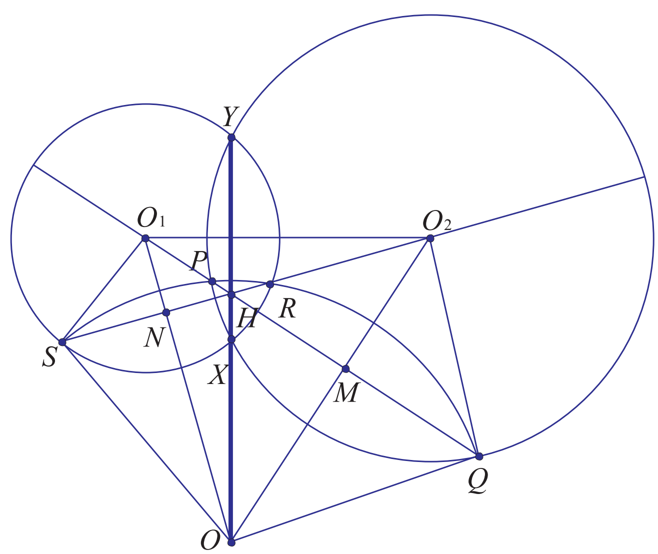
Let M and N be the intersections of lines O_2O, ℓ_1 and O_1O, ℓ_2. Because circles ω and ω_2 intersect at points P and Q, we have $PQ \perp OO_2$ (or $\ell_1 \perp OO_2$). Hence

$$OO_1^2 - OQ^2 = (OM^2 + MO_1^2) - (OM^2 + MQ^2) = (O_2M^2 + MO_1^2) - (O_2M^2 + MQ^2) = O_2O_1^2 - O_2Q^2$$

or

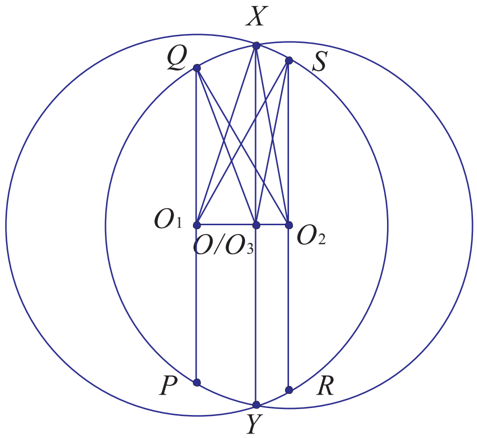
$$O_2O_1^2 + OQ^2 = OO_1^2 + O_2Q^2.$$

Likewise, we have $O_2O_1^2 + OS^2 = OO_2^2 + O_1S^2$. Because $OS = OQ$, we obtain that $OO_1^2 + O_2Q^2 = OO_2^2 + O_1S^2$, which is what to be proved.



Solution 2. We maintain the notations of the first solution. Three pairs of circles (ω, ω_1), (ω_1, ω_2), (ω_2, ω) meet at three pairs of points (R, S), (X, Y), (P, Q), respectively; that is, lines RS, XY, PQ are the respective radical axes of these pairs of circles. We consider two cases.

In the first case, we assume that these three radical axes are not parallel. They must be concurrent at the radical center, denoted by H, of these three circles. In particular, it follows that H, X, Y lie a line, denoted by ℓ, and $\ell \perp O_1O_2$. On the other hand, $O_1M \perp O_2O$ and $O_2N \perp O_1O$. Hence H is the orthocenter of triangle OO_1O_2, from which it follows that $OH \perp O_1O_2$. Therefore, O lies on ℓ; that is, X, P, Q are collinear.



In the second case, we assume that these three radical axes are parallel. We will then deduce the above configurations. Let O_3 be the midpoint of segment XY. From right triangles $O_1O_3Q, O_1O_3X, O_1O_2Q$, we have

$$O_3Q^2 = O_1Q^2 + O_1O_3^2 = O_2Q^2 - O_1O_2^2 + O_1X^2 - XO_3^2,$$

which is a expression symmetric about circles ω_1 and ω_2. Hence we can easily obtain that $O_3Q^2 = O_3S^2$ and that O_3 is the circumcenter of isosceles trapezoid $PQSR$; that is, $O_3 = O$, completing the proof.

This problem was suggested by Ian Le. The solutions were contributed by Zuming Feng.

2. The maximum size is n if n is even, and $n+1$ if n is odd, achieved by the subset

$$\{-n, \ldots, -\left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n\}.$$

Lemma. Let A, B be finite nonempty subsets of $\mathbb{Z}$. Then the set $A + B = \{a + b : a \in A, b \in B\}$ has cardinality at least $|A| + |B| - 1$.

Proof: Write $A = \{a_1, \ldots, a_l\}$ and $B = \{b_1, \ldots, b_m\}$ with $a_1 < \cdots < a_l$ and $b_1 < \cdots < b_m$. Then

$$a_1 + b_1, \ldots, a_1 + b_m, a_2 + b_m, \ldots, a_l + b_m$$

is a strictly increasing sequence of $l + m - 1$ elements of $A + B$.

Let S be a subset of $\{-n, \ldots, n\}$ with the desired property; clearly $0 \notin S$. Put $A = S \cap \{-n, \ldots, -1\}$ and $B = S \cap \{1, \ldots, n\}$. Then $A + B$ and $-S = \{-s : s \in S\}$ are disjoint subsets of $\{-n, \ldots, n\}$, so by the lemma,

$$2n + 1 \geq |A + B| + |-S| \geq |A| + |B| - 1 + |S| = 2|S| - 1,$$

or $|S| \leq n + 1$. If n is odd, we are done.

If n is even, we must still show that $|S| = n + 1$ is impossible. Since $A + B \subseteq \{-n + 1, \ldots, n - 1\}$, we cannot achieve the equality $2n + 1 = |A + B| + |-S|$ unless $-n, n \in -S$, or equivalently $-n, n \in S$. Since $-n \in S$, each of the sets $\{1, n-1\}, \ldots, \{n/2 - 1, n/2 + 1\}, \{n/2\}$ must contain an element not in B. Thus $|B| \leq n/2$, and similarly $|A| \leq n/2$, contradicting the hypothesis $|S| = n + 1$.

This problem was suggested by Kiran Kedlaya with Tewodros Amdeberhan.

3. a) We prove the first part by induction on the number n of dominoes in the tiling. The claim is clearly true for $n = 1$. So suppose we have a chessboard polygon that can be tiled by $n > 1$ dominoes. Of all the leftmost squares in the polygon, select the lowest one and label it L; assume for sake of argument that square L is black. In the given tiling, remove the domino covering L, leaving a polygon which may be tiled with $n - 1$ dominoes. By the induction hypothesis, this chessboard polygon can be tastefully tiled.

   Now replace the domino that was removed. If this domino is horizontal, then we are guaranteed that the augmented tiling is still tasteful, since square L is black and there are no squares below it. If the domino is vertical the augmented tiling may still be tasteful, but if not the trouble can only arise because there is another vertical domino directly to its right. In this case rotate the offending pair of dominoes to get two horizontal dominoes. We are not done yet, but if we now repeat this process—removing the *horizontal* domino covering L, tiling the remainder, and replacing the domino—then we will obtain a tasteful tiling.

   If square L is white we may obtain a tasteful tiling by performing a similar process. This time we only encounter difficulty if the domino covering L in the original tiling is horizontal, in which case there must be another horizontal domino directly above it. We rotate this pair, remove the now vertical domino covering L, tile the remainder tastefully using the induction hypothesis, and restore the vertical domino to finish.

   b) Suppose now that there are two tasteful tilings of a given chessboard polygon. By overlaying these two tilings we obtain chains of overlapping dominos, since every square is part of one domino from each tiling. For example, a chain of length one indicates a domino common to both tilings. A chain of length two cannot occur, since these arise when a 2×2 block is covered by horizontal dominos in one tiling and vertical dominos in the other, and one of these configurations will be distasteful.

Since the tilings are distinct a chain of length three or more must occur; let R be the region consisting of such a chain along with its interior, if any. (It is possible that such a chain may completely occupy a region, so that only some of the dominoes in the chain adjoin squares outside of R.) Note that the chain must include a horizontal domino along its lowermost row. If there are two or more overlapping horizontal dominos, then one of them will be a WB domino, i.e. have a white square on the left. Otherwise there are two adjacent vertical dominos that overlap with the single horizontal domino; since they are part of a tasteful tiling we again must have a WB domino. We will now focus on the tiling that includes this WB domino.

The two squares above the WB domino must be part of region R. Furthermore, a single horizontal domino cannot cover them both, nor can a pair of vertical dominos. (Both cases yield distasteful configurations.) Hence a horizontal domino must cover at least one of these squares, extending past the given WB domino either to the left or right. Hence we can deduce the existence of a horizontal WB domino on the next row up. We may repeat this argument until we reach a horizontal WB domino in region R for which the two squares immediately above it are not both in region R. Hence this domino must be part of the chain that defined R.

Now imagine walking along the chain, starting on the white square of the WB domino that exists along the lowest row of region R and taking the first step towards the black square of the same domino. Draw an arrow along each domino in the direction of travel all the way around the chain. Since the squares must alternate white and black, these arrows will always point from a white square to a black square. Furthermore, since the interior of the region was initially to our left when we began the loop, it will always be to our left whenever the chain follows the boundary of R.

But we now reach a contradiction. We earlier deduced the existence of a horizontal WB domino that was part of the chain and was adjacent to the boundary of R, having a square above it that was not part of R. Hence this domino must be traversed from right to left, since we leave the interior of R to our left as we traverse the loop. Hence it must contain an arrow pointing to the left, implying that it must be a BW domino instead. This contradiction completes the proof.

This problem was suggested by Sam Vandervelde.

4. Let $m = \min(a_1, a_2, \ldots, a_n)$ and $M = \max(a_1, a_2, \ldots, a_n)$. Without loss of generality, $a_1 = m$ and $a_n = M$. The Cauchy-Schwarz Inequality gives

Remark: Let $m = \min(a_1, a_2, \ldots, a_n)$ and $M = \max(a_1, a_2, \ldots, a_n)$. By symmetry, we may assume without loss of generality, $m = a_1 \le a_2 \le \cdots \le a_n = M$. We present three solutions. The first solution is a direct application of the Cauchy-Schwarz Inequality. The second solution bypasses Cauchy-Schwarz by applying one of its proofs. The third solution applies the AM-GM and AM-HM inequalities. All of them share the same finish, the case for $n = 2$.

If $n = 2$, given condition reads

$$(m+M)\left(\frac{1}{m}+\frac{1}{M}\right) \le \frac{25}{4}.$$

It follows that

$$4(m+M)^2 \le 25Mm \quad \text{or} \quad (4M-m)(M-4m) \le 0. \tag{1}$$

Because $4M - m > 0$, it must be that $M - 4m \le 0$ and thus $M \le 4m$.

We may assume from now that $n \ge 3$.

Solution 1. The Cauchy-Schwarz Inequality gives

$$\begin{aligned}\left(n+\frac{1}{2}\right)^2 &\ge (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\\ &= (m+a_2+\cdots+a_{n-1}+M)\left(\frac{1}{M}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}+\frac{1}{m}\right)\\ &\ge \left(\sqrt{\frac{m}{M}}+\underbrace{1+\cdots+1}_{n-2}+\sqrt{\frac{M}{m}}\right)^2.\end{aligned}$$

Hence

$$n+\frac{1}{2} \ge \sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}} \quad \text{or} \quad \sqrt{\frac{m}{M}}+\sqrt{\frac{M}{m}} \le \frac{5}{2}. \tag{2}$$

It follows that

$$2(m+M) \le 5\sqrt{Mm},$$

which is (1), completing our proof.

Solution 2. Consider the quadratic polynomial (in x)

$$p(x)=\frac{1}{2}\left[\left(\sqrt{a_1}x+\frac{1}{\sqrt{a_n}}\right)^2+\left(\sqrt{a_n}x+\frac{1}{\sqrt{a_1}}\right)^2+\sum_{i=2}^{n-1}\left(\sqrt{a_i}x+\frac{1}{\sqrt{a_i}}\right)^2+\left(5-2\sqrt{\frac{m}{M}}-2\sqrt{\frac{M}{m}}\right)x\right]$$
$$=\left(\frac{1}{2}\sum_{i=1}^{n}a_i\right)x^2+\frac{2n+1}{2}\cdot x+\left(\frac{1}{2}\sum_{i=1}^{n}\frac{1}{a_i}\right)$$

Its discriminant is equal to

$$\Delta=\left(n+\frac{1}{2}\right)^2-\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}\frac{1}{a_i}\right),$$

which, by the given condition is nonnegative. Thus $p(x)$ has a real root r, and

$$0=2p(r)\geq\left(5-2\sqrt{\frac{m}{M}}-2\sqrt{\frac{M}{m}}\right)r.$$

Because all of the coefficients of p are positive, we must have $r<0$, from which (2) follows.

Solution 3. We set $a=\frac{a_2+\cdots+a_{n-1}}{n-2}$. Then $m\leq a_2\leq a\leq a_{n-1}\leq M$ and $a_2+\cdots+a_{n-1}=(n-2)a$. By the AM-HM Inequality, we have

$$\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}\geq\frac{(n-2)^2}{a_2+\cdots+a_{n-1}}=\frac{n-2}{a}.$$

If follows that

$$\left(n+\frac{1}{2}\right)^2\geq(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$
$$\geq(m+(n-2)a+M)\left(\frac{1}{m}+\frac{n-2}{a}+\frac{1}{M}\right)$$
$$=(m+M)\left(\frac{1}{m}+\frac{1}{M}\right)+(n-2)^2+\frac{(n-2)(m+M)}{a}+(n-2)a\left(\frac{1}{m}+\frac{1}{M}\right)$$
$$=\frac{(m+M)^2}{mM}+(n-2)^2+\frac{(n-2)(m+M)}{mM}\cdot\left(\frac{mM}{a}+a\right)$$

By the AM-GM Inequality, we have $\frac{mM}{a}+a\geq2\sqrt{mM}$ with equality at $m\leq a=\sqrt{mM}\leq M$. We deduce that

$$\left(n+\frac{1}{2}\right)^2\geq\frac{(m+M)^2}{mM}+(n-2)^2+\frac{2(n-2)(m+M)}{\sqrt{mM}}.$$

Setting $t=\frac{m+M}{\sqrt{mM}}$ in the last inequality yields

$$\left(n+\frac{1}{2}\right)^2\geq t^2+(n-2)^2+2(n-2)t=(t+n-2)^2,$$

from which it follows that

$$n + \frac{1}{2} \geq t + n - 2.$$

Hence $t \leq 5/2$, which is (1).

This problem was suggested by Titu Andreescu. The second solution was contributed by Adam Hesterberg, and the third by Zuming Feng.

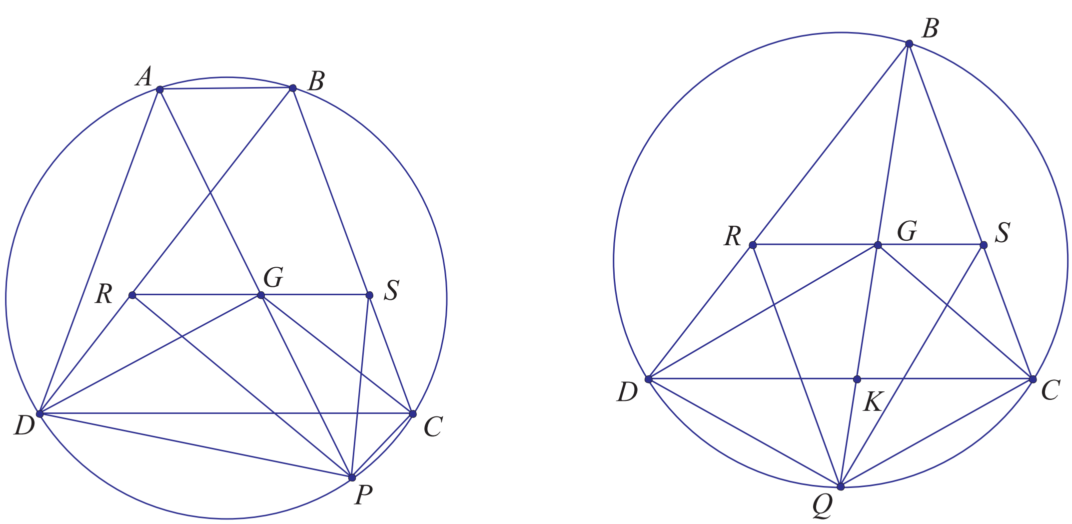
5.

Solution 1. First, we prove the "if" part by assuming that ray BG bisects $\angle CBD$; that is, we assume that $\widehat{DQ} = \widehat{CQ}$.

It is easy to see that $ABCD$ is an isosceles trapezoid with $AD = BC$. In particular, $\widehat{AD} = \widehat{BC}$ and $\widehat{AC} = \widehat{BD}$.

Because $ABCPD$ is cyclic, it follows that

$$\angle APC = \frac{\widehat{AC}}{2} = \frac{\widehat{BD}}{2} = \angle BCD = \angle SCD \quad \text{and} \quad \angle APD = \frac{\widehat{AD}}{2} = \frac{\widehat{BC}}{2} = \angle BDC = \angle RDC.$$



Because $RS \parallel DC$, it follows that $180° = \angle GRD + \angle RDC = \angle GRD + \angle APD$ and $180° = \angle GSC + \angle SCD = \angle GSC + \angle APC$; that is, both $GSCP$ and $GRDP$ are cyclic. Hence, $\angle GPR = \angle GDR$ and $\angle GPS = \angle GCS$. In particular, we hav

$$\angle RPS = \angle GPR + \angle GPS = \angle GDR + \angle GCS. \tag{3}$$

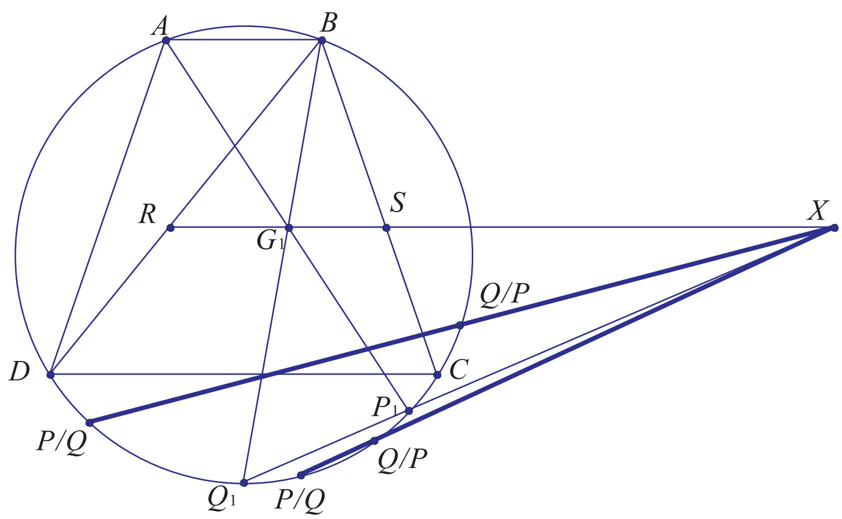
Let K be the intersection of segments BQ and CD. We have $\angle CBK = \angle QBD$ and $\angle KCB = \angle DCB = \angle DQB$; that is, triangles CBK and QBD are similar to each other. Because $RG \parallel CD$, we have $BG/GK = BR/RD$. This means that G and R are the corresponding points in the similar triangles CBK and QBD. Consequently, we have $\angle BCG = \angle BQR$. In exactly the same way, we can show that $\angle BDG = \angle BQS$. Combining the last two equations together with (3) yields

$$\angle RQS = \angle BQS + \angle BQR = \angle BDG + \angle BCG = \angle RDG + \angle SCG = \angle RPS;$$

from which it follows that $PQRS$ is cyclic.

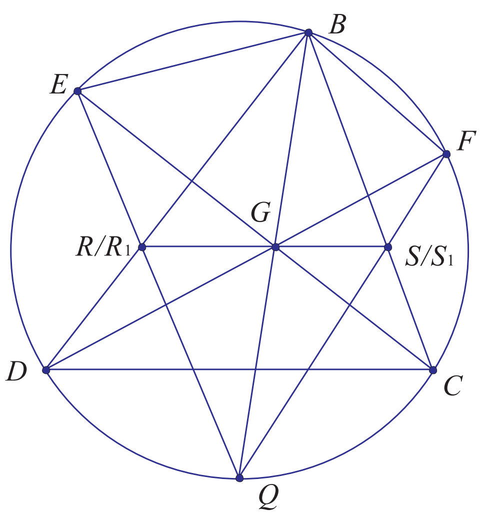
Second, we prove the "only if" part by assuming that $PQRS$ is cyclic. Let γ denote the circumcircle of $PQRS$. We approach indirectly by assuming that ray BG does not bisect $\angle CBD$. Let G_1 be the point on segment RS such that ray BG_1 bisects $\angle CBD$. Let rays AG_1 and BG_1 meet ω again at P_1 and Q_1 (other than A and B). By our proof of the "if" part, P_1Q_1RS is cyclic, and let γ_1 denote its circumcircle.

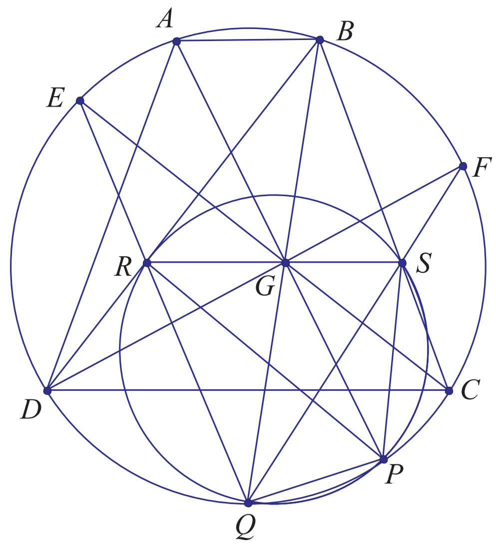
Hence lines RS, PQ, P_1Q_1 are the radical axes of pairs of circles γ and γ_1, γ and ω, γ_1 and ω, respectively. Because segments P_1 is the midpoint of arc $\widehat{CD}$ (not including A and B), lines $P_1Q_1 \not\parallel CD$, implying that lines P_1Q_1 and RS intersect, and let X denote this intersection. Thus X is the radical center of ω, γ, γ_1. In particular, line PQ also passes through X. We obtain the following configuration.



There are two possibilities for the position of line PQ, namely, (1) both P and Q lie on minor arc $\widehat{P_1Q_1}$; (2) one of P and Q lies on minor arc $\widehat{DQ_1}$ and the other lies on minor arc $\widehat{P_1B}$. If G lies on segment RG_1, then Q lies on minor arc $\widehat{DQ}$, and we must have (2). But in this case, P must lie on minor arc $\widehat{Q_1P_1}$, violating (2). If G lies on segment G_1S, then P must lie on minor arc $\widehat{P_1B}$, and again we must have (2). But in this case, Q must lie on minor arc $\widehat{Q_1C}$, violating (2). In every case, we have a contradiction. Hence our assumption was wrong, and ray BG bisects $\angle CBD$.

Solution 2. We present another approach of the "if" part.





Let rays CG and DG meet ω again at E and F, respectively. Let R_1 denote the intersection of segments BD and QE, and let S_1 denote the intersection of segments BC and QF. Applying Pascal's theorem to cyclic hexagon $BDFQEC$ shows that R_1, G, S_1 are collinear. Because

$$\angle R_1EG = \angle QEC = \frac{\widehat{CQ}}{2} = \frac{\widehat{DQ}}{2} = \angle DBQ = \angle R_1BG,$$

we deduce that $EBGR_1$ is cyclic. Because $EBGR_1$ and $EBCD$ are cyclic, we have

$$\angle BR_1S_1 = \angle BR_1G = \angle BEG = \angle BEC = \angle BDC,$$

from which it follows that $R_1S_1 \parallel CD$; that is, $R_1 = R$ and $S_1 = S$.

Therefore, (3) becomes

$$\angle RPS = \angle GDR + \angle GCS = \angle FDB + \angle BCE = \angle FQB + \angle BQE = \angle FQE = \angle RQS,$$

implying that $PQRS$ is cyclic.

This problem was suggested by Zuming Feng.

6. **Solution 1.** First, we claim there exist i, j such that $(s_i - s_j)(t_i - t_j) \neq 0$. Indeed, for any fixed i, because the sequence $s_1, s_2, \ldots$ is nonconstant, there is some j such that $s_j \neq s_i$. If $t_j \neq t_i$ the claim follows, so suppose $t_j = t_i$. Because the sequence $t_1, t_2, \ldots$ is nonconstant,

there exists k such that $t_k \neq t_i$. If $s_k \neq s_i$ the claim again follows, so suppose $s_k = s_i$. Then $(s_j - s_k)(t_j - t_k) = (s_j - s_i)(t_i - t_k) \neq 0$, and the claim is proven.

We can reorder the pairs (s_i, t_i) relative to each other without affecting either the hypothesis or the conclusion of the problem. So by a suitable reordering, we may assume that $(s_1 - s_2)(t_1 - t_2) \neq 0$.

Second, for any constants a and b, we can replace s_i by $s_i - a$ and t_i by $t_i - b$ for all i without affecting either the hypothesis or the conclusion of the problem (since all the differences $s_i - s_j$ and $t_i - t_j$ remain unchanged). In particular, by taking $a = s_1$ and $b = t_1$, we may assume that $s_1 = t_1 = 0$. So we have reduced the problem to the case $s_1 = t_1 = 0$, $s_2 \neq 0$, $t_2 \neq 0$.

Call a pair of positive rational numbers (A, B) *good* if AB is an integer, and As_j and Bt_j are also integers for all j.

Third, we show that a good pair exists.

We know that for all $i \geq 2$, $(s_i - s_1)(t_i - t_1) = s_i t_i$ is an integer; and for all $i, j \geq 2$, $(s_i - s_j)(t_i - t_j) = s_i t_i - s_i t_j - s_j t_i + s_j t_j$ is an integer, which implies $s_i t_j + s_j t_i$ is an integer.

Write the rational numbers s_j, t_j in lowest terms as $s_j = p_j/q_j$ and $t_j = u_j/v_j$. We know that, for each j, $s_j t_j = p_j u_j / q_j v_j$ is an integer. Because u_j is relatively prime to v_j, then, p_j is divisible by v_j, say $p_j = d_j v_j$ for some integer d_j. We also know that

$$s_2 t_j + s_j t_2 = \frac{p_2 u_j}{q_2 v_j} + \frac{p_j u_2}{q_j v_2} = \frac{p_2 u_j q_j v_2 + p_j u_2 q_2 v_j}{q_2 v_j q_j v_2}$$

is an integer. In particular, q_j, being a factor of the denominator, must divide the numerator. But q_j divides $p_2 u_j q_j v_2$, so it also divides the other term, $p_j u_2 q_2 v_j = d_j u_2 q_2 v_j^2$. Since q_j is relatively prime to $p_j = d_j v_j$, it must divide $u_2 q_2$. Moreover, $u_2 q_2 \neq 0$, because of our assumption $t_2 \neq 0$. So we have a positive integer $A = |u_2 q_2|$ such that As_j is an integer for all j. Analogously, we can find a positive integer B such that Bt_j is an integer for all j. This (A, B) constitute a good pair, and existence is proven.

Now we are ready to complete our proof. We know that some good pair exists. We consider a good pair for which the product AB is as small as possible. We will show that $AB = 1$.

Suppose that, for the minimal good pair, $AB > 1$; then AB has a prime factor p. If the integer As_i is divisible by p for all i, then we can divide A by p and obtain a new good pair $(A/p, B)$ having a smaller product than before — a contradiction. So for some i, As_i is not divisible by p. Then Bt_i must be divisible by p, because $s_i t_i$ is an integer and so $(As_i)(Bt_i) = (AB)(s_i t_i)$ is an integer divisible by p. Likewise, there exists some j such that Bt_j is not divisible by p, but As_j is.

Now write

$$(AB)(s_it_j + s_jt_i) - (As_j)(Bt_i) = (As_i)(Bt_j).$$

All the parenthesized factors are integers, and the left-hand side is divisible by p, but the right-hand side is not. This contradiction completes the proof that the minimal good pair satisfies $AB = 1$.

But now take the minimal good pair (A, B), and let $r = A$. We have that $s_ir = As_i$ and $t_i/r = Bt_i$ are integers for all i, from which our desired conclusion follows.

Solution 2. For p a prime, define the *p-adic norm* $\|\cdot\|_p$ on rational numbers as follows: for $r \neq 0$, $\|r\|_p$ is the unique integer n for which we can write $r = p^n a/b$ with a, b integers not divisible by p. (By convention, $\|0\|_p = +\infty$.) We will repeatedly use the well-known (or easy to prove) fact that for any rational numbers r_1, r_2, we have $\|r_1 \pm r_2\|_p \geq \min(\|r_1\|_p, \|r_2\|_p)$, with equality whenever $\|r_1\|_p \neq \|r_2\|_p$. The condition of the problem implies that

$$\|s_i - s_j\|_p \geq -\|t_i - t_j\|_p \tag{4}$$

for all i, j and all prime p.

We claim in fact that

$$\|s_i - s_j\|_p \geq -\|t_k - t_l\|_p$$

for all i, j, k, l and all prime p. Suppose otherwise; then there exist i, j, k, l, p for which $\|s_i - s_j\|_p < -\|t_k - t_l\|_p$. Since $\|s_i - s_j\|_p = \|(s_i - s_k) - (s_j - s_k)\|_p \geq \min(\|s_i - s_k\|_p, \|s_j - s_k\|_p)$, at least one of $\|s_i - s_k\|_p$ and $\|s_j - s_k\|_p$, say the former, is strictly less than $-\|t_k - t_l\|_p$. By (4), it follows that $\|t_i - t_k\|_p > \|t_k - t_l\|_p$, and thus $\|t_i - t_l\|_p = \|(t_i - t_k) + (t_k - t_l)\|_p = \|t_k - t_l\|_p$. Then by (4) again, $\|s_i - s_l\|_p \geq -\|t_k - t_l\|_p$ and $\|s_k - s_l\|_p \geq -\|t_k - t_l\|_p$, whence $\|s_i - s_k\|_p = \|(s_i - s_l) - (s_k - s_l)\|_p \geq -\|t_k - t_l\|_p$, contradicting the assumption that $\|s_i - s_k\|_p < -\|t_k - t_l\|_p$. This proves the claim.

Now for each prime p, define the integer $f(p) = \min_{i,j} \|s_i - s_j\|_p$. Choose i_0, j_0, k_0, l_0 such that $s_{i_0} \neq s_{j_0}$ and $t_{k_0} \neq t_{l_0}$; then $f(p)$ exists since it is bounded below by $-\|t_{k_0} - t_{l_0}\|_p$ (by the claim) and above by $\|s_{i_0} - s_{j_0}\|_p$. Moreover, if p does not divide the numerator or denominator of either $s_{i_0} - s_{j_0}$ or $t_{k_0} - t_{l_0}$, then $\|s_{i_0} - s_{j_0}\|_p = \|t_{k_0} - t_{l_0}\|_p = 0$ and thus $f(p) = 0$. It follows that $f(p) = 0$ for all but finitely many primes.

We can now define $r = \prod_p p^{-f(p)}$, where the product is over all primes. For any i, j, we have $\|s_i - s_j\|_p \geq f(p)$ for all p by construction, and so $(s_i - s_j)r$ is an integer. On the other hand, for any k, l and any prime p, $\|t_k - t_l\|_p \geq -\|s_i - s_j\|_p$ for all i, j by the claim, and so $\|t_k - t_l\|_p \geq -f(p)$. It follows that $(t_k - t_l)/r$ is an integer for all k, l, whence r is the desired rational number.

This problem and the first solution was suggested by Gabriel Carroll. The second solution was suggested by Lenhard Ng.

USAMO 2009 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2009 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

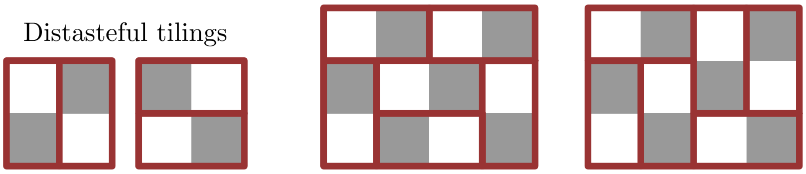
Contents

§0 Problems

1. Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

2. Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \dots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c=0$.

3. We define a *chessboard polygon* to be a simple polygon whose sides are situated along lines of the form $x=a$ or $y=b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes and colors shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



Prove that (a) if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully, and (b) such a tasteful tiling is unique.

4. For $n \ge 2$, let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \le \left(n+\frac{1}{2}\right)^2.$$

Prove that $\max(a_1,\dots,a_n) \le 4\min(a_1,\dots,a_n)$.

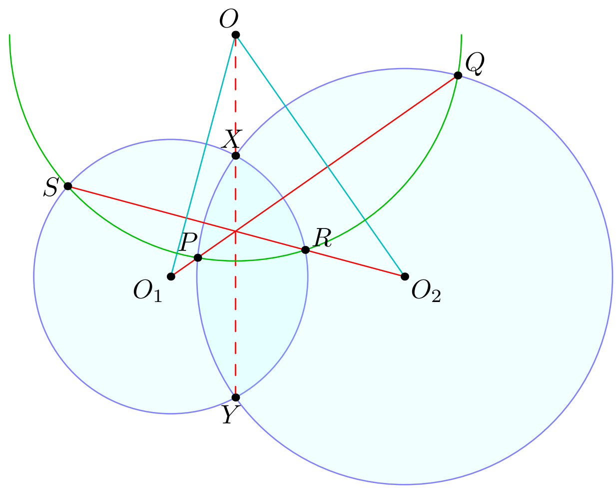
5. Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

6. Let $s_1, s_2, s_3, \dots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \dots$. Suppose that $t_1, t_2, t_3, \dots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

§1 USAMO 2009/1, proposed by Ian Le

Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

Let r_1, r_2, r_3 denote the circumradii of ω_1, ω_2, and ω_3, respectively.



We wish to show that O_3 lies on the radical axis of ω_1 and ω_2. Let us encode the conditions using power of a point. Because O_1 is on the radical axis of ω_2 and ω_3,

$$\begin{aligned} \operatorname{Pow}_{\omega_2}(O_1) &= \operatorname{Pow}_{\omega_3}(O_1) \\ \implies O_1O_2^2 - r_2^2 &= O_1O_3^2 - r_3^2. \end{aligned}$$

Similarly, because O_2 is on the radical axis of ω_1 and ω_3, we have

$$\begin{aligned} \operatorname{Pow}_{\omega_1}(O_2) &= \operatorname{Pow}_{\omega_3}(O_2) \\ \implies O_1O_2^2 - r_1^2 &= O_2O_3^2 - r_3^2. \end{aligned}$$

Subtracting the two gives

$$\begin{aligned} (O_1O_2^2 - r_2^2) - (O_1O_2^2 - r_1^2) &= (O_1O_3^2 - r_3^2) - (O_2O_3^2 - r_3^2) \\ \implies r_1^2 - r_2^2 &= O_1O_3^2 - O_2O_3^2 \\ \implies O_2O_3^2 - r_2^2 &= O_1O_3^2 - r_1^2 \\ \implies \operatorname{Pow}_{\omega_2}(O_3) &= \operatorname{Pow}_{\omega_1}(O_3) \end{aligned}$$

as desired.

§2 USAMO 2009/2, proposed by Kiran Kedlaya and Tewordos Amdeberhan

Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \dots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c=0$.

The answer is n with n even and $n+1$ with n odd; the construction is to take all odd numbers.

To prove this is maximal, it suffices to show it for n even; we do so by induction on even $n \geq 2$ with the base case being trivial. Letting A be the subset, we consider three cases:

(i) If $|A \cap \{-n, -n+1, n-1, n\}| \leq 2$, then by the hypothesis for $n-2$ we are done.

(ii) If both $n \in A$ and $-n \in A$, then there can be at most $n-2$ elements in $A \setminus \{\pm n\}$, one from each of the pairs $(1, n-1)$, $(2, n-2)$, $\dots$ and their negations.

(iii) If $n, n-1, -n+1 \in A$ and $-n \notin A$, and at most $n-3$ more can be added, one from each of $(1, n-2)$, $(2, n-3)$, $\dots$ and $(-2, -n+2)$, $(-3, -n+3)$, $\dots$. (In particular $-1 \notin A$. Analogous case for $-A$ if $n \notin A$.)

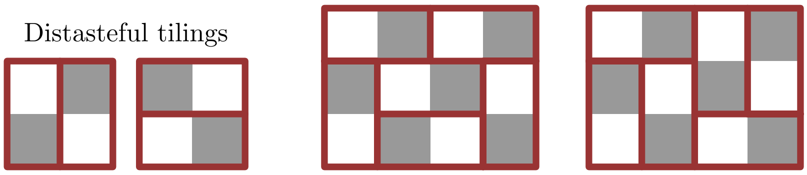
Thus in all cases, $|A| \leq n$ as needed.

Remark. Examples of equality cases:

- All odd numbers
- For n even, the set $\{1, 2, \dots, n\}$
- For $n=4$, the set $\{-3, 2, 3, 4\}$ also achieves the optimum. I suspect there are more.

§3 USAMO 2009/3, proposed by Sam Vandervelde

We define a *chessboard polygon* to be a simple polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes and colors shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



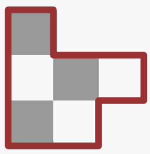
Prove that (a) if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully, and (b) such a tasteful tiling is unique.

Proof of (a): This is easier, and by induction. Let $\mathcal{P}$ denote the chessboard polygon which can be tiled by dominoes.

Consider a *lower-left* square s of the polygon, and WLOG is it black (other case similar). Then we have two cases:

- If there exists a domino tiling of $\mathcal{P}$ where s is covered by a vertical domino, then delete this domino and apply induction on the rest of $\mathcal{P}$. This additional domino will not cause any distasteful tilings.
- Otherwise, assume s is covered by a horizontal domino in *every* tiling. Again delete this domino and apply induction on the rest of $\mathcal{P}$. The resulting tasteful tiling should not have another horizontal domino adjacent to the one covering s, because otherwise we could have replaced that 2×2 square with two vertical dominoes to arrive in the first case. So this additional domino will not cause any distasteful tilings.

Remark. The second case can actually arise, for example in the following picture.



Thus one cannot just try to cover s with a vertical domino and claim the rest of $\mathcal{P}$ is tile-able. So the induction is not as easy as one might hope.

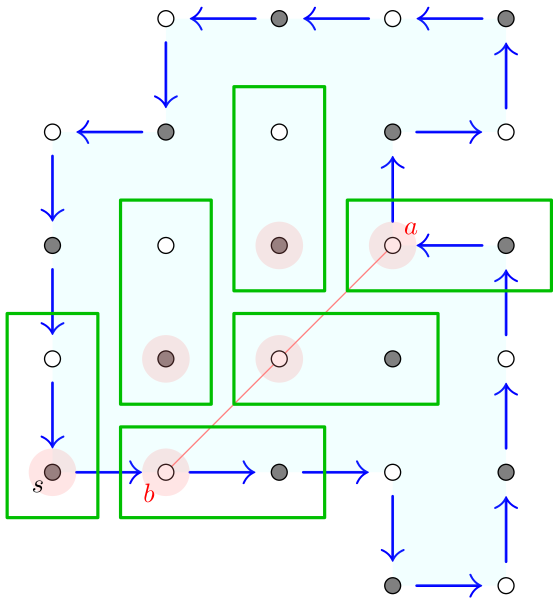
One can phrase the solution algorithmically too, in the following way: any time we see a distasteful tiling, we rotate it to avoid the bad pattern. The bottom-left corner eventually becomes stable, and an induction shows the termination of the algorithm.

Proof of (b): We now turn to proving uniqueness. Suppose for contradiction there are two distinct tasteful tilings. Overlaying the two tilings on top of each other induces several *cycles* of overlapping dominoes at positions where the tilings differ.

Henceforth, it will be convenient to work with the lattice $\mathbb{Z}^2$, treating the squares as black/white points, and we do so. Let γ be any such cycle and let s denote a lower left point, and again WLOG it is black. Orient γ counterclockwise henceforth. Restrict attention to the lattice polygon $\mathcal{Q}$ enclosed by γ (we consider points of γ as part of $\mathcal{Q}$).

In one of the two tilings of (lattice points of) $\mathcal{Q}$, the point s will be covered by a horizontal domino; in the other tiling s will be covered by a vertical domino. From now on we will focus only on the latter one. Observe that we now have a set of dominoes along γ, such that γ points from the white point to the black point within each domino.

Now impose coordinates so that $s = (0, 0)$. Consider the stair-case sequence of points $p_0 = s = (0, 0)$, $p_1 = (1, 0)$, $p_2 = (1, 1)$, $p_3 = (2, 1)$, and so on. By hypothesis, p_0 is covered by a vertical domino. Then p_1 must be covered by a horizontal domino, to avoid a distasteful tiling. Then if p_2 is in $\mathcal{Q}$, then it must be covered by a vertical domino to avoid a distasteful tiling, and so on. We may repeat this argument as long as the points p_i lie inside $\mathcal{Q}$. (See figure below; the staircase sequence is highlighted by red halos.)



The curve γ by definition should cross $y = x - 1$ at the point $b = (1, 0)$. Let a denote the first point of this sequence after p_1 for which γ *crosses* $y = x - 1$ again.

Now a is tiled by a vertical domino whose black point is to the right of ℓ. But the line segment ℓ cuts $\mathcal{Q}$ into two parts, and the orientation of γ has this path also entering from the right. This contradicts the fact that the orientation of γ points only from white to black within dominoes. This contradiction completes the proof.

> **Remark.** Note the problem is false if you allow holes (consider a 3×3 with the middle square deleted).

§4 USAMO 2009/4, proposed by Titu Andreescu

For $n \ge 2$, let $a_1, a_2, \dots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) \le \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max(a_1, \dots, a_n) \le 4 \min(a_1, \dots, a_n)$.

Assume a_1 is the largest and a_2 is the smallest. Let $M = a_1/a_2$. We wish to show $M \le 4$.

In left-hand side of given, write as $a_2 + a_1 + \dots + a_n$. By Cauchy Schwarz, one obtains

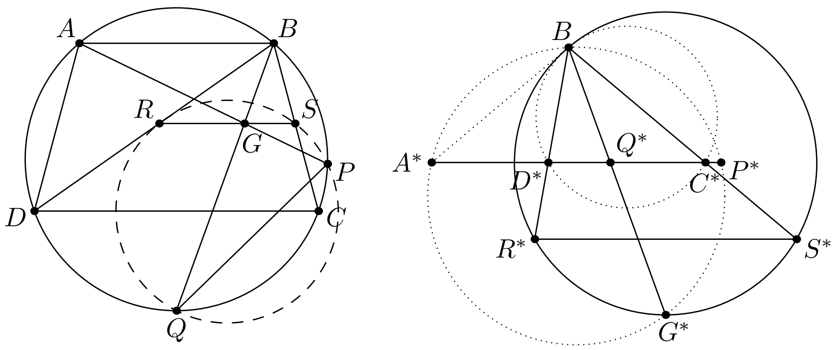
$$\begin{aligned}\left(n + \frac{1}{2}\right)^2 &\ge (a_2 + a_1 + a_3 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}\right) \\ &\ge \left(\sqrt{\frac{a_2}{a_1}} + \sqrt{\frac{a_1}{a_2}} + 1 + \dots + 1\right)^2 \\ &\ge \left(\sqrt{1/M} + \sqrt{M} + (n-2)\right)^2.\end{aligned}$$

Expanding and solving for M gives $1/4 \le M \le 4$ as needed.

§5 USAMO 2009/5, proposed by Zuming Feng

Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

Perform an inversion around B with arbitrary radius, and denote the inverse of a point Z with Z^*.



After inversion, we obtain a cyclic quadrilateral $BS^*G^*R^*$ and points C^*, D^* on $\overline{BS^*}$, $\overline{BR^*}$, such that (BC^*D^*) is tangent to $(BS^*G^*R^*)$ — in other words, so that $\overline{C^*D^*}$ is parallel to $\overline{S^*R^*}$. Point A^* lies on line $\overline{C^*D^*}$ so that $\overline{A^*B}$ is tangent to $(BS^*G^*R^*)$. Points P^* and Q^* are the intersections of (A^*BG^*) and $\overline{BG^*}$ with line C^*D^*.

Observe that $P^*Q^*R^*S^*$ is a trapezoid, so it is cyclic if and only if it isosceles.

Let X be the second intersection of line G^*P^* with (BS^*R^*). Because

$$\measuredangle Q^*P^*G^* = \measuredangle A^*BG^* = \measuredangle BXG^*$$

we find that BXS^*R^* is an isosceles trapezoid.

If G^* is indeed the midpoint of the arc then everything is clear by symmetry now. Conversely, if $P^*R^* = Q^*S^*$ then that means $P^*Q^*R^*S^*$ is a cyclic trapezoid, and hence that the perpendicular bisectors of $\overline{P^*Q^*}$ and $\overline{R^*S^*}$ are the same. Hence B, X, P^*, Q^* are symmetric around this line. This forces G^* to be the midpoint of arc R^*S^* as desired.

§6 USAMO 2009/6, proposed by Gabriel Carroll

Let $s_1, s_2, s_3, \dots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \dots$. Suppose that $t_1, t_2, t_3, \dots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

First we eliminate the silly edge case:

Claim — For some i and j, we have $(s_i - s_j)(t_i - t_j) \neq 0$.

Proof. Assume not. WLOG $s_1 \neq s_2$, then $t_1 = t_2$. Now select i such that $t_i \neq t_1 = t_2$. Then either $t_i - s_1 \neq 0$ or $t_i - s_2 \neq 0$, contradiction. □

So, WLOG (by permutation) that $n = (s_1 - s_2)(t_1 - t_2) \neq 0$. By shifting and scaling appropriately, we may assume

$$s_1 = t_1 = 0, \quad s_2 = 1, \quad t_2 = n.$$

Thus we deduce

$$s_i t_i \in \mathbb{Z}, \quad s_i t_j + s_j t_i \in \mathbb{Z} \qquad \forall i, j.$$

Claim — For any index i, $t_i \in \mathbb{Z}$, $s_i \in \frac{1}{n}\mathbb{Z}$.

Proof. We have $s_i t_i \in \mathbb{Z}$ and $t_i + n s_i \in \mathbb{Z}$ by problem condition. By looking at ν_p of this, we conclude $n s_i, t_i \in \mathbb{Z}$ (since if either as negative p-adic value, so does the other, and then $s_i t_i \notin \mathbb{Z}$). □

Last claim:

Claim — If $d = \gcd t_\bullet$, then $d s_i \in \mathbb{Z}$ for all i.

Proof. Consider a prime $p \mid n$, and let $e = \nu_p(t_j)$. We will show $\nu_p(s_i) \geq -e$ for any i.

This is already true for $i = j$, so assume $i \neq j$. Assume for contradiction $\nu_p(s_i) < -e$. Then $\nu_p(t_i) > e = \nu_p(t_k)$. Since $\nu_p(s_k) \geq -e$ we deduce $\nu_p(s_i t_k) < \nu_p(s_k t_i)$; so $\nu_p(s_i t_k) \geq 0$ and $\nu_p(s_i) \geq -e$ as desired. □

39^{th} United States of America Mathematical Olympiad 2010

Day I 12:30 PM – 5 PM EDT

April 27, 2010

1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

2. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \ldots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

3. The 2010 positive numbers $a_1, a_2, \ldots, a_{2010}$ satisfy the inequality $a_i a_j \leq i+j$ for all distinct indices i, j. Determine, with proof, the largest possible value of the product $a_1 a_2 \cdots a_{2010}$.

39^{th} United States of America Mathematical Olympiad 2010

Day II 12:30 PM – 5 PM EDT

April 28, 2010

4. Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

5. Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

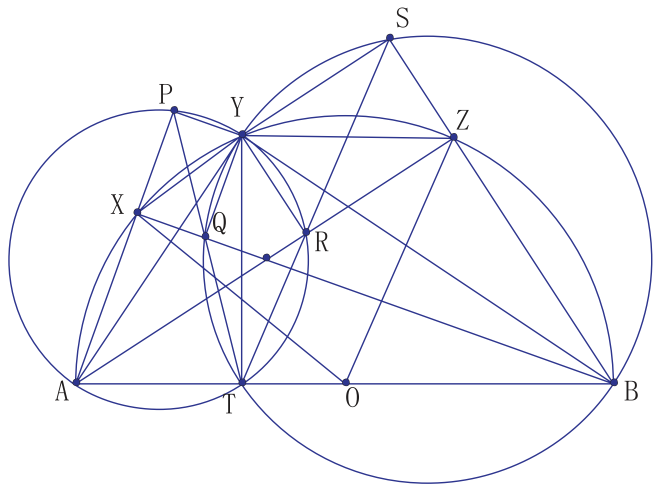
$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \ldots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

6. A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

39^{th} United States of America Mathematical Olympiad 2010

1. **Solution by Titu Andreescu:** Let T be the foot of the perpendicular from Y to line AB. We note the P, Q, T are the feet of the perpendiculars from Y to the sides of triangle ABX. Because Y lies on the circumcircle of triangle ABX, points P, Q, T are collinear, by Simson's theorem. Likewise, points S, R, T are collinear.



We need to show that $\angle XOZ = 2\angle PTS$ or

$$\angle PTS = \frac{\angle XOZ}{2} = \frac{\overset{\frown}{XZ}}{2} = \frac{\overset{\frown}{XY}}{2} + \frac{\overset{\frown}{YZ}}{2}$$
$$= \angle XAY + \angle ZBY = \angle PAY + \angle SBY.$$

Because $\angle PTS = \angle PTY + \angle STY$, it suffices to prove that

$$\angle PTY = \angle PAY \quad \text{and} \quad \angle STY = \angle SBY;$$

that is, to show that quadrilaterals $APYT$ and $BSYT$ are cyclic, which is evident, because $\angle APY = \angle ATY = 90^\circ$ and $\angle BTY = \angle BSY = 90^\circ$.

Alternate Solution from Lenny Ng and Richard Stong: Since YQ, YR are perpendicular to BX, AZ respectively, $\angle RYQ$ is equal to the acute angle between lines BX and AZ, which is $\frac{1}{2}(\widehat{AX} + \widehat{BZ}) = \frac{1}{2}(180^\circ - \widehat{XZ})$ since X, Z lie on the circle with diameter AB. Also, $\angle AXB = \angle AZB = 90^\circ$ and so $PXQY$ and $SZRY$ are rectangles, whence $\angle PQY = 90^\circ - \angle YXB = 90^\circ - \widehat{YB}/2$ and $\angle YRS = 90^\circ - \angle AZY = 90^\circ - \widehat{AY}/2$. Finally, the angle between PQ and RS is

$$\begin{aligned}\angle PQY + \angle YRS - \angle RYQ &= (90^\circ - \widehat{YB}/2) + (90^\circ - \widehat{AY}/2) - (90^\circ - \widehat{XZ}/2)\\ &= \widehat{XZ}/2\\ &= (\angle XOZ)/2,\end{aligned}$$

as desired.

This problem was proposed by Titu Andreescu.

2. **Solution from Kiran Kedlaya:** Let h_i also denote the student with height h_i. We prove that for $1 \leq i < j \leq n$, h_j can switch with h_i at most $j - i - 1$ times. We proceed by induction on $j - i$, the base case $j - i = 1$ being evident because h_i is not allowed to switch with h_{i-1}.

   For the inductive step, note that h_i, h_{j-1}, h_j can be positioned on the circle either in this order or in the order h_i, h_j, h_{j-1}. Since h_{j-1} and h_j cannot switch, the only way to change the relative order of these three students is for h_i to switch with either h_{j-1} or h_j. Consequently, any two switches of h_i with h_j must be separated by a switch of h_i with h_{j-1}. Since there are at most $j - i - 2$ of the latter, there are at most $j - i - 1$ of the former.

   The total number of switches is thus at most

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j - i - 1) = \sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} j = \sum_{i=1}^{n-1} \binom{n-i}{2} = \sum_{i=1}^{n-1} \left(\binom{n-i+1}{3} - \binom{n-i}{3} \right) = \binom{n}{3}.$$

   Note: One can also ask to prove that the number of switches before no more are possible depends only on the original ordering, or to find all initial positions for which $\binom{n}{3}$ switches are possible (the only one is when the students are sorted in increasing order).

   Alternative Solution from Warut Suksompong: For $i = 1, 2, \ldots, n-1$, let s_i be the number of students with height no more than h_{i+1} standing (possibly not directly) behind the student with height h_i and (possibly not directly) in front of the one with height h_{i+1}. Note that $s_i \leq i - 1$ for all i.

   Now we take a look what happens when two students switch places.

   - If the student with height h_n is involved in the switch, s_{n-1} decreases by 1, while all the other s_i's remain the same.

- Otherwise, suppose the students with heights h_a and h_b are switched, with $a+1 < b < n$, then s_{b-1} decreases by 1, while s_b increases by 1. All the other s_i's remain the same.

Since $s_i \le i-1$ for all $i = 1, 2, \dots, n-1$, the maximal number of switches is no more than the number of switches in the case where initially $s_i = i-1$ for all i. In that case, the number of switches is $\sum_{i=1}^{n-2} i(n-1-i) = \binom{n}{3}$.

Note: With this solution, it is also easy to see that the number of switches until no more are possible depends only on the original ordering.

This problem was proposed by Kiran Kedlaya jointly with Travis Schedler and David Speyer.

3. **Solution from Gabriel Carroll:** Multiplying together the inequalities $a_{2i-1}a_{2i} \le 4i-1$ for $i = 1, 2, \dots, 1005$, we get

$$a_1 a_2 \cdots a_{2010} \le 3 \cdot 7 \cdot 11 \cdots 4019. \tag{1}$$

The tricky part is to show that this bound can be attained.

Let

$$a_{2008} = \sqrt{\frac{4017 \cdot 4018}{4019}}, \qquad a_{2009} = \sqrt{\frac{4019 \cdot 4017}{4018}}, \qquad a_{2010} = \sqrt{\frac{4018 \cdot 4019}{4017}},$$

and define a_i for $i < 2008$ by downward induction using the recursion

$$a_i = (2i+1)/a_{i+1}.$$

We then have

$$a_i a_j = i + j \qquad \text{whenever } j = i+1 \qquad \text{or} \qquad i = 2008, j = 2010. \tag{2}$$

We will show that (2) implies $a_i a_j \le i + j$ for all $i < j$, so that this sequence satisfies the hypotheses of the problem. Since $a_{2i-1}a_{2i} = 4i-1$ for $i = 1, \dots, 1005$, the inequality (1) is an equality, so the bound is attained.

We show that $a_i a_j \le i + j$ for $i < j$ by downward induction on $i + j$. There are several cases:

- If $j = i+1$, or $i = 2008, j = 2010$, then $a_i a_j = i + j$, from (2).

- If $i = 2007, j = 2009$, then

$$a_i a_{i+2} = \frac{(a_i a_{i+1})(a_{i+2} a_{i+3})}{(a_{i+1} a_{i+3})} = \frac{(2i+1)(2i+5)}{2i+4} < 2i+2.$$

Here the second equality comes from (2), and the inequality is checked by multiplying out: $(2i+1)(2i+5) = 4i^2 + 12i + 5 < 4i^2 + 12i + 8 = (2i+2)(2i+4)$.

- If $i < 2007$ and $j = i+2$, then we have

$$a_i a_{i+2} = \frac{(a_i a_{i+1})(a_{i+2} a_{i+3})(a_{i+2} a_{i+4})}{(a_{i+1} a_{i+2})(a_{i+3} a_{i+4})} \le \frac{(2i+1)(2i+5)(2i+6)}{(2i+3)(2i+7)} < 2i+2.$$

The first inequality holds by applying the induction hypothesis for $(i+2, i+4)$, and (2) for the other pairs. The second inequality can again be checked by multiplying out: $(2i+1)(2i+5)(2i+6) = 8i^3 + 48i^2 + 82i + 30 < 8i^3 + 48i^2 + 82i + 42 = (2i+2)(2i+3)(2i+7)$.

- If $j - i > 2$, then

$$a_i a_j = \frac{(a_i a_{i+1})(a_{i+2} a_j)}{a_{i+1} a_{i+2}} \le \frac{(2i+1)(i+2+j)}{2i+3} < i+j.$$

Here we have used the induction hypothesis for $(i+2, j)$, and again we check the last inequality by multiplying out: $(2i+1)(i+2+j) = 2i^2 + 5i + 2 + 2ij + j < 2i^2 + 3i + 2ij + 3j = (2i+3)(i+j)$.

This covers all the cases and shows that $a_i a_j \le i + j$ for all $i < j$, as required.

Variant Solution by Paul Zeitz: It is possible to come up with a semi-alternative solution, after constructing the sequence, by observing that when the two indices differ by an even number, you can divide out precisely. For example, if you wanted to look at $a_3 a_8$, you would use the fact that $a_3 a_4 a_5 a_6 a_7 a_8 = (7)(11)(15)$ and $a_4 a_5 a_6 a_7 = (9)(13)$. Hence we need to check that $(7)(11)(15)/((9)(13)) < 11$, which is easy AMGM/ Symmetry.

However, this attractive method requires much more subtlety when the indices differ by an odd number. It can be pulled off, but now you need, as far as I know, either to use the precise value of a_{2010} or establish inequalities for $(a_k)^2$ for all values of k. It is ugly, but it may be attempted.

This problem was suggested by Gabriel Carroll.

4. **Solution from Zuming Feng:** The answer is *no*, it is not possible for segments AB, BC, BI, ID, CI, IE to all have integer lengths.

Assume on the contrary that these segments do have integer side lengths. We set $\alpha = \angle ABD = \angle DBC$ and $\beta = \angle ACE = \angle ECB$. Note that I is the incenter of triangle ABC, and so $\angle BAI = \angle CAI = 45°$. Applying the Law of Sines to triangle ABI yields

$$\frac{AB}{BI} = \frac{\sin(45° + \alpha)}{\sin 45°} = \sin\alpha + \cos\alpha,$$

by the addition formula (for the sine function). In particular, we conclude that $s = \sin\alpha + \cos\alpha$ is rational. It is clear that $\alpha + \beta = 45°$. By the subtraction formulas, we have

$$s = \sin(45° - \beta) + \cos(45° - \beta) = \sqrt{2}\cos\beta,$$

from which it follows that $\cos\beta$ is not rational. On the other hand, from right triangle ACE, we have $\cos\beta = AC/EC$, which is rational by assumption. Because $\cos\beta$ cannot not be both rational and irrational, our assumption was wrong and not all the segments AB, BC, BI, ID, CI, IE can have integer lengths.

Alternate Solution from Jacek Fabrykowski: Using notations as introduced in the problem, let $BD = m$, $AD = x$, $DC = y$, $AB = c$, $BC = a$ and $AC = b$. The angle bisector theorem implies

$$\frac{x}{b - x} = \frac{c}{a}$$

and the Pythagorean Theorem yields $m^2 = x^2 + c^2$. Both equations imply that

$$2ac = \frac{(bc)^2}{m^2 - c^2} - a^2 - c^2$$

and since $a^2 = b^2 + c^2$ is rational, a is rational too (observe that to reach this conclusion, we only need to assume that b, c, and m are integers). Therefore, $x = \frac{bc}{a+c}$ is also rational, and so is y. Let now (similarly to the notations above from the solution by Zuming Feng) $\angle ABD = \alpha$ and $\angle ACE = \beta$ where $\alpha + \beta = \pi/4$. It is obvious that $\cos\alpha$ and $\cos\beta$are both rational and the above shows that also $\sin\alpha = x/m$ is rational. On the other hand, $\cos\beta = \cos(\pi/4 - \alpha) = (\sqrt{2}/2)(\sin\alpha + \sin\beta)$, which is a contradiction. The solution shows that a stronger statement holds true: There is no right triangle with both legs and bisectors of acute angles all having integer lengths.

Alternate Solution from Zuming Feng: Prove an even stronger result: there is no such right triangle with AB, AC, IB, IC having rational side lengths. Assume on the contrary, that AB, AC, IB, IC have rational side lengths. Then $BC^2 = AB^2 + AC^2$ is rational. On the other hand, in triangle BIC, $\angle BIC = 135°$. Applying the law of cosines to triangle BIC yields

$$BC^2 = BI^2 + CI^2 - \sqrt{2}BI \cdot CI$$

which is irrational. Because BC^2 cannot be both rational and irrational, we conclude that our assumption was wrong and that not all of the segments AB, AC, IB, IC can have rational lengths.

This problem was proposed by Zuming Feng.

5. **Solution by Titu Andreescu:** We have

$$\begin{aligned}\frac{2}{k(k+1)(k+2)} &= \frac{(k+2)-k}{k(k+1)(k+2)} = \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \\ &= \frac{1}{k} - \frac{1}{k+1} - \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &= \frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} - \frac{3}{k+1}.\end{aligned}$$

Hence

$$\begin{aligned}2S_q &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{q} + \frac{1}{q+1} + \frac{1}{q+2}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \ldots + \frac{1}{q+1}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\frac{3p-1}{2}}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{\frac{p-1}{2}}\right),\end{aligned}$$

and so

$$\begin{aligned}1 - \frac{m}{n} = 1 + 2S_q - \frac{1}{p} &= \frac{1}{\frac{p+1}{2}} + \ldots + \frac{1}{p-1} + \frac{1}{p+1} + \ldots + \frac{1}{\frac{3p-1}{2}} \\ &= \left(\frac{1}{\frac{p+1}{2}} + \frac{1}{\frac{3p-1}{2}}\right) + \ldots + \left(\frac{1}{p-1} + \frac{1}{p+1}\right) \\ &= \frac{p}{\left(\frac{p+1}{2}\right)\left(\frac{3p-1}{2}\right)} + \ldots + \frac{p}{(p-1)(p+1)}.\end{aligned}$$

Because all denominators are relatively prime with p, it follows that $n - m$ is divisible by p and we are done.

This problem was suggested by Titu Andreescu.

6. **Solution by Zuming Feng and Paul Zeitz:** The answer is 43.

We first show that we can always get 43 points. Without loss of generality, we assume that the value of x is positive for every pair of the form (x, x) (otherwise, replace every occurrence of x on the blackboard by $-x$, and every occurrence of $-x$ by x). Consider the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ denote all the distinct absolute values of the integers written on the board.

Let $\phi = \frac{\sqrt{5}-1}{2}$, which is the positive root of $\phi^2 + \phi = 1$. We consider 2^n possible underlining strategies: Every strategy corresponds to an ordered n-tuple $s = (s_1, \ldots, s_n)$ with $s_i = \phi$

or $s_i = 1-\phi$ ($1 \le i \le n$). If $s_i = \phi$, then we underline all occurrences of a_i on the blackboard. If $s_i = 1-\phi$, then we underline all occurrences of $-a_i$ on the blackboard. The weight $w(s)$ of strategy s equals the product $\prod_{i=1}^n s_i$. It is easy to see that the sum of weights of all 2^n strategies is equal to $\sum_s w(s) = \prod_{i=1}^n [\phi + (1-\phi)] = 1$.

For every pair p on the blackboard and every strategy s, we define a corresponding cost coefficient $c(p,s)$: If s scores a point on p, then $c(p,s)$ equals the weight $w(s)$. If s does not score on p, then $c(p,s)$ equals 0. Let $c(p)$ denote the the sum of of coefficients $c(p,s)$ taken over all s. Now consider a fixed pair $p=(x,y)$:

(a) In this case, we assume that $x=y=a_j$. Then every strategy that underlines a_j scores a point on this pair. Then $c(p) = \phi\prod_{i\ne j}^n [\phi+(1-\phi)] = \phi$.

(b) In this case, we assume that $x \ne y$. We have

$$c(p) = \begin{cases} \phi^2 + \phi(1-\phi) + (1-\phi)\phi = 3\phi - 1, & (x,y) = (a_k, a_\ell); \\ \phi(1-\phi) + (1-\phi)\phi + (1-\phi)^2 = \phi, & (x,y) = (-a_k, -a_\ell); \\ \phi^2 + \phi(1-\phi) + (1-\phi)^2 = 2 - 2\phi, & (x,y) = (\pm a_k, \mp a_\ell). \end{cases}$$

By noting that $\phi \approx 0.618$, we can easily conclude that $c(p) \ge \phi$.

We let C denote the sum of the coefficients $c(p,s)$ taken over all p and s. These observations yield that

$$C = \sum_{p,s} c(p,s) = \sum_p c(p) \ge \sum_p \phi = 68\phi > 42.$$

Suppose for the sake of contradiction that every strategy s scores at most 42 points. Then every s contributes at most $42w(s)$ to C, and we get $C \le 42\sum_s w(s) = 42$, which contradicts $C > 42$.

To complete our proof, we now show that we cannot always get 44 points. Consider the blackboard contains the following 68 pairs: For each of $m = 1, \ldots, 8$, five pairs of (m,m) (for a total of 40 pairs of type (a)); For every $1 \le m < n \le 8$, one pair of $(-m,-n)$ (for a total of $\binom{8}{2} = 28$ pairs of type (b)). We claim that we cannot get 44 points from this initial stage. Indeed, assume that exactly k of the integers $1, 2, \ldots, 8$ are underlined. Then we get at most $5k$ points on the pairs of type (a), and at most $28 - \binom{k}{2}$ points on the pairs of type (b). We can get at most $5k + 28 - \binom{k}{2}$ points. Note that the quadratic function $5k + 28 - \binom{k}{2} = -\frac{k^2}{2} + \frac{11k}{2} + 28$ obtains its maximum 43 (for integers k) at $k=5$ or $k=6$. Thus, we can get at most 43 points with this initial distribution, establishing our claim and completing our solution.

This problem was suggested by Zuming Feng.

USAMO 2010 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2010 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

2. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \dots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

3. The 2010 positive real numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \le i + j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1 a_2 \dots a_{2010}$.

4. Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

5. Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \dots + \frac{1}{q(q+1)(q+2)}.$$

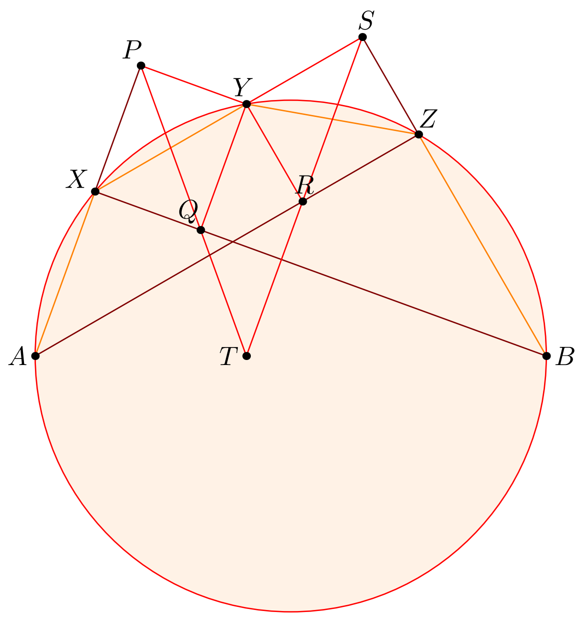
Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

6. There are 68 ordered pairs (not necessarily distinct) of nonzero integers on a blackboard. It's known that for no integer k does both (k, k) and $(-k, -k)$ appear. A student erases some of the 136 integers such that no two erased integers have sum zero, and scores one point for each ordered pair with at least one erased integer. What is the maximum possible score the student can guarantee?

§1 USAMO 2010/1, proposed by Zuming Feng

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Let T be the foot from Y to $\overline{AB}$. Then the Simson line implies that lines PQ and RS meet at T.



Now it's straightforward to see $APYRT$ is cyclic (in the circle with diameter $\overline{AY}$), and therefore

$$\angle RTY = \angle RAY = \angle ZAY.$$

Similarly,

$$\angle YTQ = \angle YBQ = \angle YBX.$$

Summing these gives $\angle RTQ$ is equal to half the measure of arc $\widehat{XZ}$ as needed.

(Of course, one can also just angle chase; the Simson line is not so necessary.)

§2 USAMO 2010/2, proposed by David Speyer

There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

The main claim is the following observation, which is most motivated in the situation $j - i = 2$.

Claim — The students with heights h_i and h_j switch at most $|j - i| - 1$ times.

Proof. By induction on $d = |j - i|$, assuming $j > i$. For $d = 1$ there is nothing to prove.

For $d \geq 2$, look at only students h_j, h_{i+1} and h_i ignoring all other students. After h_j and h_i switch the first time, the relative ordering of the students must be $h_i \to h_j \to h_{i+1}$. Thereafter h_j must always switch with h_{i+1} before switching with h_i, so the inductive hypothesis applies to give the bound $1 + j - (i + 1) - 1 = j - i - 1$. □

Hence, the number of switches is at most

$$\sum_{1 \leq i < j \leq n} (|j - i| - 1) = \binom{n}{3}.$$

§3 USAMO 2010/3, proposed by Gabriel Carroll

The 2010 positive real numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \le i + j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1 a_2 \dots a_{2010}$.

The answer is $3 \times 7 \times 11 \times \cdots \times 4019$, which is clearly an upper bound (and it's not too hard to show this is the lowest number we may obtain by multiplying 1005 equalities together; this is essentially the rearrangement inequality). The tricky part is the construction. Intuitively we want $a_i \approx \sqrt{2i}$, but the details require significant care.

Note that if this is achievable, we will require $a_n a_{n+1} = 2n + 1$ for all odd n. Here are two constructions:

- One can take the sequence such that $a_{2008} a_{2010} = 4028$ and $a_n a_{n+1} = 2n + 1$ for all $n = 1, 2, \dots, 2009$. This can be shown to work by some calculation. As an illustrative example,
$$a_1 a_4 = \frac{a_1 a_2 \cdot a_3 a_4}{a_2 a_3} = \frac{3 \cdot 7}{5} < 5.$$
- In fact one can also take $a_n = \sqrt{2n}$ for all even n (and hence $a_{n-1} = \sqrt{2n} - \frac{1}{\sqrt{2n}}$ for such even n).

Remark. This is a chief example of an "abstract" restriction-based approach. One can motivate it in three steps:

- The bound $3 \cdot 7 \cdot \dots \cdot 4019$ is provably best possible upper bound by pairing the inequalities; also the situation with 2010 replaced by 4 is constructible with bound 21.
- We have $a_n \approx \sqrt{2n}$ heuristically; in fact $a_n = \sqrt{2n}$ satisfies inequalities by AM-GM.
- So we are most worried about $a_i a_j \le i + j$ when $|i - j|$ is small, like $|i - j| = 1$.

I then proceeded to spend five hours on various constructions, but it turns out that the right thing to do was just require $a_k a_{k+1} = 2k + 1$, to make sure these pass: and the problem almost solves itself.

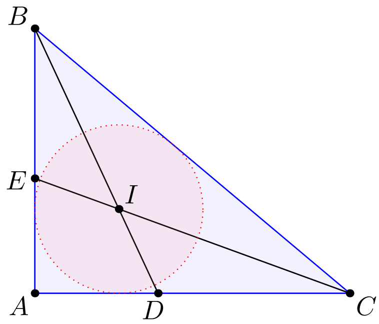
Remark. When 2010 is replaced by 4 it is not too hard to manually write an explicit example: say $a_1 = \frac{\sqrt{3}}{1.1}$, $a_2 = 1.1\sqrt{3}$, $a_3 = \frac{\sqrt{7}}{1.1}$ and $a_4 = 1.1\sqrt{7}$. So this is a reason one might guess that $3 \times 7 \times \cdots \times 4019$ can actually be achieved in the large case.

Remark. Victor Wang says: I believe we can actually prove that WLOG (!) assume $a_i a_{i+1} = 2i + 1$ for all i (but there are other ways to motivate that as well, like linear programming after taking logs), which makes things a bit simpler to think about.

§4 USAMO 2010/4, proposed by Zuming Feng

Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135^\circ$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90^\circ}{2} = 45^\circ.$$

So, $\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 135^\circ$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 - BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

which is irrational, and this produces the desired contradiction.

§5 USAMO 2010/5, proposed by Titu Andreescu

Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m-n$ is divisible by p.

By partial fractions, we have

$$\frac{2}{(3k-1)(3k)(3k+1)} = \frac{1}{3k-1} - \frac{2}{3k} + \frac{1}{3k+1}.$$

Thus

$$\begin{aligned}
2S_q &= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \cdots + \left(\frac{1}{q} - \frac{2}{q+1} + \frac{1}{q+2}\right) \\
&= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{q+2}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{q+1}\right) \\
&= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\
\implies 2S_q - \frac{1}{p} + 1 &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{p+1} + \frac{1}{p+2} \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right)
\end{aligned}$$

Now we are ready to take modulo p. The given says that $q - p + 2 = \frac{q+1}{3}$, so

$$\begin{aligned}
2S_q - \frac{1}{p} + 1 &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\
&\equiv \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{q-p+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\
&= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1} \\
&\equiv 0 \pmod p.
\end{aligned}$$

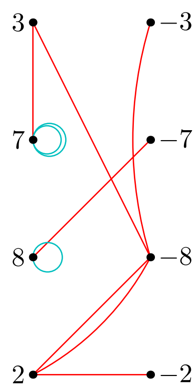
So $\frac{1}{p} - 2S_q \equiv 1 \pmod p$ which is the desired.

§6 USAMO 2010/6, proposed by Zuming Feng and Paul Zeitz

There are 68 ordered pairs (not necessarily distinct) of nonzero integers on a blackboard. It's known that for no integer k does both (k,k) and $(-k,-k)$ appear. A student erases some of the 136 integers such that no two erased integers have sum zero, and scores one point for each ordered pair with at least one erased integer. What is the maximum possible score the student can guarantee?

The answer is 43.

The structure of this problem is better understood as follows: we construct a multigraph whose vertices are the entries, and the edges are the 68 ordered pairs on the blackboard. To be precise, construct a multigraph G with vertices $a_1, b_1, \dots, a_n, b_n$, with $a_i = -b_i$ for each i. The ordered pairs then correspond to 68 edges in G, with self-loops allowed (WLOG) only for vertices a_i. The student may then choose one of $\{a_i, b_i\}$ for each i and wishes to maximize the number of edges adjacent to the set of chosen vertices.



First we use the probabilistic method to show $N \ge 43$. We select the real number $p = \frac{\sqrt{5}-1}{2} \approx 0.618$ satisfying $p = 1 - p^2$. For each i we then select a_i with probability p and b_i with probability $1-p$. Then

- Every self-loop (a_i, a_i) is chosen with probability p.
- Any edge (b_i, b_j) is chosen with probability $1 - p^2$.

All other edges are selected with probability at least p, so in expectation we have $68p \approx 42.024$ edges scored. Hence $N \ge 43$.

For a construction showing 43 is optimal, we let $n = 8$, and put five self-loops on each a_i, while taking a single K_8 on the b_i's. The score achieved for selecting m of the a_i's and $8 - m$ of the b_i's is

$$5m + \left(\binom{8}{2} - \binom{m}{2} \right) \le 43$$

with equality when either $m = 5$ and $m = 6$.

> **Remark** (Colin Tang)**.** Here is one possible motivation for finding the construction. In equality case we probably want all the edges to either be a_i loops or $b_i b_j$ edges. Now if b_i and b_j are not joined by an edge, one can "merge them together", also combining the corresponding a_i's, to get another multigraph with 68 edges whose optimal score is at most the original ones. So by using this smoothing algorithm, we can reduce to a situation where the b_i and b_j are all connected to each other.
>
> It's not unnatural to assume it's a clique then, at which point fiddling with parameters gives the construction. Also, there is a construction for $\lceil 2/3n \rceil$ which is not too difficult to

find, and applying this smoothing operation to this construction could suggest a clique of at least 8 vertices too.

Remark (David Lee)**.** One could consider changing the probability $p(n)$ to be a function of the number n of non-loops (hence there are $68 - n$ loops); we would then have

$$\mathbb{E}[\text{points}] = (68 - n)p(n) + n(1 - p(n)^2).$$

The optimal value of $p(n)$ is then

$$p(n) = \begin{cases} \frac{68-n}{2n} = \frac{34}{n} - \frac{1}{2} & n \geq 23 \\ 1 & n < 22. \end{cases}$$

For $n > 23$ we then have

$$\begin{aligned} E(n) =& (68 - n)\left(\frac{34}{n} - \frac{1}{2}\right) + n\left(1 - \left(\frac{34}{n} - \frac{1}{2}\right)^2\right) \\ =& \frac{5n}{4} + \frac{34^2}{n} - 34 \end{aligned}$$

which has its worst case at around $5n^2 = 68^2$, so at $n = 30$ and $n = 31$. Indeed, one can find

$$\begin{aligned} E(30) &= 42.033 \\ E(31) &= 42.040. \end{aligned}$$

This gives another way to get the lower bound 43, and gives a hint about approximately how many non-loops one would want in order to achieve such a bound.

40^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 27, 2011

USAMO 1. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3\,.$$

USAMO 2. An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

USAMO 3. In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

40^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 28, 2011

USAMO 4. Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

USAMO 5. Let P be a given point inside quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP, \quad \angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

USAMO 6. Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which equality holds.

40th United States of America Mathematical Olympiad

1. The given condition is equivalent to $a^2 + b^2 + c^2 + ab + bc + ca \le 2$. We will prove that

$$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \ge 6\,.$$

Indeed, we have

$$\frac{2ab+2}{(a+b)^2} \ge \frac{2ab + a^2 + b^2 + c^2 + ab + bc + ca}{(a+b)^2} = 1 + \frac{(c+a)(c+b)}{(a+b)^2}.$$

Adding the last inequality with its cyclic analogous forms yields

$$\frac{2ab+2}{(a+b)^2} + \frac{2bc+2}{(b+c)^2} + \frac{2ca+2}{(c+a)^2} \ge 3 + \frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2}$$

Hence it remains to prove that

$$\frac{(c+a)(c+b)}{(a+b)^2} + \frac{(a+b)(a+c)}{(b+c)^2} + \frac{(b+c)(b+a)}{(c+a)^2} \ge 3.$$

But this follows directly from the AM–GM inequality. Equality holds if and only if $a+b = b+c = c+a$, which together with the given condition, shows that it occurs if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

OR

Set $2x = a + b$, $2y = b + c$, and $2z = c + a$; that is, $a = z + x - y$, $b = x + y - z$, and $c = y + z - x$. Hence

$$\frac{ab+1}{(a+b)^2} = \frac{(z+x-y)(x+y-z)+1}{4x^2} = \frac{x^2-(y-z)^2+1}{4x^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2}.$$

On the other hand, the given condition is equivalent to $2a^2+2b^2+2c^2+2ab+2bc+2ca \le 4$ or $(a+b)^2+(b+c)^2+(c+a)^2 \le 4$; that is, $x^2+y^2+z^2 \le 1$ or $1-y^2-z^2 \ge x^2$. It follows that

$$\frac{ab+1}{(a+b)^2} = \frac{x^2+2yz+1-y^2-z^2}{4x^2} \ge \frac{x^2+2yz+x^2}{4x^2} = \frac{1}{2} + \frac{yz}{2x^2}.$$

Likewise, we have

$$\frac{bc+1}{(b+c)^2} = \frac{1}{2} + \frac{zx}{2y^2} \quad \text{and} \quad \frac{ca+1}{(c+a)^2} = \frac{1}{2} + \frac{xy}{2z^2}.$$

Adding the last three inequalities gives

$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2}\geq\frac{3}{2}+\frac{yz}{2x^2}+\frac{zx}{2y^2}+\frac{xy}{2z^2}\geq 3,$$

by the AM–GM inequality. Equality holds if and only if $x=y=z$ or $a=b=c=\frac{1}{\sqrt{3}}$.

2. Let a_1, a_2, a_3, a_4 and a_5 represent the integers at vertices v_1 to v_5 (in order around the pentagon) at the start of the game. We will first show that the game can be won at only one of the vertices. Observe that the quantity $a_1+2a_2+3a_3+4a_4 \bmod 5$ is an invariant of the game. For instance, one move involves replacing a_1, a_3 and a_5 by a_1-m, a_3+2m and a_5-m. Thus the quantity $a_1+2a_2+3a_3+4a_4$ becomes

$$(a_1-m)+2a_2+3(a_3+2m)+4a_4=a_1+2a_2+3a_3+4a_4+5m,$$

which is unchanged mod 5. The other moves may be checked similarly. Now suppose that the game may be won at vertex v_j. The value of the invariant at the winning position is $2011j$. If the initial value of the invariant is n, then we must have $2011j\equiv n \bmod 5$, or $j\equiv n \bmod 5$. Hence the game may only be won at vertex v_j, where j is the least positive residue of $n \bmod 5$.

By renumbering the vertices, we may assume without loss of generality that the winning vertex is v_5. We will show that the game can be won in four moves by adding a suitable amount $2m_j$ at vertex v_j (and subtracting m_j from the opposite vertices) on the jth turn for $j=1, 2, 3, 4$. The net change at vertex v_1 after these four moves is $2m_1-m_3-m_4$, which must equal $-a_1$ if we are to finish with 0 at v_1. In this fashion we find that

$$\begin{aligned}
2m_1-m_3-m_4 &= -a_1\\
2m_2-m_4 &= -a_2\\
2m_3 \qquad -m_1 &= -a_3\\
2m_4-m_1-m_2 &= -a_4\\
-m_2-m_3 &= -a_5+2011.
\end{aligned}$$

The sum of the first four equations is the negative of the fifth equation, so it is redundant. Multiplying the first four equations by -1, 3, -3, 1 and adding them yields $5m_2-5m_3=a_1-3a_2+3a_3-a_4$. But

$$a_1-3a_2+3a_3-a_4\equiv a_1+2a_2+3a_3+4a_4\equiv n\equiv 5\equiv 0 \bmod 5,$$

since we are assuming v_5 is the winning vertex. Therefore we may divide by 5 to obtain $m_2 - m_3 = \frac{1}{5}(a_1 - 3a_2 + 3a_3 - a_4)$. We also know that $m_2 + m_3 = a_1 + a_2 + a_3 + a_4$, and one easily confirms that the right-hand sides of these equations are integers with the same parity. Hence the system admits an integral solution for m_2 and m_3. The second and third equations then quickly give integer values for m_1 and m_4 as well, so it is indeed possible to win the game at vertex v_5.

3. We first give a recipe for constructing hexagons as in the problem statement. Let ACE be a triangle, with all angles less than $2\pi/3$. Let D be the reflection of A across CE; let F be the reflection of C across EA; let B be the reflection of E across AC. Then, $\angle BAF = \angle BAC + \angle CAE + \angle EAF = 3\angle CAE = 3\angle CDE$, and similarly for the other angle equalities. Also, $AB = AE = DE$, and similarly for the other side equalities. Thus, the hexagon satisfies the equations in the problem statement. The diagonals AD, BE, CF are simply the altitudes of the triangle ACE, so they are concurrent at the orthocenter.

   Now we show that the only possible hexagons meeting the conditions of the problem statement are the ones constructed in this manner. This will suffice to complete the solution.

   Given the hexagon $ABCDEF$ as in the problem statement, let β, δ, ϕ be the measures of its angles B, D, F. Since $4(\angle B + \angle D + \angle F) = \angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 4\pi$, we must have $\beta + \delta + \phi = \pi$. Also, the fact that opposite sides are not parallel implies that $\pi + 2\beta = \angle D + \angle E + \angle F \neq 2\pi$, so $\beta \neq \pi/2$; likewise $\delta, \phi \neq \pi/2$.

   We can construct a hexagon $A_1B_1C_1D_1E_1F_1$ meeting the angle and side equality conditions, with angles $\angle B_1 = \beta, \angle D_1 = \delta, \angle F_1 = \phi$, by taking $A_1C_1E_1$ to be a triangle with angles β, δ, ϕ, and reflecting each vertex across the opposite site as above. We wish to show that $ABCDEF \sim A_1B_1C_1D_1E_1F_1$.

   Treat the positions of A, B as fixed, and treat β, δ, ϕ as fixed; these are enough to uniquely determine the orientation of each edge of the hexagon, given the known angles. Let $x = AB = DE$, $y = BC = EF$, $z = CD = FA$. Our goal is to show that these lengths are uniquely determined (up to scale) by the given angles.

   Let a, b, c, d, e, f be unit vectors in the directions of the edges from A to B, B to C, C to D, D to E, E to F, and F to A, respectively. Then the vector identity

$$x(a + d) + y(b + e) + z(c + f) = 0 \qquad (1)$$

holds. Without loss of generality, assume the vertices of $ABCDEF$ are labeled in counterclockwise order. The respective orientations of vectors b, c, d, e, f, measured counterclockwise relative to a, are

$$\begin{array}{rl} b: & \pi - \beta \\ c: & -\beta - 3\phi \\ d: & -2\phi \\ e: & \pi + 2\delta - \beta \\ f: & 2\delta - \phi - \beta \end{array}$$

(These angles are given modulo 2π; we have made liberal use of the identity $\beta + \delta + \phi = \pi$.)

Now, whenever two unit vectors point in directions θ and ψ, which do not differ by π, then their sum is a nonzero vector pointing in direction $(\theta + \psi)/2$ or $(\theta + \psi)/2 + \pi$. It follows that vectors $a + d, b + e, c + f$ are all nonzero and point in the following directions (modulo π):

$$\begin{array}{rl} a + d: & -\phi \\ b + e: & \delta - \beta \\ c + f: & \delta - 2\phi - \beta \end{array}$$

None of the differences between these angles are multiples of π. (This follows from the fact that $\beta, \delta, \phi \neq \pi/2$.) Thus, $a+d, b+e, c+f$ are not collinear, nonzero vectors. Consequently, the equation (1) determines the coefficients x, y, z uniquely up to scale, as required.

It follows that $ABCDEF$ and $A_1B_1C_1D_1E_1F_1$ are similar to each other, as required, and this completes the proof.

4. The assertion is false, and the smallest n for which it fails is $n = 25$. Given $n \geq 2$, let r be the remainder when 2^n is divided by n. Then $2^n = kn + r$ where k is a positive integer and $0 \leq r < n$. It follows that

$$2^{2^n} = 2^{kn+r} \equiv 2^r \mod 2^n - 1,$$

and $2^r < 2^n - 1$ so 2^r is the remainder when 2^{2^n} is divided by $2^n - 1$. If r is even then 2^r is power of 4. Hence to disprove the assertion, it is enough to find an n for which the corresponding r is odd.

If n is even then so is $r = 2^n - kn$.

If n is an odd prime then $2^n \equiv 2 \pmod n$ by Fermat's Little Theorem; hence $r \equiv 2^n \equiv 2 \mod n$ and $r = 2$.

There remains the case in which n is odd and composite. In the first three instances $n = 9$, 15, 21 there is no contradiction to the assertion:

$$n = 9: 2^6 \equiv 1 \mod 9 \Rightarrow 2^9 \equiv 2^6 \cdot 2^3 \equiv 8 \mod 9$$
$$n = 15: 2^4 \equiv 1 \mod 15 \Rightarrow 2^{15} \equiv (2^4)^3 \cdot 2^3 \equiv 8 \mod 15$$
$$n = 21: 2^6 \equiv 1 \mod 21 \Rightarrow 2^{21} \equiv (2^6)^3 \cdot 2^3 \equiv 8 \mod 21$$

However,

$$2^{10} = 1024 \equiv -1 \Rightarrow 2^{20} \equiv 1 \Rightarrow 2^{25} \equiv 2^5 \equiv 7 \mod 25,$$

so 7 is the remainder when 2^{25} is divided by 25 and 2^7 is the remainder when $2^{2^{25}}$ is divided by $2^{25} - 1$.

5. We will prove that the lines $\overline{AB}$, $\overline{CD}$, and $\overline{Q_1Q_2}$ are either concurrent or all parallel. Let X and Y denote the reflections of P across the lines $\overline{AB}$ and $\overline{CD}$. We first claim that $XQ_1 = YQ_1$ and $XQ_2 = YQ_2$. Indeed, let Z be the reflection of Q_1 across BC. Then $XB = PB$, $BQ_1 = BZ$, and

$$\angle XBQ_1 = \angle XBA + \angle ABQ_1 = \angle ABC = \angle PBC + \angle CBZ = \angle PBZ,$$

whence $\triangle XBQ_1 \cong \triangle PBZ$ and thus $XQ_1 = PZ$. Similarly $YQ_1 = PZ$, and so $XQ_1 = YQ_1$. In exactly the same way, we see that $XQ_2 = YQ_2$, establishing the claim.

We conclude that the line $\overline{Q_1Q_2}$ is the perpendicular bisector of the segment $\overline{XY}$. If $\overline{AB} \parallel \overline{CD}$, then $\overline{XY} \perp \overline{AB}$ and it follows that $\overline{Q_1Q_2} \parallel \overline{AB}$, as desired. If the lines $\overline{AB}$ and $\overline{CD}$ are not parallel, then let R denote their intersection. Since $RX = RP = RY$, R lies on the perpendicular bisector of $\overline{XY}$ and thus R, Q_1, Q_2 are collinear, as desired.

OR

This solution uses isogonal conjugates. Recall that two points S, T are isogonal conjugates with respect to $\triangle ABC$ if $\angle SAB = \angle CAT$, $\angle SBC = \angle ABT$, and $\angle SCA = \angle BCT$, with any two of these equalities implying the third.

If $\overline{AB} \parallel \overline{CD}$, then there is nothing to prove; thus we assume $\overline{AB}$ intersects $\overline{CD}$ in a point R. Then Q_1 and P are isogonal conjugates with respect to $\triangle RBC$, whence $\angle Q_1RB = \angle CRP$, and Q_2 and P are isogonal conjugates with respect to $\triangle RAD$, whence $\angle Q_2RA = \angle DRP$. Therefore $\angle Q_1RB = \angle Q_2RA = \angle Q_2RB$ and the lines $\overline{AB}$, $\overline{CD}$, $\overline{Q_1Q_2}$ all intersect at R.

Remark: Although not needed for the problem as stated, here is an alternate proof that if $\overline{AB} \parallel \overline{CD}$, then $\overline{Q_1Q_2}$ is parallel to both. Extend $\overline{BQ_1}$ and $\overline{BP}$ to meet $\overline{CD}$ at points E and F, respectively. Then $\angle BCP = \angle Q_1CE$ and $\angle PBC = \angle ABQ_1 = \angle CEQ_1$, and so $\triangle PBC \sim \triangle Q_1EC$, whence $PC/PB = Q_1C/Q_1E$. Similarly $\triangle Q_1BC \sim \triangle PFC$ and $PC/PF = Q_1C/Q_1B$. We conclude that $Q_1B/Q_1E = PF/PB$. Similarly, extend $\overline{AQ_2}$ and $\overline{AP}$ to meet $\overline{CD}$ at G and H; then $Q_2A/Q_2G = PH/PA = PF/PB = Q_1B/Q_1E$, and it follows that $\overline{Q_1Q_2} \parallel \overline{AB} \parallel \overline{CD}$.

6. Let S be the complement of $A_1 \cup A_2 \cup \cdots \cup A_{11}$ in A; we wish to prove that $|S| \le 60$. For $\ell \ge 0$, define

$$\theta(\ell) = \left(1 - \frac{\ell}{2}\right)\left(1 - \frac{\ell}{3}\right) = 1 - \frac{2}{3}\ell + \frac{1}{3}\binom{\ell}{2}.$$

Note that $\theta(0) = 1$ and $\theta(\ell) \ge 0$ for any integer $\ell > 0$. Therefore, since S is the intersection of the complements of the A_i,

$$|S| \le \sum_{n \in A} \theta(\ell(n)).$$

On the other hand,

$$\sum_{n\in A} \theta(\ell(n)) = \sum_{n \in A}\left(1 - \frac{2}{3}\ell(n) + \frac{1}{3}\binom{\ell(n)}{2}\right) = |A| - \frac{2}{3}\sum_i |A_i| + \frac{1}{3}\sum_{i<j} |A_i \cap A_j|.$$

Consequently,

$$|S| \le 225 - \frac{2}{3}\cdot 11 \cdot 45 + \frac{1}{3}\cdot \binom{11}{2} \cdot 5 = 60,$$

and therefore $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \ge 165$.

We construct an example to show that this lower bound is best possible. Let $p_1, p_2, \ldots, p_{11}$ be a set of 11 distinct primes, and let A' denote the set of all products of three of these primes. Furthermore, let $A'' = \{q_1, q_2, q_3, \ldots, q_{60}\}$ be a set of 60 distinct positive integers that are all prime to $p_1, \ldots, p_{11}$. Set $A = A' \cup A''$, and define

$$A_i = \{n \in A' : p_i \,|\, n\}.$$

Then $|A_i| = \binom{10}{2} = 45$, $|A_i \cap A_j| = \binom{9}{1} = 9$, and

$$|A_1 \cup A_2 \cup \cdots \cup A_{11}| = |A'| = \binom{11}{3} = 165.$$

Finally, $|A| = |A'| + |A''| = 165 + 60 = 225$.

Remark: To get an upper bound for $|S|$, one could replace $\theta(\ell)$ by any function of the form

$$\left(1 - \frac{\ell}{r}\right)\left(1 - \frac{\ell}{r+1}\right)$$

for any positive integer r. The choice $r = 2$ is optimal for the stated problem. The choice $r = 1$ yields

$$|S| \leq |A| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j|,$$

which is a familiar truncated inclusion-exclusion inequality, known in number theory as "Brun's Pure Sieve" and in probability as "Bonferroni's Inequality."

USAMO 2011 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2011 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that
$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

2. An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

3. In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

4. Consider the assertion that for each positive integer $n \ge 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

5. Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that
$$\angle Q_1BC = \angle ABP, \qquad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \qquad \angle Q_2DA = \angle CDP.$$
Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

6. Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \dots, A_{11}$ of A such that $|A_i| = 45$ for $1 \le i \le 11$ and $|A_i \cap A_j| = 9$ for $1 \le i < j \le 11$. Prove that $|A_1 \cup A_2 \cup \dots \cup A_{11}| \ge 165$, and give an example for which equality holds.

§1 USAMO 2011/1

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \le 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \ge 3.$$

The condition becomes $2 \ge a^2 + b^2 + c^2 + ab + bc + ca$. Therefore,

$$\begin{aligned}
\sum_{\text{cyc}} \frac{2ab + 2}{(a + b)^2} &\ge \sum_{\text{cyc}} \frac{2ab + (a^2 + b^2 + c^2 + ab + bc + ca)}{(a + b)^2} \\
&= \sum_{\text{cyc}} \frac{(a + b)^2 + (c + a)(c + b)}{(a + b)^2} \\
&= 3 + \sum_{\text{cyc}} \frac{(c + a)(c + b)}{(a + b)^2} \\
&\ge 3 + 3\sqrt[3]{\prod_{\text{cyc}} \frac{(c + a)(c + b)}{(a + b)^2}} = 3 + 3 = 6
\end{aligned}$$

with the last line by AM-GM. This completes the proof.

§2 USAMO 2011/2, proposed by Sam Vandervelde

An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

Call the vertices 0, 1, 2, 3, 4 in order. First, notice that the quantity $N_1+2N_2+3N_3+4N_4 \pmod 5$ is invariant, where N_i is the amount at vertex i. This immediately implies that at most one vertex can win, since in a winning situation all N_i are 0 except for one, which is 2011.

Now we prove we can win on this unique vertex. Let a_i, x_i denote the number initially at i and x_i denote $\sum m$ over all m where vertex i gains $2m$. WLOG the possible vertex is 0, meaning $a_1 + 2a_2 + 3a_3 + 4a_4 \equiv 0 \pmod 5$. Moreover we want

$$\begin{aligned} 2011 &= a_0 + 2x_0 - x_2 - x_3 \\ 0 &= a_1 + 2x_1 - x_3 - x_4 \\ 0 &= a_2 + 2x_2 - x_4 - x_0 \\ 0 &= a_3 + 2x_3 - x_0 - x_1 \\ 0 &= a_4 + 2x_4 - x_1 - x_2. \end{aligned}$$

We can ignore the first equation since its the sum of the other four. Moreover, we can WLOG shift $x_0 \to 0$ by shifting each x_i by a fixed amount. Then

$$x_4 = 2x_2 + a_2 \text{ and } x_1 = 2x_3 + a_3.$$

We let p and q denote x_2 and x_3 (noting that $p, q \in \mathbb{Z} \implies x_1, x_4 \in \mathbb{Z}$). Anyways the system now expands as

$$2p - 3q = 2a_3 + a_1 - a_2 \text{ and } 2q - 3p = 2a_2 + a_4 - a_3$$

whence we have a two-var system, easy! We compute

$$p - q = \frac{1}{5}\left[a_1 - 3a_2 + 3a_3 - a_4\right].$$

This is an integer by the condition, whence so are p and q, QED.

§3 USAMO 2011/3

In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

We present the official solution. We say a hexagon is *satisfying* if it obeys the six conditions; note that intuitively we expect three degrees of freedom for satisfying hexagons.

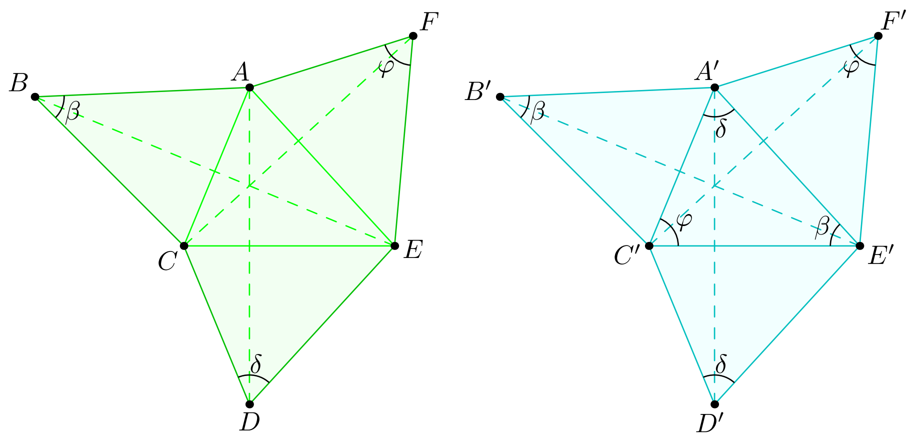
Main idea:

Claim — In a satisfying hexagon, B, D, F are reflections of A, C, E across the sides of $\triangle ACE$.

(This claim looks plausible because every excellent hexagon is satisfying, and both configuration spaces are three-dimensional.) Call a hexagon of this shape "excellent"; in a excellent hexagon the diagonals clearly concur (at the orthocenter).

Set $\beta = \angle B$, $\delta = \angle D$, $\varphi = \angle F$.

Now given a satisfying hexagon $ABCDEF$, construct a "phantom hexagon" $A'B'C'D'E'F'$ with the same angles which is excellent (see figure). This is possible since $\beta + \delta + \varphi = 180°$.



Then it would suffice to prove that:

Lemma

A satisfying hexagon is uniquely determined by its angles up to similarity. That is, at most one hexagon (up to similarity) has angles β, δ, γ as above.

Proof. Consider any two satisfying hexagons $ABCDEF$ and $A'B'C'D'E'F'$ (not necessarily as constructed above!) with the same angles. We show they are similar.

To do this, consider the unit complex numbers in the directions $\overrightarrow{BA}$ and $\overrightarrow{DE}$ respectively and let $\vec{x}$ denote their sum. Define $\vec{y}$, $\vec{z}$ similarly. Note that the condition $\overline{AB} \nparallel \overline{DE}$ implies $\vec{x} \neq 0$, and similarly. Then we have the identities

$$AB \cdot \vec{x} + CD \cdot \vec{y} + EF \cdot \vec{z} = A'B' \cdot \vec{x} + C'D' \cdot \vec{y} + E'F' \cdot \vec{z} = 0.$$

So we would obtain $AB : CD : EF = A'B' : C'D' : E'F'$ if only we could show that $\vec{x}$, $\vec{y}$, $\vec{z}$ are not multiples of each other (linear dependency reasons). This is a tiresome computation with arguments, but here it is.

First note that none of β, δ, φ can be 90°, since otherwise we get a pair of parallel sides. Now work in the complex plane, fix a reference such that $\vec{A} - \vec{B}$ has argument 0, and assume $ABCDEF$ are labelled counterclockwise. Then

- $\vec{B} - \vec{C}$ has argument $\pi - \beta$
- $\vec{C} - \vec{D}$ has argument $-(\beta + 3\varphi)$
- $\vec{D} - \vec{E}$ has argument $\pi - (\beta + 3\varphi + \delta)$
- $\vec{E} - \vec{F}$ has argument $-(4\beta + 3\varphi + \delta)$

So the argument of $\vec{x}$ has argument $\frac{\pi-(\beta+3\varphi+\delta)}{2} \pmod{\pi}$. The argument of $\vec{y}$ has argument $\frac{\pi-(5\beta+3\varphi+\delta)}{2} \pmod{\pi}$. Their difference is $2\beta \pmod{\pi}$, and since $\beta \neq 90^\circ$, it follows that $\vec{x}$ and $\vec{y}$ are not multiples of each other; the other cases are similar. □

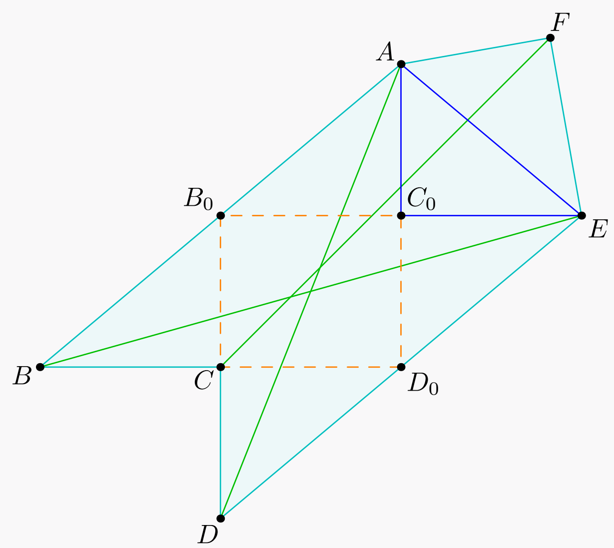
Then the lemma implies $ABCDEF \sim A'B'C'D'E'F$ and we're done.

Remark. This problem turned out to be known already. It appears in this reference:

Nikolai Beluhov, *Matematika*, 2008, issue 6, problem 3.

It was reprinted as Kvant, 2009, issue 2, problem M2130; the reprint is available at http://kvant.ras.ru/pdf/2009/2009-02.pdf.

Remark. The vector perspective also shows the condition about parallel sides cannot be dropped. Here is a counterexample from Ryan Kim in the event that it is.



By adjusting the figure above so that the triangles are right isosceles (instead of just right), one also finds an example of a hexagon which is satisfying and whose diagonals are concurrent, but which is *not* excellent.

§4 USAMO 2011/4, proposed by Zuming Feng

Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

We claim $n = 25$ is a counterexample. Indeed, note that

$$2^{2^{25}} \equiv 2^{2^{25} \pmod{25}} \equiv 2^7 \pmod{2^{25}}$$

which isn't a power of 4, and is actually the remainder since $2^7 < 2^{25}$.

§5 USAMO 2011/5

Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \qquad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \qquad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

If $\overline{AB} \parallel \overline{CD}$ there is nothing to prove. Otherwise let $X = \overline{AB} \cap \overline{CD}$. Then the Q_i are isogonal conjugates of P with respect to triangles XAD, XBC. Thus X, Q_1, Q_2 are collinear, on the isogonal of $\overline{XY}$ with respect to $\angle DXA = \angle CXB$.

§6 USAMO 2011/6

Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \dots, A_{11}$ of A such that $|A_i| = 45$ for $1 \le i \le 11$ and $|A_i \cap A_j| = 9$ for $1 \le i < j \le 11$. Prove that $|A_1 \cup A_2 \cup \dots \cup A_{11}| \ge 165$, and give an example for which equality holds.

Ignore the 225 — it is irrelevant.

Denote the elements of $A_1 \cup \dots \cup A_{11}$ by $a_1, \dots, a_n$, and suppose that a_i appears x_i times among A_i for each $1 \le i \le n$ (so $1 \le x_i \le 11$). Then we have

$$\sum_{i=1}^{11} x_i = \sum_{i=1}^{11} |A_i| = 45 \cdot 11$$

and

$$\sum_{i=1}^{11} \binom{x_i}{2} = \sum_{1 \le i < j \le 11} |A_i \cap A_j| = \binom{11}{2} \cdot 9.$$

Therefore, we deduce that $\sum x_i = 495$ and $\sum_i x_i^2 = 1485$. Now, by Cauchy Schwarz

$$n \left(\sum_i x_i^2 \right) \ge \left(\sum x_i \right)^2$$

which implies $n \ge \frac{495^2}{1485} = 165$.

Equality occurs if we let A consist of the 165 three-element subsets of $\{1, \dots, 11\}$ (plus 60 of your favorite reptiles). Then we let A_i denote those subsets containing i, of which there are $\binom{10}{2} = 45$, and so that $|A_i \cap A_j| = \binom{9}{1} = 9$.

41^{st} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 24, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 1. Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, ..., a_n$ with

$$\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n),$$

there exist three that are the side lengths of an acute triangle.

USAMO 2. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

USAMO 3. Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

41^{st} United States of America Mathematical Olympiad

Day II   12:30 PM – 5 PM EDT

April 25, 2012

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 4. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

USAMO 5. Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB, respectively. Prove that A', B', C' are collinear.

USAMO 6. For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = 1\,.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For what choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

41$^{\text{st}}$ United States of America Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 24-25, 2012

USAMO 1. First we prove that any $n \geq 13$ is a solution of the problem. Suppose that $a_1, a_2, ..., a_n$ satisfy $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, and that we cannot find three that are the side-lengths of an acute triangle. We may assume that $a_1 \leq a_2 \leq ... \leq a_n$. Then $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ for all $i \leq n-2$. Let (F_n) be the Fibonacci sequence, with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. It is easy to check that $F_n < n^2$ for $n \leq 11$, $F_{12} = 12^2$ and $F_n > n^2$ for $n > 12$ (the last inequality follows by an immediate induction, while the first one can be checked by hand). The inequality $a_{i+2}^2 \geq a_i^2 + a_{i+1}^2$ and the fact that $a_1 \leq a_2 \leq ... \leq a_n$ imply that $a_i^2 \geq F_i \cdot a_1^2$ for all $i \leq n$. Hence, if $n \geq 13$, we obtain $a_n^2 > n^2 \cdot a_1^2$, contradicting the hypothesis. This shows that any $n \geq 13$ is a solution of the problem.

By taking $a_i = \sqrt{F_i}$ for $1 \leq i \leq n$, we have $\max(a_1, a_2, ..., a_n) \leq n \cdot \min(a_1, a_2, ..., a_n)$, for any $n < 13$, but it is easy to see that no three a_i's can be the side-lengths of an acute triangle. Hence the answer to the problem is: all $n \geq 13$.

This problem and solution were suggested by Titu Andreescu.

USAMO 2. Let R, G, B, Y denote the sets of Red, Green, Blue, Yellow points, respectively, and let r, g, b, y denote a generic Red, Green, Blue, Yellow point, respectively. For integers $0 \leq k \leq 431$, let $\mathcal{T}_k$ denote the counterclockwise rotation of $\left(\frac{360k}{432}\right)$ degree around the center of the circle. Furthermore, for a set S, let $|S|$ denote the number of elements in S.

First, we claim that there is some index i_1 such that $|\mathcal{T}_{i_1}(R) \cap G| \geq 28$. Indeed, for each k, set $\mathcal{T}_k(R) \cap G$ consists of all Green points that are the images of Red points under the rotation $\mathcal{T}_k$. Hence the sum

$$s_1 = |\mathcal{T}_0(R) \cap G| + |\mathcal{T}_1(R) \cap G| + \cdots + |\mathcal{T}_{431}(R) \cap G|$$

is equal to the number of pairs of points (r, g) such that $g = \mathcal{T}_k(r)$ for some k. On the other hand, for each r and each g, there is a unique rotation $\mathcal{T}_k$ with $\mathcal{T}_k(r) = g$, form which it follows that $s_1 = 108^2 = 11664$. Clearly, $|\mathcal{T}_0(R) \cap G| = |R \cap G| = 0$ (because $R \cap G = \emptyset$). By the Pigeonhole principle, there is some index i_1 such that

$$|\mathcal{T}_{i_1}(R) \cap G| \geq \left\lceil \frac{s_1}{431} \right\rceil = \left\lceil \frac{11664}{431} \right\rceil = \lceil 27.06\ldots \rceil = 28,$$

establishing our claim. Let RG denote the set $\mathcal{T}_{i_1}(R) \cap G$, and let rg denote a generic point in RG.

Second, we claim that there is some index i_2 such that $|\mathcal{T}_{i_2}(RG) \cap B| \geq 8$. Again, for each k, set $\mathcal{T}_k(RG) \cap B$ consists of all Blue points that are the images of the points in RG under the rotation $\mathcal{T}_k$. Hence the sum

$$s_2 = |\mathcal{T}_0(RG) \cap B| + |\mathcal{T}_1(RG) \cap B| + \cdots + |\mathcal{T}_{431}(RG) \cap B|$$

is equal to the number of pairs of points (rg, b) such that $b = \mathcal{T}_k(rg)$ for some k. On the other hand, for each rg and each b, there is a unique rotation $\mathcal{T}_k$ with $\mathcal{T}_k(rg) = b$, form which it follows that $s_2 \geq 28 \cdot 108 = 3024$. Clearly, RG is a subset of B, which is disjoint with B, so $|\mathcal{T}_0(RG) \cap B| = 0$. Furthermore, $\mathcal{T}_{432-i_1}(\mathcal{T}_{i_1})$ is the identity transformation, implying that $\mathcal{T}_{432-i_1}(\mathcal{T}_{i_1}(R)) = R$ and $\mathcal{T}_{432-i_1}(RG)$ is a subset of R which is disjoint with B. In particular, $|\mathcal{T}_{432-i_1}(RG) \cap B| = 0$. By the Pigeonhole principle, there is some index i_2 such that

$$|\mathcal{T}_{i_2}(RG) \cap B| \geq \left\lceil \frac{s_2}{430} \right\rceil \geq \left\lceil \frac{3024}{430} \right\rceil = \lceil 7.0325\ldots \rceil = 8,$$

establishing our claim. Let RGB denote the set $\mathcal{T}_{i_2}(RG) \cap B$, and let rgb denote a generic point in RGB.

Third, we claim that there is some index i_3 such that $|\mathcal{T}_{i_3}(RGB) \cap Y| \geq 3$. We repeated our previous process one more time. We note that

$$s_3 = |\mathcal{T}_0(RGB) \cap Y| + |\mathcal{T}_1(RGB) \cap Y| + \cdots + |\mathcal{T}_{431}(RGB) \cap Y| \geq 8 \cdot 108 = 864$$

and

$$|\mathcal{T}_0(RGB) \cap Y| = |\mathcal{T}_{432-i_2}(RGB) \cap Y| = |\mathcal{T}_{432-i_2-i_1}(RGB) \cap Y| = 0.$$

By the Pigeonhole principle, there is some index i_3 such that

$$|\mathcal{T}_{i_3}(RGB) \cap Y| \geq \left\lceil \frac{s_3}{429} \right\rceil \geq \left\lceil \frac{864}{429} \right\rceil = \lceil 2.01\ldots \rceil = 3,$$

establishing our claim.

Let y_1, y_2, y_3 be three points in $\mathcal{T}_{i_2}(RGB) \cap Y$. Then

$$\begin{aligned}(y_1, y_2, y_3), (b_1, b_2, b_3) &= \mathcal{T}_{432-i_3}(y_1, y_2, y_3), (g_1, g_2, g_3) \\ &= \mathcal{T}_{432-i_3-i_2}(y_1, y_2, y_3), (r_1, r_2, r_3) \\ &= \mathcal{T}_{432-i_3-i_2-i_1}(y_1, y_2, y_3)\end{aligned}$$

are twelve points that we are looking for.

This problem and solution were suggested by Gregory Galperin.

USAMO 3. We will show that the sequence exists for all $n \geq 3$.

For $n = 2$, the sequence cannot exist: If it existed, we would have $a_k = -2a_{2k}$ for all k, from which $a_1 = (-2)^r a_{2^r}$ for all r by induction. Then a_1 would have to be divisible by 2^r for all r, which is impossible for $a_1 \neq 0$.

Now fix $n \geq 3$. We will show that the desired sequence exists. The construction is basically a repeated application of the Chinese Remainder Theorem, but the details require substantial care.

First we prove two lemmas.

Lemma 1 It is possible to partition the positive integers into subsets $S_1, S_2, S_3, \ldots$ so that for every positive integer k,

(i) the numbers $(n-1)k$ and nk are in the same subset, and

(ii) the numbers $k, 2k, \ldots, (n-2)k$ are all in strictly earlier subsets than $(n-1)k$.

Proof To show this, define a function f from positive integers to positive reals as follows. Let P_n be the set of primes dividing n. No element of P_n divides $n-1$. For any number k, write its prime factorization $k = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, and then define

$$f(k) = \prod_{p_i \notin P_n} p_i^{e_i} \cdot \prod_{p_i \in P_n} (p_i^{e_i})^{\log_n(n-1)}.$$

Notice that for every positive integer k,

$$f((n-1)k) = (n-1)f(k) = f(nk) \tag{1}$$

whereas for each $t = 1, 2, \ldots, n-2$,

$$f(tk) \leq tf(k) < f((n-1)k). \tag{2}$$

Also notice that for each k, $f(k) \geq k^{\log_n(n-1)}$, which implies that for any fixed C, there can only be finitely many values of k with $f(k) < C$. The latter fact means that the elements of the image of f can be arranged in increasing order, $x_1 < x_2 < x_3 < \cdots$. Now just let $S_i = f^{-1}(x_i)$ for each i. The sets S_i are a partition of the positive integers, and (1) and (2) ensure that they satisfy (i) and (ii) respectively.

Lemma 2 Let p, q be relatively prime positive integers and $t_1, t_2, \ldots, t_r$ arbitrary integers. Then it is possible to choose nonzero integers $b_1, b_2, \ldots, b_{r+1}$ such that

$$pb_i + qb_{i+1} = t_i \qquad \text{for } i = 1, 2, \ldots, r. \tag{3}$$

Proof We first prove existence of a sequence of integers satisfying (3) for each i, by induction on r. If $r = 1$, then since p, q are relatively prime, we can find c, d such that $pc + qd = 1$. Then, $b_1 = ct_1$ and $b_2 = dt_1$ satisfy (3). Now suppose we have $b_1, \ldots, b_r$ satisfying (3) for $i = 1, 2, \ldots, r-1$. If we choose any integer k, and replace each b_i with $b_i' = b_i + (-1)^i p^{i-1} q^{r-i} k$, then (3) still holds for $i = 1, 2, \ldots, r-1$, and $pb_r' = pb_r + (-1)^r p^{r-1} k$. Since p, q are relatively prime, we can choose k so as to make pb_r' congruent to t_r modulo q, and then we take $b_{r+1} = (t_r - pb_r')/q$. Then the numbers $b_1', b_2', \ldots, b_r', b_{r+1}$ satisfy (3) for $i = 1, 2, \ldots, r$.

This shows that we can find $b_1, b_2, \ldots, b_{r+1}$ satisfying (3), but they may not all be nonzero. However, once again, we can make the replacements $b_i' = b_i + (-1)^i p^{i-1} q^{r+1-i} k$ for any integer k, and the new sequence still satisfies (3). By an appropriate choice of k, we can ensure each b_i' is nonzero.

Now both lemmas are proven, and we resume the main proof. We will construct terms of the sequence inductively, but not in the order $a_1, a_2, \ldots$.

Suppose S is any set of positive integers, and we have chosen nonzero integers a_k for each $k \in S$. Say that there is a conflict in S if there exists some k such that $k, 2k, \ldots, nk$ are all in S, and

$$a_k + 2a_{2k} + \cdots + na_{nk} \neq 0.$$

Let $S_1, S_2, \ldots$ be as given by Lemma 1 We will inductively define our sequence as follows:

(a) Step 1: Choose nonzero values a_k for all $k \in S_1$ simultaneously, without creating a conflict in S_1.

(b) Step $t \geq 1$: Given the values of a_k for $k \in S_1 \cup \cdots \cup S_{t-1}$ chosen at previous steps, choose nonzero integers a_k for all $k \in S_t$ simultaneously, without creating a conflict in $S_1 \cup \cdots \cup S_t$.

If we can show that each step of this process can indeed be carried out, then it will eventually define a_k for all positive integers k, meeting the required condition

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0 \tag{4}$$

for all k (since no conflicts are created).

For step 1, Lemma 1 implies we can choose a_k arbitrarily for $k \in S_1$ without creating any conflicts, since $(n-1)k, nk \notin S_1$ for all k. Now for step $t \geq 1$, suppose a_k have been assigned already for all $k \in S_1 \cup S_2 \cup \cdots \cup S_{t-1}$. We need to assign a_k for $k \in S_t$ to avoid creating any new conflicts. This just requires that the new assignments satisfy (4) for all integers k such that $(n-1)k$ and nk are in S_t: for any other value k, either $\{k, 2k, \ldots, nk\} \not\subseteq S_1 \cup \cdots \cup S_t$ so no conflict can be created, or else Lemma 1 implies $\{k, 2k, \ldots, nk\} \subseteq S_1 \cup \cdots \cup S_{t-1}$ so that the corresponding constraint (4) has been dealt with at an earlier step.

Thus for each k such that $(n-1)k, nk \in S_t$, we have a constraint

$$(n-1)a_{(n-1)k} + na_{nk} = X_k, \tag{5}$$

where $X_k = -(a_k + \cdots + (n-2)a_{(n-2)k})$ is determined by the assignments made at previous steps. We just need to show that it is possible to choose a_k for all $k \in S_t$ such that all these constraints are satisfied.

Form a directed graph whose vertices are the elements of S_t, with an edge leading from $(n-1)k$ to nk whenever both numbers are in S_t. Then every component of this graph is either a single vertex or a (directed) path. We wish to show that nonzero integer values can be assigned to elements of S_t so that for each edge, the corresponding constraint (5) is satisfied. It suffices to show this for each component of the graph. If the component is a single vertex, any nonzero value works. Otherwise, it is a path $k_1, k_2, \ldots, k_{r+1}$, and Lemma 2 ensures that we can choose nonzero integer values for $a_{k_1}, a_{k_2}, \ldots, a_{k_{r+1}}$ so as to satisfy (5) for each edge.

This shows that each step of our constructive process can indeed be performed successfully, and iterating eventually constructs every term of the sequence.

This problem and solution were suggested by Gabriel Carroll.

USAMO 4. There are three solutions: the constant functions $1, 2$ and the identity function. Let us prove that these are the only ones. Consider such a function f and suppose first of all that there exists $a > 2$ such that $f(a) = a$. Then $a!, (a!)!, \ldots$ are all fixed points of f. So there is an increasing sequence $(a_n)_{n \geq 0}$ of fixed points. If n is any positive integer, $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for all k, and so it also divides $f(n) - n$ for all k. Thus $f(n) = n$ and since it holds for any n, we are done in this case.

Now suppose that f has no fixed points greater than 2. Let $p > 3$ be a prime and observe that $(p-2)! \equiv 1 \pmod p$ (by Wilson's theorem), thus $f(p-2)! - f(1) = f((p-2)!) - f(1)$ is a multiple of p. Clearly $f(1)$ is 1 or 2. As $p > 3$, the fact that p divides $f(p-2)! - f(1)$ implies that $f(p-2) < p$. Since $(p-1)! - f(1)$ is not a multiple of p (again by Wilson), we deduce that actually $f(p-2) \le p-2$. On the other hand, $p-3$ divides $f(p-2) - f(1) \le f(p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p-2 > 2$, the last case is excluded and so $f(p-2) = f(1)$ and this for all primes $p > 3$. Taking n any positive integer, we deduce that $p-2-n$ divides $f(1) - f(n)$ and this holds for all large primes p. Thus $f(n) = f(1)$ and f is constant. The conclusion is now clear.

This problem and solution were suggested by Gabriel Dospinescu.

USAMO 5. **Solution 1:** The proof is split into two cases.

Case 1: P is on the circumcircle of ABC. Then P is the Miquel point of A', B', C' with respect to ABC. Indeed, because $\angle A'B'C' = \angle CBA = \angle CPA = \angle A'PC'$, points P, A', B', C' are concyclic, and the same can be said for P, A, B', C' and P, A', B', C. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so $A'B'C'$ are collinear.

Case 2: P is not on the circumcircle of ABC. Let Q be isogonal conjugate of P with respect to ABC (which is not degenerate).

Claim. Let Q' be the isogonal conjugate of P with respect to $AB'C'$. Then $Q = Q'$.

Proof of the claim. Note that

$$\begin{aligned}\angle BQC &= \angle BAC + \angle CPB \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } ABC\text{)}\\ &= \angle C'AB' + \angle B'PC'\\ &= \angle C'Q'B' \quad \text{(because } P \text{ and } Q \text{ are isogonal conjugates in } AB'C'\text{)}.\end{aligned}$$

Let X, Y, Z denote the reflections of P in sides BC, CA, AB, respectively, and let X' denote P's reflection in side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because QC is orthogonal to XY and QB is orthogonal to XZ), whereas $\angle ZX'Y' = \angle C'Q'B'$ because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal to $X'Z$ and $Q'C'$ is orthogonal to $X'Z$, so since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. It follows that X, Y, Z, X' are concyclic. The center of the XYZ-circle is Q while the center of the $X'Y'Z$-circle is Q'. Thus $Q = Q'$.

Note. We have made use of the well-known fact that the circumcenter of the triangle determined by the reflections of a point across the sidelines of another given triangle is precisely the isogonal conjugate of the point with respect to that triangle. For a proof see R. A. Johnson, *Advanced Euclidean Geometry*, 1929 ed., reprinted by Dover, 2007.

Similar arguments show that Q is also the isogonal point of P with respect to triangles $A'BC'$ and $A'B'C$. Therefore,

$$\begin{aligned}\angle BC'A' &= \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A'\\ &= \angle QC'B' + \angle PC'Q + \angle BC'P\\ &= \angle BC'B' = \angle AC'B'.\end{aligned}$$

This means that A', B', C' are collinear. ■

This problem and solution were suggested by Titu Andreescu and Cosmin Pohoata.

Solution 2: It's easy to see (say, by law of sines) that

$$\frac{AC'}{BC'} = \frac{AP\sin\angle APC'}{BP\sin\angle BPC'}, \quad \frac{BA'}{CA'} = \frac{BP\sin\angle BPA'}{CP\sin\angle CPA'}, \quad \frac{CB'}{AB'} = \frac{CP\sin\angle CPB'}{AP\sin\angle APB'}.$$

The construction of A', B', C' by reflections implies that

$$\sin\angle APC' = \sin\angle CPA', \quad \sin\angle BPC' = \sin\angle CPB', \quad \sin\angle BPC' = \sin\angle CPB'.$$

Hence,

$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'} = 1,$$

and the proof is complete by Menelaus' theorem.

This second solution was suggested by Li Zhou, Polk State College, Winter Haven FL.

USAMO 6. This problem is a form of Chebyshev's inequality for random variables. For each set $A \subseteq \{1, 2, \ldots, n\}$, define

$$\Delta_A = 2S_A = \sum_{i\in A} x_i \quad - \sum_{i\in\{1,2,\ldots,n\}\setminus A} x_i \quad = \quad \sum_{i=1}^{n} \epsilon_A(i)x_i,$$

where $\epsilon_A(i) = 1$ if $i \in A$ and -1 otherwise. Squaring, we have

$$\Delta_A^2 = \sum_{i=1}^{n} x_i^2 + \sum_{\substack{i,j\in\{1,\ldots,n\}\\ i\neq j}} \epsilon_A(i)\epsilon_A(j)x_ix_j. \tag{6}$$

Now sum the Δ_A^2's over all 2^n possible choices of A. For each pair $i \neq j$, there are 2^{n-2} sets A with $i, j \in A$, and another 2^{n-2} sets with $i, j \notin A$; these sets each contributes a term of $+x_ix_j$ to the sum in (6). There are also 2^{n-2} sets A with $i \in A$, $j \notin A$, and 2^{n-2} sets with $i \notin A$, $j \in A$. Each of these sets each contributes a term of $-x_ix_j$ to (6). Hence, x_ix_j appears 2^{n-1} times with a + sign and 2^{n-1} times with a − sign. Therefore all of these terms cancel, and we find

$$\sum_{A\subseteq\{1,2,\ldots,n\}} \Delta_A^2 = 2^n(x_1^2 + \cdots + x_n^2) = 2^n. \tag{7}$$

Now let $\lambda > 0$. There cannot be more than $2^{n-2}/\lambda^2$ terms Δ_A^2 whose value greater than or equal to $4\lambda^2$. If this were not the case, then the sum of these terms would be greater than 2^n, so the sum in (7) would exceed 2^n. Hence, there can be at most $2^{n-2}/\lambda^2$ sets A such that $|S_A| \geq \lambda$. (Recall that $\Delta_A = 2S_A$). Moreover, these sets can be arranged into complementary pairs because $S_A = -S_{\{1,\ldots,n\}\setminus A}$. In each of these pairs, exactly one of the two members is positive. Therefore there are at most $2^{n-3}/\lambda^2$ sets A with $S_A \geq \lambda$.

For equality to hold, it must be the case that all positive values of Δ_A^2 are equal to $4\lambda^2$; otherwise we would again have a contradiction because the sum of all Δ_A^2 would exceed 2^n. In particular, all positive values of Δ_A^2 must be the same. Thus all positive values of x_A must be the same. This will be the case only if at most one of the x_i is positive and at most one of the x_i is negative. Because we must have at least one of each, there must be exactly one positive term and one negative term. Thus it must be the case that one $x_k = \sqrt{2}/2$ for some k, one is $x_j = -\sqrt{2}/2$ for some $j \neq k$, and all other $x_i = 0$. Then the assumption that every positive $\Delta_A^2 = 4\lambda^2$ yields $\lambda = \sqrt{2}/2$.

Conversely, with the x_i and λ as described, we have exactly $2^{n-2} = 2^{n-3}/\lambda^2$ sets A such that $x_A \geq \lambda$ (namely, those sets A that contain the $\sqrt{2}/2$ term and do not contain the $-\sqrt{2}/2$ term.) Thus this is indeed the equality case.

This problem and solution were suggested by Gabriel Carroll.

USAMO 2012 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2012 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, ..., a_n with
$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$
there exist three that are the side lengths of an acute triangle.

2. A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

3. Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, ... of nonzero integers such that the equality
$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$
holds for every positive integer k.

4. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

5. Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

6. For integer $n \geq 2$, let x_1, x_2, ..., x_n be real numbers satisfying
$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$
For each subset $A \subseteq \{1, 2, \dots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of x_1, x_2, ..., x_n, λ does equality hold?

§1 USAMO 2012/1, proposed by Titu Andreescu

Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Claim — For positive integers m, we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the main problem. The hypothesis $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$ will be denoted by (†).

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \geq F_i a_1^2$. In particular, $a_n^2 \geq F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \leq n \cdot a_1$. Therefore we have $F_n \leq n^2$, and so $n \leq 12$, contradiction!

Proof that no $n \leq 12$ have the property. Assume that $n \leq 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \leq 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \geq F_j + F_i = a_j^2 + a_i^2$.

§2 USAMO 2012/2, proposed by Gregory Galperin

A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

First, consider the 431 possible non-identity rotations of the red points, and count overlaps with green points. If we select a rotation randomly, then each red point lies over a green point with probability $\frac{108}{431}$; hence the expected number of red-green incidences is

$$\frac{108}{431} \cdot 108 > 27$$

and so by pigeonhole, we can find a red 28-gon and a green 28-gon which are rotations of each other.

Now, look at the 430 rotations of this 28-gon (that do not give the all-red or all-green configuration) and compare it with the blue points. The same approach gives

$$\frac{108}{430} \cdot 28 > 7$$

incidences, so we can find red, green, blue 8-gons which are similar under rotation.

Finally, the 429 nontrivial rotations of this 8-gon expect

$$\frac{108}{429} \cdot 8 > 2$$

incidences with yellow. So finally we have four monochromatic 3-gons, one of each color, which are rotations of each other.

§3 USAMO 2012/3, proposed by Gabriel Carroll

Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, … of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

Answer: all $n > 2$.

For $n = 2$, we have $a_k + 2a_{2k} = 0$, which is clearly not possible, since it implies $a_{2^k} = \frac{a_1}{2^{k-1}}$ for all $k \geq 1$.

For $n \geq 3$ we will construct a *completely multiplicative* sequence (meaning $a_{ij} = a_i a_j$ for all i and j). Thus (a_i) is determined by its value on primes, and satisfies the condition as long as $a_1 + 2a_2 + \cdots + na_n = 0$. The idea is to take two large primes and use Bezout's theorem, but the details require significant care.

We start by solving the case where $n \geq 9$. In that case, by Bertrand postulate there exists primes p and q such that

$$\lceil n/2 \rceil < q < 2\lceil n/2 \rceil \quad \text{and} \quad \frac{1}{2}(q-1) < p < q-1.$$

Clearly $p \neq q$, and $q \geq 7$, so $p > 3$. Also, $p < q < n$ but $2q > n$, and $4p \geq 4\left(\frac{1}{2}(q+1)\right) > n$. We now stipulate that $a_r = 1$ for any prime $r \neq p, q$ (in particular including $r = 2$ and $r = 3$). There are now three cases, identical in substance.

- If $p, 2p, 3p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$6p \cdot a_p + q \cdot a_q = 6p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(6p, q) = 1$.
- Else if $p, 2p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$3p \cdot a_p + q \cdot a_q = 3p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(3p, q) = 1$.
- Else if $p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$p \cdot a_p + q \cdot a_q = p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(p, q) = 1$. (This case is actually possible in a few edge cases, for example when $n = 9$, $q = 7$, $p = 5$.)

It remains to resolve the cases where $3 \leq n \leq 8$. We enumerate these cases manually:

- For $n = 3$, let $a_n = (-1)^{\nu_3(n)}$.
- For $n = 4$, let $a_n = (-1)^{\nu_2(n)+\nu_3(n)}$.
- For $n = 5$, let $a_n = (-2)^{\nu_5(n)}$.
- For $n = 6$, let $a_n = 5^{\nu_2(n)} \cdot 3^{\nu_3(n)} \cdot (-42)^{\nu_5(n)}$.
- For $n = 7$, let $a_n = (-3)^{\nu_7(n)}$.
- For $n = 8$, we can choose $(p, q) = (5, 7)$ in the prior construction.

This completes the constructions for all $n > 2$.

§4 USAMO 2012/4, proposed by Gabriel Dospinescu

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

By putting $n = 1$ and $n = 2$ we give $f(1), f(2) \in \{1, 2\}$. Also, we will use the condition

$$m! - n! \text{ divides } f(m)! - f(n)!.$$

We consider four cases on $f(1)$ and $f(2)$, and dispense with three of them.

- If $f(2) = 1$ then for all $m \geq 3$ we have $m! - 2$ divides $f(m)! - 1$, so $f(m) = 1$ for modulo 2 reasons. Then clearly $f(1) = 1$.
- If $f(1) = f(2) = 2$ we first obtain $3! - 1 \mid f(3)! - 2$, which implies $f(3) = 2$. Then $m! - 3 \mid f(m)! - 2$ for $m \geq 4$ implies $f(m) = 2$ for modulo 3 reasons.

Hence we are left with the case where $f(1) = 1$ and $f(2) = 2$. Continuing, we have

$$3! - 1 \mid f(3)! - 1 \quad \text{and} \quad 3! - 2 \mid f(3)! - 2 \implies f(3) = 3.$$

Continuing by induction, suppose $f(1) = 1, \dots, f(k) = k$.

$$k! \cdot k = (k+1)! - k! \mid f(k+1)! - k!$$

and thus we deduce that $f(k+1) \geq k$, and hence

$$k \mid \frac{f(k+1)!}{k!} - 1.$$

Then plainly $f(k+1) \leq 2k$ for mod k reasons, but also $f(k+1) \equiv 1 \pmod{k}$ so we conclude $f(k) = k + 1$.

Remark. Shankar Padmanabhan gives the following way to finish after verifying that $f(3) = 3$. Note that if

$$M = ((((3!)!)!)! \dots)!$$

for any number of iterated factorials then $f(M) = M$. Thus for any n, we have

$$M - n \mid f(M) - f(n) = M - f(n) \implies M - n \mid n - f(n)$$

and so taking M large enough implies $f(n) = n$.

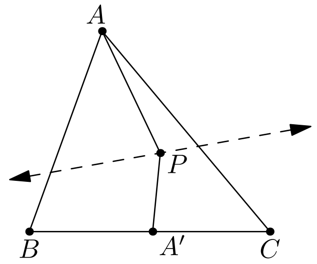
§5 USAMO 2012/5, proposed by Titu Andreescu and Cosmin Pohoata

Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present two solutions.

First solution (complex numbers) Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\overline{a}$. So, we get

$$a' = \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})\overline{a} - (b - c)a}.$$



Note that

$$\overline{a}' = \frac{a(b\overline{c} - \overline{b}c)}{(b - c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \begin{vmatrix} \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-\overline{b}c)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1 \\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1 \\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{vmatrix}.$$

This is equivalent to

$$0 = \begin{vmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b} - \overline{c})\overline{a} - (b - c)a \\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c} - \overline{a})\overline{b} - (c - a)b \\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a} - \overline{b})\overline{c} - (a - b)c \end{vmatrix}.$$

Evaluating the determinant gives

$$\sum_{\text{cyc}} ((\overline{b} - \overline{c})\overline{a} - (b - c)a) \cdot - \begin{vmatrix} b & \overline{b} \\ c & \overline{c} \end{vmatrix} \cdot (\overline{c}a - c\overline{a}) \left(\overline{a}b - a\overline{b}\right)$$

or, noting the determinant is $b\overline{c} - \overline{b}c$ and factoring it out,

$$(\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{\text{cyc}} \left(ab - ac + \overline{ca} - \overline{ba}\right) = 0.$$

Second solution (Desargues involution) We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see τ coincides with reflection about ℓ, hence conclude $C'' = C$.

§6 USAMO 2012/6, proposed by Gabriel Carroll

For integer $n \geq 2$, let x_1, x_2, $\dots$, x_n be real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \dots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of x_1, x_2, $\dots$, x_n, λ does equality hold?

Let ε_i be a coin flip of 0 or 1. Then we have

$$\begin{aligned} \mathbb{E}[S_A^2] = \mathbb{E}\left[\left(\sum \varepsilon_i x_i\right)^2\right] &= \sum_i \mathbb{E}[\varepsilon_i^2] x_i^2 + \sum_{i<j} \mathbb{E}[\varepsilon_i \varepsilon_j] 2 x_i x_j \\ &= \frac{1}{2} \sum x_i^2 + \frac{1}{2} \sum x_i x_j = \frac{1}{2} + \frac{1}{2} \sum_{i<j} x_i x_j = \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

In other words, $\sum_A S_A^2 = 2^{n-2}$. Since can always pair A with its complement, we conclude

$$\sum_{S_A > 0} S_A^2 = 2^{n-3}.$$

Equality holds iff $S_A \in \{\pm\lambda, 0\}$ for every A. This occurs when $x_1 = 1/\sqrt{2}$, $x_2 = -1/\sqrt{2}$, $x_3 = \cdots = 0$ (or permutations), and $\lambda = 1/\sqrt{2}$.

42^{nd} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 30, 2013

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let $\omega_A, \omega_B, \omega_C$ denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects $\omega_A, \omega_B, \omega_C$ again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

USAMO 2. For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

USAMO 3. Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

42^{nd} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

May 1, 2013

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 4. Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

USAMO 5. Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

USAMO 6. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

42^{nd} United States of America Mathematical Olympiad

Day I, II 12:30 PM – 5 PM EDT

April 30 - May 1, 2013

USAMO 1. **First Solution:** Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

$$\angle QSR = 360^\circ - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180^\circ - \angle BAC;$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)

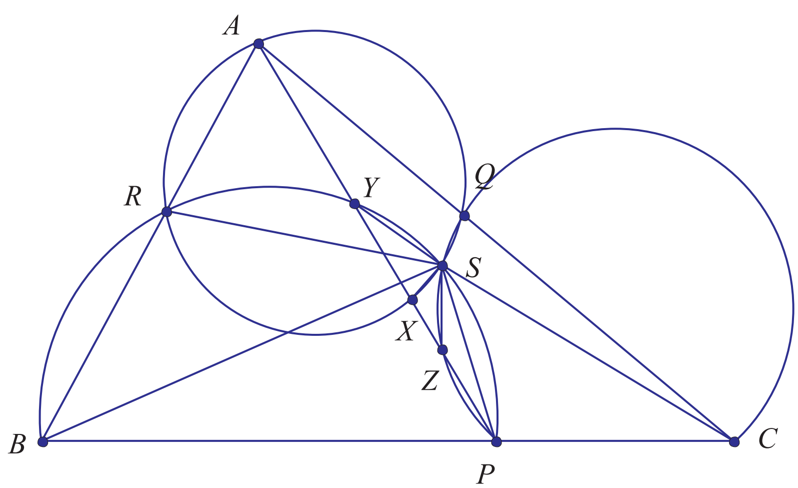
Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similar, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.



This problem and solution were suggested by Zuming Feng.

Second Solution: Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt with similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

$$\angle QSR = 360^\circ - \angle RSP - \angle PSQ = \angle PBR + \angle QCP = \angle CBA + \angle ACB = 180^\circ - \angle BAC;$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)

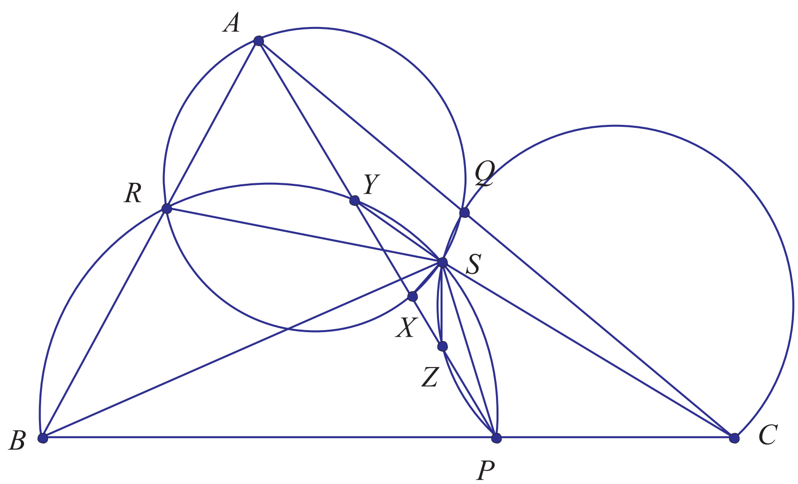
Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similar, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.



We consider the configuration shown in the above diagram. (We can adjust the proof below easily for other configurations. In particular, our proof is carried with directed angles modulo 180°.)

Line RY intersects ω_A again at T_Y (other than R). Because $BPYR$ is cyclic, $\angle T_Y YX = \angle T_Y YP = \angle RBP = \angle ABP$. Because $ARXT_Y$ is cyclic, $\angle XT_Y Y = \angle XAR = \angle PAB$. Hence triangles $T_Y YX$ and ABP are similar to each other. In particular,

$$\angle YXT_Y = \angle BPA \quad \text{and} \quad \frac{YX}{BP} = \frac{XT_Y}{PA}. \tag{1}$$

Likewise, if line QZ intersect ω_A again at T_Z (other than R), we can show that triangles $T_Z ZX$ and ACP are similar to each other and that

$$\angle T_Z XZ = \angle APC \quad \text{and} \quad \frac{XT_Z}{PA} = \frac{XZ}{PC}. \tag{2}$$

In the light of the second equations (on lengths proportions) in (1) and (2), it suffices to show that $T_Z = T_Y$. On the other hand, the first equations (on angles) in (1) and (2) imply that X, T_Y, T_Z lie on a line. But this line can only intersect ω_A twice with X being one of them. Hence we must have $T_Y = T_Z$, completing our proof.

Comment: The result remains to be true if segment AP is replaced by line AP. The current statement is given to simplify the configuration issue. Also, a very common mistake in attempts following the second solution is assuming line RY and QZ meet at a point on ω_A.

This solution was suggested by Zuming Feng.

USAMO 2. **First Solution.** We will show that $a_n = \frac{1}{3}(2^{n+1} + (-1)^n)$. This would be sufficient, since then we would have

$$a_{n-1} + a_n = \tfrac{1}{3}(2^n + (-1)^{n-1}) + \tfrac{1}{3}(2^{n+1} + (-1)^n) = \tfrac{1}{3}(2^n + 2\cdot 2^n) = 2^n.$$

We will need the fact that for all positive integers n

$$\sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} 2^k = \tfrac{1}{3}(2^{n+1} + (-1)^n).$$

This may be established by strong induction. To begin, the cases $n = 1$ and $n = 2$ are quickly verified. Now suppose that $n \geq 3$ is odd, say $n = 2m + 1$. We find that

$$\begin{aligned}\sum_{k=0}^{m} \binom{2m+1-k}{k} 2^k &= 1 + \sum_{k=1}^{m} \binom{2m-k}{k} 2^k + \sum_{k=1}^{m} \binom{2m-k}{k-1} 2^k \\ &= \sum_{k=0}^{m} \binom{2m-k}{k} 2^k + 2\sum_{k=0}^{m-1} \binom{2m-1-k}{k} 2^k \\ &= \tfrac{1}{3}(2^{2m+1} + 1) + \tfrac{2}{3}(2^{2m} - 1) \\ &= \tfrac{1}{3}(2^{2m+2} - 1),\end{aligned}$$

using the induction hypothesis for $n = 2m$ and $n = 2m - 1$. For even n the computation is similar, so we omit the steps. This proves the claim.

We now determine the number of ways to advance around the circle twice, organizing our count according to the points visited both times around the circle. It is straight-forward to check that no two such points may be adjacent, and that there are exactly two sequences of moves leading from any such point to the next. (These sequences involve only moves of length two except possibly at the endpoints.) Hence given $k \geq 1$ points around the circle, no two adjacent and not including point A, there would appear to be 2^k ways to traverse the circle twice without repeating a move. However, half of these options lead to repeating the same route twice, giving 2^{k-1} ways in actuality. There are $\binom{n-k}{k}$ ways to select k nonadjacent points on the circle not including A (add an extra point behind each of k chosen points), for a total contribution of

$$\sum_{k=1}^{\lfloor n/2\rfloor} \binom{n-k}{k} 2^{k-1} = \frac{1}{2}\left[-1 + \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} 2^k\right] = \tfrac{1}{6}(2^{n+1} + (-1)^n) - \tfrac{1}{2}.$$

On the other hand, if the $k \geq 1$ nonadjacent points do include point A then there are $\binom{n-k-1}{k-1}$ ways to choose them around the circle. (Select A but not the next point, then add an extra point after each of $k-1$ selected points.) But now there are actually 2^k ways to circle twice, since we can choose either move at A and the subsequent points, then select the other options the second time around. Hence the contribution in this case is

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} 2^k = 2 \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-k}{k} 2^k = \tfrac{2}{3}(2^{n-1} + (-1)^n).$$

Finally, if n is odd then there is one additional way to circle in which no point is visited twice by using only steps of length two, giving a contribution of $\frac{1}{2}(1-(-1)^n)$. Therefore the total number of paths is

$$\tfrac{1}{6}(2^{n+1} + (-1)^n) - \tfrac{1}{2} + \tfrac{2}{3}(2^{n-1} + (-1)^n) + \tfrac{1}{2}(1 - (-1)^n),$$

which simplifies to $\frac{1}{3}(2^{n+1} + (-1)^n)$, as desired.

This problem and solution were suggested by Sam Vandervelde.

Second Solution: We give a bijective proof of the identity

$$a_n = a_{n-1} + 2a_{n-2},$$

which immediately implies that $a_n + a_{n-1} = 2(a_{n-1} + a_{n-2})$. Since trivially $a_0 = a_1 = 1$ (or alternatively $a_1 = 1$, $a_2 = 3$), the desired identity will then follow by induction on n.

To construct the bijection, it is convenient to introduce some alternate representations for the sequences we are counting. Label the points $P_0, \ldots, P_{n-1}$ in order, and define $P_{i+n} = P_i$. One can then represent the sequences to be counted by listing the sequence of vertices $P_{i_0} P_{i_1} \ldots P_{i_m}$ visited by the marker, with the conventions that $i_0 = 0$, $i_m = 2n$, and $i_{j+1} - i_j \in \{1, 2\}$ for $j = 0, \ldots, m-1$. One can represent such sequences of vertices in turn by $2 \times (n+1)$ matrices A by setting

$$A_{ij} = \begin{cases} 1 & P_{ni+j} \text{ is visited} \\ 0 & P_{ni+j} \text{ is not visited} \end{cases} \qquad (i = 0, 1; j = 0, \ldots, n).$$

Such a matrix A corresponds to a valid sequence if and only if $A_{00} = A_{1n} = 1$ (so the sequence of steps starts and ends at P_0), $A_{0n} = A_{n0}$ (so the sequence of steps is well-defined at P_n), and there are no submatrices of any of the forms

$$\begin{pmatrix} 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(to exclude steps of length greater than 2, duplication of a length 2 step, and duplication of a length 1 step). For example, the valid sequences for $n = 3$ are represented by the matrices

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Let S_n be the set of valid $2 \times (n+1)$ matrices. The correspondence $S_{n-2} \sqcup S_{n-2} \sqcup S_{n-1} \cong S_n$ can then be described by replacing the right end of the matrix in the following fashion, where $\cdots$ represents any row of length $n-2$.

$$\begin{array}{c|c}
\begin{pmatrix} \cdots & 1 \\ \cdots & 1 \end{pmatrix} & \begin{pmatrix} \cdots & 1 & 1 & 1 \\ \cdots & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cdots & 1 & 0 & 1 \\ \cdots & 1 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 0 \\ \cdots & 1 \end{pmatrix} & \begin{pmatrix} \cdots & 0 & 1 & 0 \\ \cdots & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cdots & 0 & 1 & 0 \\ \cdots & 1 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 0 & 1 \\ \cdots & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 1 \\ \cdots & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 0 \\ \cdots & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} \cdots & 1 & 0 \\ \cdots & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{array}$$

From this description, it is easy to see that passing from one side to the other preserves the boundary condition and the excluded submatrix conditions (because every submatrix whose entries are not all shown remains unchanged). We thus have the claimed bijection.

This solution was suggested by Kiran Kedlaya.

Third Solution: This solution uses some of the same notation as the second solution.

We first solve a related but simpler counting problem. Let S_n be the set of sequences of steps of lengths 1 or 2 of total length n. For each sequence $s \in S_n$, let $b(s)$ be the number of steps of length 2 in s and define $f_n = \sum_{s \in S_n} 2^{b(s)}$. It is clear that $f_0 = f_1 = 1$. For $n \geq 2$, we also have

$$f_n = f_{n-1} + 2f_{n-2}$$

by counting sequences of length n according to whether they end in a step of length 1 or 2. Thus

$$f_n + f_{n-1} = 2(f_{n-1} + f_{n-2}),$$

from which it follows by induction on n that $f_n + f_{n-1} = 2^n$ for $n \geq 1$. By induction on n, we also have

$$f_n = \frac{2^n + (-1)^n}{3}.$$

We now write a_n in terms of f_n. Label the points of the circle as in the previous solution. We may separate sequences of moves into three types.

1. Sequences that visit P_n but not P_{n-1}. Such a sequence starts with some $s \in S_{n-2}$ followed by a step of length 2. The number of complements for s (i.e., the number of ways to complete it to a full sequence) can be seen to be $2^{b(s)}$ as follows. If we decide in order whether to skip each of $P_{n+1}, \dots, P_{2n}$, then the choice for P_{n+i} is uniquely forced if $A_{0(i-1)} = 1$ and unrestricted if $A_{0(i-1)} = 0$. In the notation of the previous solution, we may see this by noting that

$$\begin{pmatrix} A_{0(i-1)} & A_{0i} \\ A_{1(i-1)} & A_{1i} \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

(This logic does not apply to P_{2n}: we have $A_{0(n-1)} = 0$ but must take $A_{1(2n)} = 1$.) We thus get f_{n-2} sequences of this type.

2. Sequences that visit P_{n-1} but not P_n. Such a sequence starts with some $s \in S_{n-1}$ followed by a step of length 2. There are f_{n-1} sequences of this type.
3. Sequences that visit both P_{n-1} and P_n. Such a sequence starts with some $s \in S_{n-1}$ followed by a step of length 1. Here the count is complicated by the constraint that we must skip P_{2n-1}, so the final step of length 2 does not create an option. Therefore, s contributes $2^{b(s)-1}$ complements if $b(s) > 0$. The only case where $b(s) = 0$ is when s consists of only steps of length 1, in which case we get 1 complement if n is even and 0 complements if n is odd.

Putting this together, we get

$$\begin{aligned} a_n &= f_{n-2} + f_{n-1} + \frac{1}{2}(f_{n-1} + (-1)^n) \\ &= \frac{2^{n-2} + (-1)^{n-2}}{3} + \frac{2^{n-1} + (-1)^{n-1}}{3} + \frac{2^{n-1} + (-1)^{n-1}}{6} + \frac{(-1)^n}{2} \\ &= \frac{2^n + (-1)^n}{3} \end{aligned}$$

and so $a_{n-1} + a_n = 2^n$ as desired.

Remark. The sequence a_n is known as the Jacobsthal sequence and has many other combinatorial interpretations. See sequence `A001045` in the Online Encyclopedia of Integer Sequences: `http://oeis.org`.

This solution was suggested by Kiran Kedlaya.

USAMO 3. For $n = 1$ the answer is clearly 1, since there is only one configuration other than the initial one, and that configuration takes 1 step to get to. From now on we will consider $n \geq 2$.

Note that there are $3n$ possible operations in total, since we can select $3n$ lines to perform an operation on (n lines parallel to each side of the triangle.) Performing an operation twice on the same line is equivalent to doing nothing. Hence, we will describe any combination of operations as a triple of n-tuples $((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$, where each element a_i, b_i, c_i is either 0 or 1 (0 means no operation, 1 means the opposite), each tuple of the triple denotes operating on a line parallel to one of the sides, and the indices, i.e. $1, 2, \ldots, n$, denote the number of marks in the row of operation. Let A denote the set of all such $3n$-tuples. Hence $|A| = 2^{3n}$.

Let B denote the set of all admissible configurations. Let $N = \dfrac{n(n+1)}{2}$. We will describe each element of B by an N-tuple $(z_1, z_2, \ldots, z_N)$, where each element is either 0 or 1 (0 means black, 1 means white). (Which element refers to which position is not important.)

For each element $a \in A$, let $b = f(a)$ be the element of B that is the result of applying the operations in a. Then $f(a + a') = f(a) + f(a')$ for all $a, a' \in A$, where addition is considered in modulo 2. Let K be the set of all $a \in A$ such that $f(a)$ is the all-black configuration. The following eight elements are easily seen to be in K.

- $((0,\, 0,\, \ldots,\, 0),\, (0,\, 0,\, \ldots,\, 0),\, (0,\, 0,\, \ldots,\, 0)) = \mathrm{id}$
- $((0,\, 0,\, \ldots,\, 0),\, (1,\, 1,\, \ldots,\, 1),\, (1,\, 1,\, \ldots,\, 1)) = x$
- $((1,\, 1,\, \ldots,\, 1),\, (1,\, 1,\, \ldots,\, 1),\, (0,\, 0,\, \ldots,\, 0)) = y$
- $((1,\, 1,\, \ldots,\, 1),\, (0,\, 0,\, \ldots,\, 0),\, (1,\, 1,\, \ldots,\, 1)) = x + y$
- $((0,\, 1,\, 0,\, 1,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots)) = z$
- $((0,\, 1,\, 0,\, 1,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots)) = x + z$
- $((1,\, 0,\, 1,\, 0,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots)) = y + z$
- $((1,\, 0,\, 1,\, 0,\, \ldots),\, (0,\, 1,\, 0,\, 1,\, \ldots),\, (1,\, 0,\, 1,\, 0,\, \ldots)) = x + y + z$

We will show that they are the only elements of K.

Suppose $L = ((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$ is in K. Then $a_i + b_j + c_k = 0$ whenever $i + j + k = 2n + 1$ (why this is is left as an exercise for the reader.) By adding x and/or y if necessary, we will assume that $b_n = c_n = 0$. Since $a_2 + b_{n-1} + c_n = a_2 + b_n + c_{n-1} = 0$, we have that $b_{n-1} = c_{n-1}$. There are two cases:

(a) $b_{n-1} = c_{n-1} = 0$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that all the b_i's and c_i's are 0, from which we deduce that $L = \mathrm{id}$.

(b) $b_{n-1} = c_{n-1} = 1$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that $(b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n) = (\ldots, 1, 0, 1, 0)$, from which we deduce that either $L = z$ or $L = x + z$.

Hence L is one of the eight elements listed above. It follows that the 2^{3n} elements of A form 2^{3n-3} sets, each set corresponding to an element of B. For each element $a \in A$, let x_1 be the number of $a_1, a_3, \ldots$ that are 1, and let x_2 be the number of $a_2, a_4, \ldots$ that are 1. Define y_1, y_2, z_1, and z_2 similarly with the b_i's and c_i's. We want to find the element in the set containing a that has the smallest value of $T = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$. The maximum of this value over all the sets is the desired answer.

We observe that an element $a \in A$ has the minimal value of T in its set if and only if it satisfies the following inequalities:

(a) $x_1 + x_2 + y_1 + y_2 \le n$

(b) $x_1 + x_2 + z_1 + z_2 \le n$

(c) $y_1 + y_2 + z_1 + z_2 \le n$

(d) $x_2 + y_2 + z_2 \le \left\lfloor \dfrac{3\lfloor n/2 \rfloor}{2} \right\rfloor = V$

(e) $x_1 + y_1 + z_2 \le \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(f) $x_2 + y_1 + z_1 \le \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(g) $x_1 + y_2 + z_1 \le \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

We wish to find the maximal value of T that an element satisfying all these inequalities can have. Adding the last four inequalities and dividing by 4, we obtain $T \le \left\lfloor \dfrac{V+3W}{2} \right\rfloor$. We consider four cases:

(a) $n = 4k$. $V = W = 3k$, and so $T \le 6k$. We can choose $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = k$ to attain the bound.

(b) $n = 4k+1$. $V = 3k$ and $W = 3k+1$, and so $T \le 6k+1$. We can choose $x_1 = x_2 = y_1 = y_2 = z_2 = k$ and $z_1 = k+1$ to attain the bound.

(c) $n = 4k+2$. $V = 3k+1$ and $W = 3k+1$, and so $T \le 6k+2$. We can choose $x_1 = x_2 = y_1 = y_2 = k$ and $z_1 = z_2 = k+1$ to attain the bound.

(d) $n = 4k+3$. $V = 3k+1$ and $W = 3k+2$, and so $T \le 6k+3$. We can choose $x_1 = x_2 = y_2 = k$ and $y_1 = z_1 = z_2 = k+1$ to attain the bound.

This concludes our proof.

This problem and solution were suggested by Warut Suksompong.

USAMO 4. **First Solution:** Let a, b, c be nonnegative real numbers such that $x = 1 + a^2$, $y = 1 + b^2$ and $z = 1 + c^2$. We may assume that $c \le a, b$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy-Schwarz inequality yields

$$(a + b + c)^2 \le (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1 + a^2)(1 + b^2) \le (a + b)^2,$$

which can also be written $(ab - 1)^2 \le 0$. Hence $ab = 1$ and the Cauchy-Schwarz inequality must be an equality, that is, $c(a + b) = 1$. Conversely, if $ab = 1$ and $c(a + b) = 1$, then the relation in the statement of the problem holds, since $c = \frac{1}{a+b} < \frac{1}{b} = a$ and similarly $c < b$.

Thus the solutions of the problem are

$$x = 1 + a^2, \quad y = 1 + \frac{1}{a^2}, \quad z = 1 + \left(\frac{a}{a^2 + 1}\right)^2$$

for some $a > 0$, as well as permutations of this. (Note that we can actually assume $a \ge 1$ by switching x and y if necessary.)

This problem and solution were suggested by Titu Andreescu.

Second Solution: We maintain the notations in the first solution and again consider the equation

$$(a+b+c)^2 = 1 + c^2 + (1+a^2)(1+b^2)(1+c^2).$$

Expanding both sides of the equation yields

$$a^2+b^2+c^2+2ab+2bc+2ca = 1+c^2+1+a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2+a^2b^2c^2$$

or

$$a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2-2ab-2bc-2ca+c^2+2 = 2(ab+bc+ca).$$

Setting $(u,v,w)=(ab,bc,ca)$, we can write the above equation as

$$uvw+u^2+v^2+w^2-2u-2v-2w+\frac{vw}{u}+2 = 2(u+v+w).$$

which is the equality case of the sum of the following three special cases of the AM-GM inequality:

$$uvw+\frac{vw}{u}\ge 2vw, v^2+w^2+2vw+1 = 2(v+w)\ge 0, \quad u^2+1\ge 2u.$$

Hence we must have the equality cases these AM-GM inequalities; that is, $ab=u=1$ and $a(b+c)=v+w=1$. We can then complete our solution as we did in the first solution.

This solution was suggested by Zuming Feng.

USAMO 5. **First Solution:** For a given positive integer k, write $10^k m - n = 2^r 5^s t$, where $\gcd(t,10)=1$. For large enough values of k the number of times 2 and 5 divide the left-hand side is at most the number of times they divide n, hence by choosing k large we can make t arbitrarily large. Choose k so that t is larger than either m or n.

Since t is relatively prime to 10 there is a smallest exponent b for which $t\,|\,(10^b-1)$. Thus b is the number of digits in the repeating portion of the decimal expansion for $\frac{1}{t}$. More precisely, if we write $tc=(10^b-1)$, then the repeating block is the b-digit decimal representation of c, obtained by prepending extra initial zeros to c as necessary. Since t is larger than m or n, the decimal expansions of $\frac{m}{t}$ and $\frac{n}{t}$ will consist of repeated b-digit representations of cm and cn, respectively. Rewriting the identity in the first line as

$$10^k\left(\frac{m}{t}\right) = 2^r5^s + \frac{n}{t},$$

we see that the decimal expansion of $\frac{n}{t}$ is obtained from that of $\frac{m}{t}$ by shifting the decimal to the right k places and removing the integer part. Thus the b-digit representations of cm and cn are cyclic shifts of one another. In particular, they have the same number of occurrences of each nonzero digit. (Because they may have different numbers of leading zeros as b-digit numbers, the number of zeros in their decimal expansions may differ.)

This problem and solution were suggested by Richard Stong.

Second Solution: Suppose without loss of generality that $m\ge n$. Note that if the desired conclusion holds for the pair (km,kn) for some k, then it also holds for (m,n). Write $n=2^a5^bl$ for some l relatively prime to 10, and note that it suffices to show the desired statement for the pair $(2^b5^am, 2^b5^an) = (2^b5^am, 10^{a+b}l)$. Further, because $10^{a+b}l$ ends with a string of $a+b$ trailing 0's it suffices to show the desired for the pair $(2^b5^am, l)$, where $\gcd(l, 10) = 1$. Thus, from now on we assume that $\gcd(n, 10) = 1$.

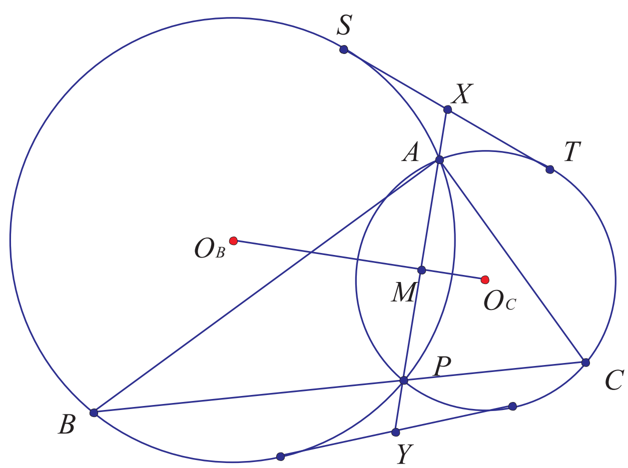
For such a pair (m, n), we see that $\gcd(10m - n, 10) = 1$, so we may find some k and some c so that $c(10m - n) = 10^k - 1$, which implies that $10cm = (10^k - 1) + cn$. We observe that $cn \equiv 1 \pmod{10}$, hence $cn = 10y + 1$ for some y which satisfies $10y < cn < 10^k$. Substituting in, we find that

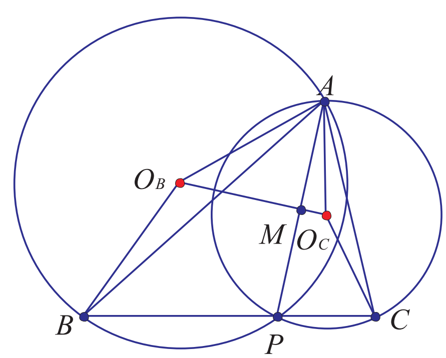
$$10cm = 10^k - 1 + cn = 10^k + 10y,$$

which implies that the non-zero digits of cm are exactly those of y with an additional 1. But the non-zero digits of cn are those of y with an additional 1, so the non-zero digits of cn and cm coincide, as needed.

This solution was suggested by Xiaodong Zhou.

USAMO 6. We consider the left-hand side configuration shown below. Let O_B and ω_B (O_C and ω_C) denote the circumcenter and circumcircle of triangle ABP (ACP) respectively. Line ST, with S on ω_B and T on ω_C, is one of the common tangent lines of the two circumcircles. Point X lies on segment ST. Point Y lies on the other common tangent line.





We will start with the following simple and well known geometry facts.

Let M be the intersection of segments XY and O_BO_C. By symmetry, M is the midpoint of both segments AP and XY, and line O_BO_C is the perpendicular bisector of segments XY and AP. By the power-of-a-point theorem,

$$XS^2 = XA \cdot XP = XT^2 \quad \text{and} \quad X \text{ is the midpoint of segment } ST. \tag{3}$$

Triangles ABC and AO_BO_C are similar to each other, which is the so called *Salmon theorem*. Indeed, $\angle ABC = \angle MO_BA = \angle O_CO_BA$, because each angle is equal to half of the angular size of arc $\widehat{AP}$ of ω_B. Likewise, $\angle O_BO_CA = \angle C$. In particular, we have

$$\frac{AB}{AO_B} = \frac{BC}{O_BO_C} = \frac{CA}{O_CA} \tag{4}$$

Set $AB = c$, $BC = a$, and $CA = b$. We will establish the following key fact in two approaches.

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{BC^2}{(AB+AC)^2} = \frac{a^2}{(b+c)^2}. \tag{5}$$

With this fact, the given condition in the problem becomes

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{a^2}{(b+c)^2} \quad \text{or} \quad PB \cdot PC = \frac{a^2bc}{(b+c)^2}. \tag{6}$$

There are precisely two points P_1 and P_2 (on segment BC) satisfying (6): AP_1 is the bisector of $\angle BAC$ and P_2 is the reflection of P_1 across the midpoint of segment BC. Indeed, by the angle-bisector theorem, $P_2C = P_1B = \frac{ac}{b+c}$ and $P_2B = P_1C = \frac{ab}{b+c}$, from which (6) follows.

In order to settle the question, it remains to show that we can't have more than two points satisfying (6). We just write (6) as

$$\frac{a^2bc}{(b+c)^2} = PB \cdot PC = PB \cdot (a - PB).$$

This a quadratic equation in PB, which can have at most two solutions.

Solution 1. Rays O_BX and O_CT meet in W. Because of (3) and $O_BS \parallel O_CT$, triangles O_BSX and WTX are congruent to each other. Hence $O_BX = XW$ and triangles O_BXO_C and WXO_C have the same area. Note that XM and XT are altitudes in triangles O_BXO_C and WXO_C respectively. Hence

$$\frac{XY \cdot O_BO_C}{4} = \frac{XM \cdot O_BO_C}{2} = \frac{XT \cdot O_CW}{2} = \frac{ST \cdot (O_CT + TW)}{4} = \frac{ST \cdot (O_CT + O_BS)}{4}.$$

By (4), we can write the above equation as

$$\frac{XY}{ST} = \frac{O_CT + O_BS}{O_BO_C} = \frac{O_CA + O_BA}{O_BO_C} = \frac{AB + AC}{BC} \quad \text{or} \quad \frac{XY^2}{ST^2} = \frac{(b+c)^2}{a^2}. \tag{7}$$

Note that O_BSTO_C is a right trapezoid. Let U be the foot of the perpendicular from O_C on O_BS. We have

$$ST^2 = UO_C^2 = O_BO_C^2 - O_SU^2 = O_BO_C^2 - (O_BS - O_CT)^2 = O_BO_C^2 - (O_BA - O_CA)^2.$$

By (4), we can write the above equation as

$$ST^2 = \frac{O_BO_C^2}{BC^2}(BC^2 - (BA - CA)^2) = \frac{O_BO_C^2}{BC^2}(a^2 - (b-c)^2) = \frac{O_BO_C^2}{BC^2}(a+b-c)(a-b+c). \tag{8}$$

Multiplying (7) and (8) together gives

$$XY^2 = \frac{O_BO_C^2}{BC^2} \cdot \frac{(a+b-c)(a-b+c)(b+c)^2}{a^2}. \tag{9}$$

Let h_a denote length of the altitude from A to side BC in triangle ABC. Then h_a and AM are corresponding parts in similar triangles ABC and AO_BO_C, and so

$$\frac{O_BO_C^2}{BC^2} = \frac{AM^2}{h_a^2} = \frac{AM^2}{4h_a^2}. \tag{10}$$

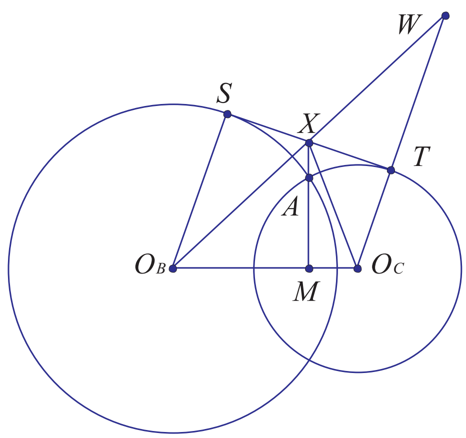
Multiplying (9) and (10) together gives

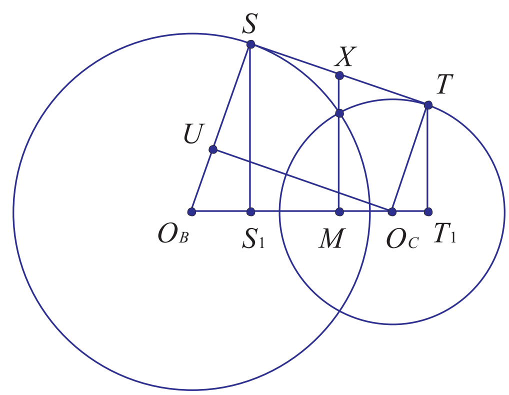
$$XY^2 = \frac{AP^2}{4h_a^2} \cdot \frac{(a+b-c)(a-b+c)(b+c)^2}{a^2}$$

By Heron's formula, we have

$$\frac{AP^2}{XY^2} = \frac{4h_a^2a^2}{(a+b-c)(a-b+c)(b+c)^2} = \frac{(a+b+c)(b+c-a)}{(b+c)^2} = \frac{(b+c)^2-a^2}{(b+c)^2} = 1-\frac{a^2}{(b+c)^2},$$

from which (5) follows.





Solution 2. By the power-of-a-point theorem, we have $XA \cdot XP = XS^2$. Therefore,

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{XY^2 - PA^2}{XY^2} = \frac{(XY+PA)(XY-PA)}{XY^2} = \frac{4XA \cdot XP}{XY^2} = \frac{4XS^2}{XY^2}\frac{ST^2}{XY^2}. \tag{11}$$

Let S_1 and T_1 be the feet of the perpendiculars from S and T to line O_BO_C. It is easy to see that right triangles $O_BSS_1, O_CTT_1, O_SO_CU$ are similar to each other. Note also that XM is the midline of right trapezoid S_1STT_1 (because of (3)). Therefore, we have

$$\frac{ST}{O_BO_C} = \frac{UO_C}{O_BO_C} = \frac{S_1S}{O_BS} = \frac{T_1T}{O_CT} = \frac{S_1S + T_1T}{O_BS + O_CT} = \frac{2XM}{O_BS + O_CT} = \frac{XY}{O_BS + O_CT},$$

or, by (4),

$$\frac{ST}{XY} = \frac{O_BO_C}{O_BS + O_CT} = \frac{O_BO_C}{O_BA + O_CA} = \frac{BC}{BA + CA} = \frac{a}{b+c}. \tag{12}$$

It is clear that (5) follows from (11) and (12).

This problem and Solution 1 were suggested by Titu Andreescu and Cosmin Pohoata. Solution 2 was suggested by Zuming Feng.

USAMO 2013 Solution Notes

Compiled by Evan Chen

May 2, 2020

This is an compilation of solutions for the 2013 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

2. For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

3. Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n tokens. Initially, each token has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the token on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

4. Find all real numbers $x, y, z \geq 1$ satisfying
$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

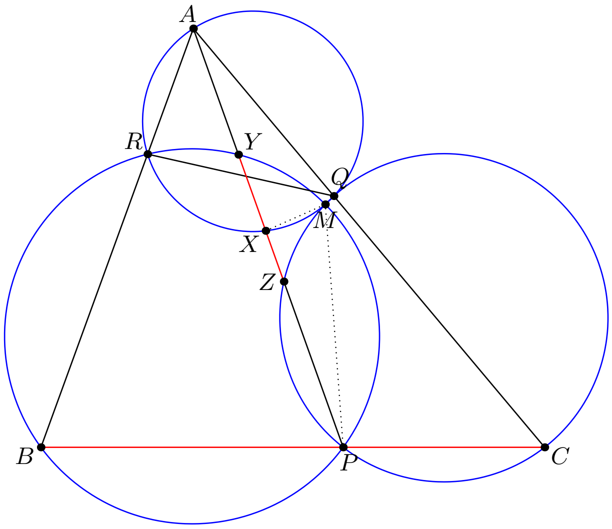
5. Let m and n be positive integers. Prove that there exists a positive integer c such that cm and cn have the same nonzero decimal digits.

6. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then
$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

§1 USAMO 2013/1, proposed by Zuming Feng

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

so this follows.

§2 USAMO 2013/2, proposed by Kiran Kedlaya

For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

Imagine the counter is moving along the set $S = \{0, 1, \dots, 2n\}$ instead, starting at 0 and ending at $2n$, in jumps of length 1 and 2. We can then record the sequence of moves as a matrix of the form

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_{n-1} & p_n \\ p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{2n} \end{bmatrix}$$

where $p_i = 1$ if the point i was visited by the counter, and $p_i = 0$ if the point was not visited by the counter. Note that $p_0 = p_{2n} = 1$ and the upper-right and lower-left entries are equal. Then, the problem amounts to finding the number of such matrices which avoid the contiguous submatrices

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which correspond to forbidding jumps of length greater than 2, repeating a length 2 jump and repeating a length 1 jump.

We will for now ignore the boundary conditions. Instead we say a $2 \times m$ matrix is *silver* ($m \geq 2$) if it avoids the three shapes above. We consider three types of silver matrices (essentially doing casework on the last column):

- *type B matrices*, of the shape $\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix}$
- *type C matrices*, of the shape $\begin{bmatrix} 1 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}$.
- *type D matrices*, of the shape $\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 1 \end{bmatrix}$.

We let b_m, c_m, d_m denote matrices of each type, of size $2 \times m$, and claim the following two recursions for $m \geq 4$:

$$\begin{aligned} b_m &= c_{m-1} + d_{m-1} \\ c_m &= b_{m-1} + d_{m-1} \\ d_m &= b_{m-1} + c_{m-1}. \end{aligned}$$

Indeed, if we delete the last column of a type B matrix and consider what used to be the second-to-last column, we find that it is either type C or type D. This establishes the first recursion and the others are analogous.

Note that $b_2 = 0$ and $c_2 = d_2 = 1$. So using this recursion, the first few values are

m	2	3	4	5	6	7	8
b_m	0	2	2	6	10	22	42
c_m	1	1	3	5	11	21	43
d_m	1	1	3	5	11	21	43

and a calculation gives $b_m = \frac{2^{m-1}+2(-1)^{m-1}}{3}$, $c_m = d_m = \frac{2^{m-1}+(-1)^{m-1}}{3}$.

We now relate a_n to b_m, c_m, d_m. Observe that a matrix as described in the problem is a silver matrix of one of two forms:

$$\begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 0 \\ 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 1 \\ 1 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix}.$$

There are c_{n+1} matrices of the first form. Moreover, there are $2d_n$ matrices of the second form (to see this, delete the first column; we either get a type-D matrix or an upside-down type-D matrix). Thus we get

$$a_n = c_{n+1} + 2d_n = \frac{2^{n+1} + (-1)^{n+1}}{3}.$$

This easily implies the result.

§3 USAMO 2013/3, proposed by Warut Suksompong

Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n tokens. Initially, each token has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the token on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

The answer is

$$\max_C f(C) = \begin{cases} 6k & n = 4k \\ 6k+1 & n = 4k+1 \\ 6k+2 & n = 4k+2 \\ 6k+3 & n = 4k+3. \end{cases}$$

The main point of the problem is actually to determine all linear dependencies among the $3n$ possible moves (since the moves commute and applying a move twice is the same as doing nothing). In what follows, assume $n > 1$ for convenience.

To this end, we consider sequences of operations as additive vectors in $v \in \mathbb{F}_2^{3n}$, with the linear map $T \colon \mathbb{F}_2^{3n} \to \mathbb{F}_2^{\frac{1}{2}n(n+1)}$ denoting the result of applying a vector v. We in particular focus on the following four vectors.

- Three vectors x, y, z are defined by choosing all n lines parallel to one axis. Note $T(x) = T(y) = T(z) = \mathbf{1}$ (i.e. these vectors flip all tokens).
- The vector θ which toggles all lines with an even number of tokens. One can check that $T(\theta) = \mathbf{0}$. (Easiest to guess from $n = 2$ and $n = 3$ case.) One amusing proof that this works is to use Vivani's theorem: in an equilateral triangle ABC, the sum of distances from an interior point P to the three sides is equal.

The main claim is:

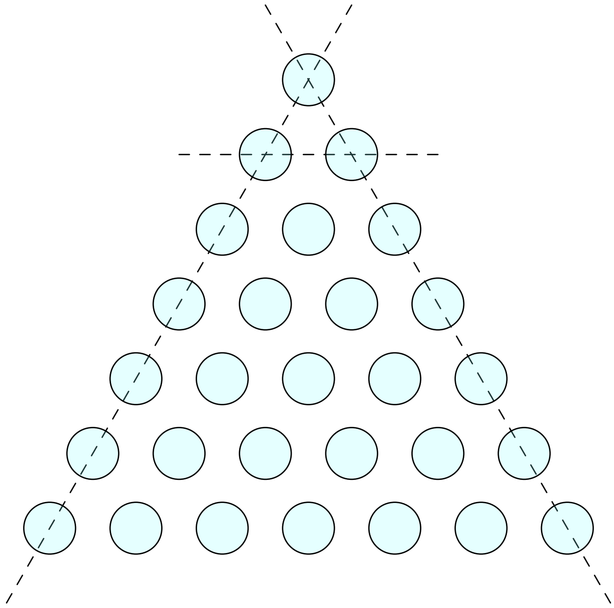
Claim — For $n \geq 2$, the kernel of T has exactly eight elements, namely $\{\mathbf{0}, x+y, y+z, z+x, \theta, \theta+x+y, \theta+y+z, \theta+z+x\}$.

Proof. Suppose $T(v) = 0$.

- If v uses the y-move of length n, then we replace v with $v + (x+y)$ to obtain a vector in the kernel not using the y-move of length n.
- If v uses the z-move of length n, then we replace v with $v + (x+z)$ to obtain a vector in the kernel not using the z-move of length n.
- If v uses the x-move of length 2, then
  - if n is odd, replace v with $v + \theta$;
  - if n is even, replace v with $v + (\theta + y + z)$

  to obtain a vector in the kernel not using the x-move of length 2.

A picture is shown below, with the unused rows being dotted.



Then, it is easy to check inductively that v must now be the zero vector, after the replacements. The idea is that for each token t, if two of the moves involving t are unused, so is the third, and in this way we can show all rows are unused. Thus the original v was in the kernel we described.

(An alternative proof by induction is feasible too; as a sequence of movings which does not affect the top n rows also does not affect the to $n-1$ rows.) □

Then problem is a coordinate bash, since given any v we now know exactly which vectors w have $T(v) = T(w)$, so given any admissible configuration C one can exactly compute $f(C)$ as the minimum of eight values.

§4 USAMO 2013/4, proposed by Titu Andreescu

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Set $x = 1+a$, $y = 1+b$, $z = 1+c$ which eliminates the $x, y, z \geq 1$ condition. Then the given equation rewrites as

$$\sqrt{(1+a)\left(1+(1+b)(1+c)\right)} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

In fact, we are going to prove the left-hand side always exceeds the right-hand side, and then determine the equality cases. We have:

$$\begin{aligned}(1+a)\left(1+(1+b)(1+c)\right) &= (a+1)\left(1+(b+1)(1+c)\right) \\ &\leq (a+1)\left(1+\left(\sqrt{b}+\sqrt{c}\right)^2\right) \\ &\leq \left(\sqrt{a}+\left(\sqrt{b}+\sqrt{c}\right)\right)\end{aligned}$$

by two applications of Cauchy-Schwarz.

Equality holds if $bc = 1$ and $1/a = \sqrt{b} + \sqrt{c}$. Letting $c = t^2$ for $t \geq 1$, we recover $b = t^{-2} \leq t^2$ and $a = \frac{1}{t+1/t} \leq t^2$.

Hence the solution set is

$$(x, y, z) = \left(1 + \left(\frac{t}{t^2+1}\right)^2, 1 + \frac{1}{t^2}, 1 + t^2\right)$$

and permutations, for any $t > 0$.

§5 USAMO 2013/5, proposed by Richard Stong

Let m and n be positive integers. Prove that there exists a positive integer c such that cm and cn have the same nonzero decimal digits.

One-line spoiler: 142857. More verbosely, the idea is to look at the decimal representation of $1/D$, m/D, n/D for a suitable denominator D, which have a "cyclic shift" property in which the digits of n/D are the digits of m/D shifted by 3.

> **Remark** (An example to follow along)**.** Here is an example to follow along in the subsequent proof If $m = 4$ and $n = 23$ then the magic numbers $e = 3$ and $D = 41$ obey
>
> $$10^3 \cdot \frac{4}{41} = 97 + \frac{23}{41}.$$
>
> The idea is that
>
> $$\frac{1}{41} = 0.\overline{02439}$$
> $$\frac{4}{41} = 0.\overline{09756}$$
> $$\frac{23}{41} = 0.\overline{56097}$$
>
> and so $c = 2349$ works; we get $4c = 9756$ and $23c = 56097$ which are cyclic shifts of each other by 3 places (with some leading zeros appended).

Here is the one to use:

Claim — There exists positive integers D and e such that $\gcd(D, 10) = 1$, $D > \max(m, n)$, and moreover

$$\frac{10^e m - n}{D} \in \mathbb{Z}.$$

Proof. Suppose we pick some exponent e and define the number

$$A = 10^e n - m.$$

Let $r = \nu_2(m)$ and $s = \nu_5(m)$. As long as $e > \max(r, s)$ we have $\nu_2(A) = r$ and $\nu_5(A) = s$, too. Now choose any $e > \max(r, s)$ big enough that $A > 2^r 5^s \max(m, n)$ also holds. Then the number $D = \frac{A}{2^r 5^s}$ works; the first two properties hold by construction and

$$10^e \cdot \frac{n}{D} - \frac{m}{D} = \frac{A}{D} = 2^r 5^s$$

is an integer. □

> **Remark** (For people who like obscure theorems)**.** Kobayashi's theorem implies we can actually pick D to be prime.

Now we take c to be the number under the bar of $1/D$ (leading zeros removed). Then the decimal representation of $\frac{m}{D}$ is the decimal representation of cm repeated (possibly including leading zeros). Similarly, $\frac{n}{D}$ has the decimal representation of cm repeated (possibly including leading zeros). Finally, since

$$10^e \cdot \frac{m}{D} - \frac{n}{D} \text{ is an integer}$$

it follows that these repeating decimal representations are rotations of each other by e places, so in particular they have the same number of nonzero digits.

Remark. Many students tried to find a D satisfying the stronger hypothesis that $1/D$, $2/D$, ..., $(D-1)/D$ are cyclic shifts of each other. For example, this holds in the famous $D = 7$ case.

The official USAMO 2013 solutions try to do this by proving that 10 is a primitive root modulo 7^e for each $e \geq 1$, by Hensel lifting lemma. I think this argument is actually *incorrect*, because it breaks if either m or n are divisible by 7. Put bluntly, $\frac{7}{49}$ and $\frac{8}{49}$ are not shifts of each other.

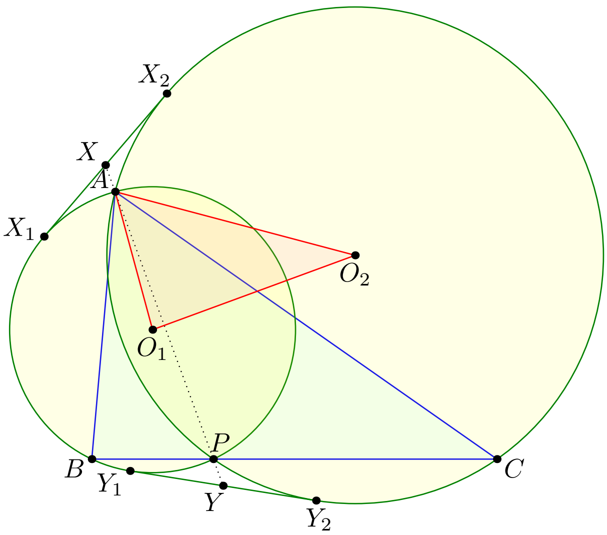
One may be tempted to resort to using large primes D rather than powers of 7 to deal with this issue. However it is an open conjecture (a special case of Artin's primitive root conjecture) whether or not $10 \pmod p$ is primitive infinitely often, which is the necessary conjecture so this is harder than it seems.

§6 USAMO 2013/6, proposed by Titu Andreescu and Cosmin Pohoata

Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Let O_1 and O_2 denote the circumcenters of PAB and PAC. The main idea is to notice that $\triangle ABC$ and $\triangle AO_1O_2$ are spirally similar.



Claim — We have $\triangle AO_1B \overset{+}{\sim} \triangle AO_2C$. Hence $\triangle ABC \overset{+}{\sim} \triangle AO_1O_2$.

Proof. Assume without loss of generality that $\angle APB \le 90°$. Then

$$\angle AO_1B = 2\angle ABP$$

but

$$\angle AO_2C = 2\,(180 - \angle APC) = 2\angle ABP.$$

Hence $\angle AO_1B = \angle AO_2C$. Moreover, both triangles are isosceles, establishing first part of claim. The second part follows from spiral similarities coming in pairs. □

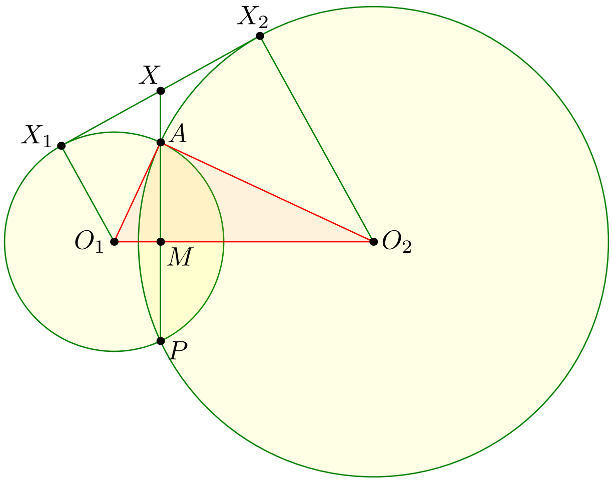
Claim — We always have

$$\left(\frac{PA}{XY}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2.$$

(In particular, this does not depend on P.)

Proof. We now delete the points B and C and remember only the fact that $\triangle AO_1O_2$ has angles A, B, C. The rest is a computation and several approaches are possible.

Without loss of generality A is closer to X than Y, and let the common tangents be $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$. We'll perform the main calculation with the convenient scaling $O_BO_C = a$, $AO_C = b$, and $AO_B = c$. Let B_1 and C_1 be the tangency points of X, and let $h = AM$ be the height of $\triangle AO_BO_C$.



Note that by Power of a Point, we have $XX_1^2 = XX_2^2 = XM^2 - h^2$. Also, by Pythagorean theorem we easily obtain $X_1X_2 = a^2 - (b-c)^2$. So putting these together gives

$$XM^2 - h^2 = \frac{a^2 - (b-c)^2}{4} = \frac{(a+b-c)(a-b+c)}{4} = (s-b)(s-c).$$

Therefore, we have

Then

$$\begin{aligned} \frac{XM^2}{h^2} &= 1 + \frac{(s-b)(s-c)}{h^2} = 1 + \frac{a^2(s-b)(s-c)}{a^2h^2} \\ &= 1 + \frac{a^2(s-b)(s-c)}{4s(s-a)(s-b)(s-c)} = 1 + \frac{a^2}{4s(s-a)} \\ &= 1 + \frac{a^2}{(b+c)^2 - a^2} = \frac{(b+c)^2}{(b+c)^2 - a^2}. \end{aligned}$$

Thus

$$\left(\frac{PA}{XY}\right)^2 = \left(\frac{h}{XM}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2. \qquad \square$$

To finish, note that when P is the foot of the $\angle A$-bisector, we necessarily have

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{\left(\frac{b}{b+c}a\right)\left(\frac{c}{b+c}a\right)}{bc} = \left(\frac{a}{b+c}\right)^2.$$

Since there are clearly at most two solutions as $\frac{PA}{XY}$ is fixed, these are the only two solutions.

43^{rd} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 29, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 1. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

USAMO 2. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

USAMO 3. Prove that there exists an infinite set of points

$$\dots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \dots$$

in the plane with the following property: For any three distinct integers a, b and c, points P_a, P_b and P_c are collinear if and only if $a + b + c = 2014$.

43^{rd} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 30, 2014

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet any of these requirements will result in a 1-point automatic deduction.

USAMO 4. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent spaces in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum exists.

USAMO 5. Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

USAMO 6. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

43^{rd} United States of America Mathematical Olympiad

Day I, II   12:30 PM – 5 PM EDT

April 29 - April 30, 2014

USAMO 1. Using Vieta's identities we have:

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 - x_1x_2x_3x_4 \geq 5$$

and so

$$x_1(x_2 + x_3 + x_4 - x_2x_3x_4) + 1(x_2x_3 + x_2x_4 + x_3x_4 - 1) \geq 4.$$

It follows that

$$4^2 \leq [x_1(x_2 + x_3 + x_4 - x_2x_3x_4) + 1(x_2x_3 + x_2x_4 + x_3x_4 - 1)]^2,$$

and by the Cauchy-Schwarz Inequality,

$$4^2 \leq (x_1^2 + 1)[(x_2 + x_3 + x_4 - x_2x_3x_4)^2 + (x_2x_3 + x_2x_4 + x_3x_4 - 1)^2]$$
$$= (x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1).$$

The equality holds if and only if

$$x_1(x_2x_3 + x_2x_4 + x_3x_4 - 1) = 1(x_2 + x_3 + x_4 - x_2x_3x_4),$$

which is equivalent to

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = x_1 + x_2 + x_3 + x_4,$$

that is, $a = c$. Taking $x_1 = ... = x_4 = 1$ we obtain $b - d = 5$ and that the smallest value of the product in question is 16.

An alternative, shorter argument runs as follows: we have

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = P(i)P(-i) =$$

$$((1 - b + d) + i(c - a))(1 - b + d - i(c - a)) = (b - d - 1)^2 + (c - a)^2 \geq 16,$$

with equality if and only if $b - d = 5$ and $a = c$, both attained if $x_1 = ... = x_4 = 1$.

This problem and solutions were suggested by Titu Andreescu.

USAMO 2. Let f be a solution of the problem. Let p be a prime. Since p divides $f(p)^2$, p divides $f(p)$ and so p divides $\frac{f(p)^2}{p}$. Taking $y=0$ and $x=p$, we deduce that p divides $f(0)$. As p is arbitrary, we must have $f(0)=0$. Next, take $y=0$ to obtain $xf(-x)=\frac{f(x)^2}{x}$. Replacing x by $-x$, and combining the two relations yields $f(x)=0$ or $f(x)=x^2$ for all x.

Suppose now that there exists $x_0\neq 0$ such that $f(x_0)=0$. Taking $y=x_0$, we obtain $xf(-x)+x_0^2f(2x)=\frac{f(x)^2}{x}$, yielding $x_0^2f(2x)=0$ for all x and so f vanishes on even numbers. Assume that there exists an odd number y_0 such that $f(y_0)\neq 0$, so $f(y_0)=y_0^2$. Taking $y=y_0$, we obtain

$$xf(2y_0^2-x)+y_0^2f(2x-y_0^2)=\frac{f(x)^2}{x}+f(y_0^3).$$

Choosing x even, we deduce that $y_0^2f(2x-y_0^2)=f(y_0^3)$. This forces $f(y_0^3)=0$, as otherwise we would have $f(2x-y_0^2)=(2x-y_0^2)^2$ for all even x and so $y_0^2(2x-y_0^2)^2=f(y_0^3)$ for all such x, obviously impossible. Thus $f(2x-y_0^2)=0$ for all even numbers x, that is f vanishes on numbers of the form $4k+3$. But since $x^2f(-x)=f(x)^2$, f also vanishes on all x such that $-x\equiv -1 \pmod 4$, that is on $4\mathbb{Z}+1$. Thus f also vanishes on all odd numbers, contradicting the choice of y_0. Hence, if f is not the zero map, then f does not vanish outside 0 and so $f(x)=x^2$ for all x.

In conclusion, $f(x)=0$ for all $x\in\mathbb{Z}$ and $f(x)=x^2$ for all $x\in\mathbb{Z}$ are the only possible solutions. The first function clearly satisfies the given relation, while the second also satisfies the Sophie Germaine identity

$$x(2y^2-x)^2+y^2(2x-y^2)^2=x^3+y^6$$

for all $x,y\in\mathbb{Z}$.

OR

$f(0)=0$: If $f(0)\neq 0$, set $x=2f(0)$ to obtain

$$2(f(0))^2=\frac{(f(2f(0)))^2}{2f(0)}+f(0)$$

that is

$$2(f(0))^2(2f(0)-1)=f(2f(0))^2.$$

But $2(2f(0)-1)$ cannot be a perfect square since it is of the form $4k+2$. So $f(0)=0$.

This problem and the solutions were suggested by Titu Andreescu and Gabriel Dospinescu.

USAMO 3. We claim that defining P_n to be the point with coordinates $(n, n^3-2014n^2)$ will satisfy the conditions of the problem. Recall that points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

Therefore we examine the determinant

$$\begin{vmatrix} a & a^3-2014a^2 & 1 \\ b & b^3-2014b^2 & 1 \\ c & c^3-2014c^2 & 1 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} - 2014 \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}.$$

The first determinant on the right is a homogenous polynomial of degree four divisible by $(a-b)(b-c)(c-a)$. The remaining factor has degree one, is symmetric, and yields an ab^3 term when the product is expanded, hence must be $(a+b+c)$. The second determinant is a homogenous polynomial of degree three divisible by $(a-b)(b-c)(c-a)$, and comparing coefficients of the ab^2 term we see that this is the desired polynomial. Thus

$$\begin{vmatrix} a & a^3-2014a^2 & 1 \\ b & b^3-2014b^2 & 1 \\ c & c^3-2014c^2 & 1 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c-2014).$$

It follows that for distinct a, b and c this expression will equal zero if and only if $a+b+c = 2014$, as desired.

This solution was suggested by Razvan Gelca.

OR

First, note that the translation $x \mapsto x - 671$ in the indices allows us to replace 2014 in the statement by 1. Now it comes natural to look for a polynomial pattern $(P(x), Q(x))$ in the coordinates of a point. The collinearity condition translates, in coordinates, into

$$P(a)Q(b) + P(b)Q(c) + P(c)Q(a) - P(a)Q(c) - P(b)Q(a) - P(c)Q(b) = 0.$$

This should happen only when $a + b + c - 1 = 0$ or when two of a, b, c are equal. Hence the left-hand side should be of the form $(a + b + c - 1)(b - a)(c - b)(a - c)R(a, b, c)$. We can try the simplest case $R = 1$ so that the dominant coefficients of both $P(x)$ and $Q(x)$ are 1. $P(x)$ and $Q(x)$ cannot both have even degree because then the 4th degree terms on the left cancel out, while on the right there are clearly 4th degree terms. Hence one of the polynomials $P(x)$ and $Q(x)$ has degree 3, the other has degree 1. By a translation we can turn the degree 1 polynomial into x, thus we may assume that $P(x) = x$. Thus we should have

$$\begin{aligned} &(c-b)Q(a) + (a-c)Q(b) + (b-a)Q(c) \\ &\quad = (a+b+c-1)(b-a)(c-b)(a-c). \end{aligned}$$

So we let $Q(x) = x^3 + \alpha x^2 + \beta x + \gamma$. Note that we are free to choose β and γ any way we want, since they cancel out. So we let $Q(x) = x^3 + \alpha x^2$.

For $a = 0$, $b = -1$, $c = 1$ the above identity yields $-2Q(0) - Q(-1) - Q(1) = 2$, and hence $\alpha = -1$.

Returning to the case of the problem with 2014 instead of 1, we have the points $P_n = (n - 671, (n - 671)^3 - (n - 671)^2)$. But we can simplify this since we can replace $P(x)$ by x and ignore the linear part of $Q(x)$. We thus obtain the simpler infinite family of points

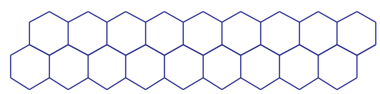
$$P_n = (n, n^3 - 3 \cdot 671n^2 - n^2) = (n, n^3 - 2014n^2)$$

satisfying the conditions of the problem.

This problem and the second solution was suggested by Sam Vandervelde.

USAMO 4. The answer is $k = 6$. First we show that A cannot win for $k \geq 6$. Color the grid in three colors so that no two adjacent spaces have the same color, and arbitrarily pick one color C. B will play by always removing a counter from a space colored C that A just played. If there is no such counter, B plays arbitrarily. Because A cannot cover two spaces colored C simultaneously, it is possible for B to play in this fashion. Now note that any line of six consecutive squares contains two spaces colored C. For A to win he must cover both, but B's strategy ensures at most one space colored C will have a counter at any time.

Now we show that A can obtain 5 counters in a row. Take a set of cells in the grid forming the shape shown below. We will have A play counters only in this set of grid cells until this is no longer possible. Since B only removes one counter for every two A places, the number of counters in this set will increase each turn, so at some point it will be impossible for A to play in this set anymore. At that point any two adjacent grid spaces in the set have at least one counter between them.



Consider only the top row of cells in the set, and take the lengths of each consecutive run of cells. If there are two adjacent runs that have a combined length of at least 4, then A gets 5 counters in a row by filling the space in between. Otherwise, a bit of case analysis shows that there exists a run of 1 counter which is neither the first nor last run. This single counter has an empty space on either side of it on the first row. As a result, the four spaces of the second row touching these two empty spaces all must have counters. Then A can play in the 5th cell on either side of these 4 to get 5 counters in a row. So in all cases A can win with $k \leq 5$.

This problem and solution was suggested by Palmer Mebane.

USAMO 5. It is well-known that the reflection H' of the orthocenter H in the line AC lies on the circumcircle of triangle ABC. Hence, the circumcenter of triangle CAH' coincides with the circumcenter of triangle ABC. But since H' is the reflection of H in the line AC, the triangles ACH and CAH' are symmetric with respect to BC, and the circumcenter O' of triangle ACH must be the reflection of the circumcenter of triangle CAH' in the line BC, i. e. the reflection of the circumcenter of triangle ABC in the line CA.

Now since the quadrilateral $AHPC$ is cyclic and since H, Y are the orthocenters of triangles ABC, and APC, respectively, we have that

$$\angle ABC = 180^\circ - \angle AHC = 180^\circ - \angle APC = \angle AYC.$$

Hence the point Y lies on the circumcircle of triangle ABC, and therefore $OC = OY = R$, where R denotes the circumradius of triangle ABC.

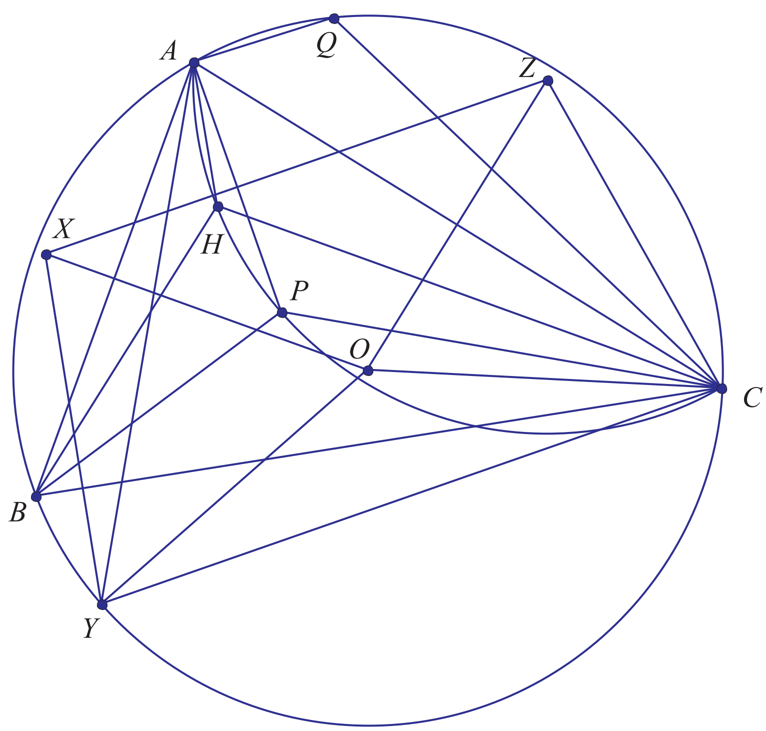
On the other hand, note that the lines OX, XO', $O'O$ are the perpendicular bisectors of the segments AB, AP, and AC, respectively, we get

$$\angle OXO' = \angle BAP = \angle PAC = m(\angle XO'O.$$

Thus $OO' = OX$. Combining this with $OC = OY$ and with the parallelism of the lines XO' and YC (note that these two lines are both perpendicular to AP), we conclude that the trapezoid $XYCO'$ is isosceles, and therefore $XY = O'C = OC = R$. This completes our proof. □

Remark. If ABC is right-angled at A, then the statement is trivially true if we convene that the circumcenter of AB is the midpoint of AB and that the orthocenter of AC is the midpoint of AC. Then, we have that $XY = \frac{1}{2}BC = R$.

OR



Because ABC is acute, H lies inside the triangle. We consider the configuration show above. (For other possible configurations, it is not difficult to adjust our proof properly.)

Let O and Z denote the circumcenters of triangles ABC and APC respectively. Let ω and r denote the circumcircle and the circumradius of triangle ABC respectively. We will show that

$$XYCZ \text{ is an isoscelees trapezoid with } XY = CZ = r. \tag{1}$$

Because X and Z are the circumcenters of triangle APB and APC, line XZ is the perpendicular bisector of segment AP. Because Y is the orthocenter of triangle APC, $CY \perp AP$. Hence both lines XZ and CY are perpendicular to line AP, implying that $XYZC$ is a trapezoid with $XZ \parallel CY$.

Because X and O are the circumcenters of triangles APB and ABC, line XO is the perpendicular bisector of segment AB. Because $XO \perp AB$ and $XZ \perp AP$, the acute angles formed by lines XO and XZ is equal to the acute angle formed by lines AP and AB; that is, $\angle OXZ = \angle BAP$. Likewise, we can can show that $\angle OZX = \angle CAP$. Therefore, we have $\angle OXZ = \angle BAP = \angle CAP = \angle OZX$, implying that $OX = OZ$; that is, O lies on the perpendicular bisector of segment XZ.

Because H is the orthocenter of acute triangle ABC, $\angle AHC = 180° - \angle ABC$. Because $APHC$ is cyclic, we have $\angle APC = \angle AHC = 180° - \angle ABC$. Now in obtuse triangle APC, $\angle AYC = 180° - \angle APC = \angle ABC$. (This relates to the fact of orthocenter group: if one point is the orthocenter of the triangle formed by the other three points, then any of the four point is the orthocenter of the triangle formed by the other three.) In particular, this means that Y lies on ω; that is, $OY = OC = r$.

Note that in trapezoid $XYCZ$, the perpendicular bisectors of the bases YC and XZ share a common point O. Thus, these two bisectors must coincide; that is, $XYCZ$ is an isosceles trapezoid with $XY = CZ$, establishing the first part of (**??**).

To complete our proof, it suffices to show that $CZ = r$. Let Q be the reflection of H across line AC. It is well known that Q lies ω (because $\angle ACQ = \angle ACH = 90° - \angle BAC = \angle ABH = \angle ABQ$.) We note that triangle AQC and its circumcenter O and triangle AHC and its circumcenter Z are respective images of each other across line AC. In particular, we conclude that $CZ = CO = r$, completing our proof.

This problem and solutions were suggested by Titu Andreescu and Cosmin Pohoata.

USAMO 6. Let a, b, n be positive integers as in the statement of the problem. Let P_n be the set of prime numbers not exceeding n. We will need the following

There is a positive integer n_0 such that for all $n \geq n_0$ we have

$$\sum_{p \in P_n} \left(\frac{n}{p} + 1\right)^2 < \frac{2}{3} n^2.$$

Proof. Expanding and dividing by n^2, and observing that $|P_n| \leq n$, it suffices to prove the inequality

$$\sum_{p \in P_n} \frac{1}{p^2} + \frac{2}{n} \sum_{p \in P_n} \frac{1}{p} + \frac{1}{n} < \frac{2}{3}.$$

Since

$$\frac{2}{n} \sum_{p \in P_n} \frac{1}{p} < \frac{2}{n} \sum_{i=2}^{n} \frac{1}{i} < \frac{2}{n} \log n,$$

it suffices to prove the existence of a constant $r < \frac{2}{3}$ such that $\sum_{p \in P_n} \frac{1}{p^2} < r$. But

$$\sum_{p \in P_n} \frac{1}{p^2} \quad \leq \quad \frac{1}{4} + \frac{1}{9} + \sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{1}{4} + \frac{1}{9} + \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$$
$$= \frac{1}{4} + \frac{1}{9} + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right) < \frac{1}{4} + \frac{1}{9} + \frac{1}{6} < \frac{1}{3}$$

and we can take $r = \frac{1}{4} + \frac{1}{9} + \frac{1}{6}$. $\square$

From now on we fix such n_0, and we prove the statement assuming $n \geq n_0$. Note that for any $p \in P_n$ there are at most $\frac{n}{p} + 1$ numbers $i \in \{0, 1, \ldots, n-1\}$ such that $p \mid a + i$, and likewise for $j \in \{0, 1, \ldots, n-1\}$ such that $p \mid b + j$. Thus there are at most $\left(\frac{n}{p} + 1\right)^2$ pairs (i, j) such that $p \mid \gcd(a+i, b+j)$. Using the previous lemma, we deduce that there are less than $\frac{2}{3}n^2$ pairs (i, j) with $i, j \in \{0, 1, \ldots, n-1\}$ such that $p \mid \gcd(a+i, b+j)$ for some $p \in P_n$.

Let N be the least integer greater than or equal to $\frac{n^2}{3}$. By the above, there are at least N pairs (i, j) with $i, j \in \{0, 1, \ldots, n-1\}$ such that $\gcd(a+i, b+j)$ is not divisible by any prime in P_n. Call these pairs (i_s, j_s) for $s = 1, 2, \ldots, N$. For each pair, choose a prime p_s that divides $\gcd(a + i_s, b + j_s)$ (since, by hypothesis, $\gcd(a + i_s, b + j_s) > 1$); thus $p_s > n$. The map $s \mapsto p_s$ is injective, for if $p_s = p_{s'}$, then $p_s \mid i_s - i_{s'}$, implying $i_s = i_{s'}$, and similarly $j_s = j_{s'}$, hence $s = s'$.

We conclude that $\prod_{i=0}^{n-1}(a + i)$ is a multiple of $\prod_{s=1}^{N} p_s$. Since the p_s are distinct prime numbers greater than n, then,

$$(a+n)^n > \prod_{i=0}^{n-1}(a+i) \geq \prod_{s=1}^{N} p_s \geq \prod_{i=1}^{N}(n + 2i - 1).$$

Let X be this last product. Then

$$X^2 = \prod_{i=1}^{N}[(n+2i-1)(n+2(N+1-i)-1)] > \prod_{i=1}^{N}(2Nn) = (2Nn)^N,$$

where the inequality holds because

$$(n+2i-1)(n+2(N+1-i)-1) > n(2(N+1-i)-1) + (2i-1)n = 2Nn.$$

Finally

$$(a+n)^n > (2Nn)^{\frac{N}{2}} \geq \left(\frac{2n^3}{3}\right)^{\frac{n^2}{6}}.$$

Thus,

$$a \geq \left(\frac{2}{3}\right)^{\frac{1}{6}\cdot n} \cdot n^{\frac{n}{2}} - n,$$

which is larger than $c^n \cdot n^{\frac{n}{2}}$ when n is large enough, for any constant $c < \left(\frac{2}{3}\right)^{\frac{1}{6}}$. Similarly, the same inequality holds for b.

This shows that $\min\{a, b\} \geq c^n \cdot n^{\frac{n}{2}}$ as long as n is large enough. By shrinking c sufficiently, we can ensure the inequality holds for all n.

One can see that the argument is not sharp, so that the factor $n^{\frac{n}{2}}$ can be improved to n^{rn} for some constant r slightly larger than $\frac{1}{2}$. Consequently, for *any* $c > 0$, the inequality in the problem holds if n is large enough.

This problem and solution was suggested by Titu Andreescu and Gabriel Dospinescu.

USAMO 2014 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2014 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

2. Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf\left(2f(y) - x\right) + y^2 f\left(2x - f(y)\right) = \frac{f(x)^2}{x} + f\left(yf(y)\right)$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

3. Prove that there exists an infinite set of points

$$\dots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \dots$$

in the plane with the following property: For any three distinct integers a, b, and c, points P_a, P_b, and P_c are collinear if and only if $a + b + c = 2014$.

4. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In her move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

5. Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of $\angle BAC$. Let X be the circumcenter of triangle APB and let Y be the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

6. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > (cn)^{n/2}.$$

§1 USAMO 2014/1, proposed by Titu Andreescu

Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

The answer is $\boxed{16}$. This can be achieved by taking $x_1 = x_2 = x_3 = x_4 = 1$, whence the product is $2^4 = 16$, and $b - d = 5$.

Now, we prove this is a lower bound. Let $i = \sqrt{-1}$. The key observation is that

$$\prod_{j=1}^{4} \left(x_j^2 + 1\right) = \prod_{j=1}^{4} (x_j - i)(x_j + i) = P(i)P(-i).$$

Consequently, we have

$$\begin{aligned}\left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)\left(x_1^2 + 1\right) &= (b - d - 1)^2 + (a - c)^2 \\ &\geq (5 - 1)^2 + 0^2 = 16.\end{aligned}$$

This proves the lower bound.

§2 USAMO 2014/2, proposed by Titu Andreescu

Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf\left(2f(y) - x\right) + y^2 f\left(2x - f(y)\right) = \frac{f(x)^2}{x} + f\left(yf(y)\right)$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

The answer is $f(x) \equiv 0$ and $f(x) \equiv x^2$. Check that these work.

Now let's prove these are the only solutions. Put $y = 0$ to obtain

$$xf\left(2f(0) - x\right) = \frac{f(x)^2}{x} + f(0).$$

Now we claim $\boxed{f(0) = 0}$. If not, select a prime $p \nmid f(0)$ and put $x = p \neq 0$. In the above, we find that $p \mid f(p)^2$, so $p \mid f(p)$ and hence $p \mid \frac{f(p)^2}{p}$. From here we derive $p \mid f(0)$, contradiction. Hence

$$f(0) = 0.$$

The above then implies that

$$x^2 f(-x) = f(x)^2$$

holds for all nonzero x, but also for $x = 0$. Let us now check that f is an even function. In the above, we may also derive $f(-x)^2 = x^2 f(x)$. If $f(x) \neq f(-x)$ (and hence $x \neq 0$), then subtracting the above and factoring implies that $f(x) + f(-x) = -x^2$; we can then obtain by substituting the relation

$$\left[f(x) + \frac{1}{2}x^2\right]^2 = -\frac{3}{4}x^4 < 0$$

which is impossible. This means $f(x)^2 = x^2 f(x)$, thus

$$f(x) \in \{0, x^2\} \qquad \forall x.$$

Now suppose there exists a nonzero integer t with $f(t) = 0$. We will prove that $f(x) \equiv 0$. Put $y = t$ in the given to obtain that

$$t^2 f(2x) = 0$$

for any integer $x \neq 0$, and hence conclude that $f(2\mathbb{Z}) \equiv 0$. Then selecting $x = 2k \neq 0$ in the given implies that

$$y^2 f(4k - f(y)) = f(yf(y)).$$

Assume for contradiction that $f(m) = m^2$ now for some odd $m \neq 0$. Evidently

$$m^2 f(4k - m^2) = f(m^3).$$

If $f(m^3) \neq 0$ this forces $f(4k - m^2) \neq 0$, and hence $m^2(4k - m^2)^2 = m^6$ for arbitrary $k \neq 0$, which is clearly absurd. That means

$$f(4k - m^2) = f(m^2 - 4k) = f(m^3) = 0$$

for each $k \neq 0$. Since m is odd, $m^2 \equiv 1 \pmod 4$, and so $f(n) = 0$ for all n other than $\pm m^2$ (since we cannot select $k = 0$).

Now $f(m) = m^2$ means that $m = \pm 1$. Hence either $f(x) \equiv 0$ or

$$f(x) = \begin{cases} 1 & x = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

To show that the latter fails, we simply take $x = 5$ and $y = 1$ in the given.

Hence, the only solutions are $f(x) \equiv 0$ and $f(x) \equiv x^2$.

§3 USAMO 2014/3, proposed by Razvan Gelca

Prove that there exists an infinite set of points

$$\dots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \dots$$

in the plane with the following property: For any three distinct integers a, b, and c, points P_a, P_b, and P_c are collinear if and only if $a+b+c=2014$.

The construction

$$P_n = \left(n - \frac{2014}{3}, \left(n - \frac{2014}{3}\right)^3\right)$$

works fine, and follows from the following claim:

Claim — If x, y, z are distinct real numbers then the points (x, x^3), (y, y^3), (z, z^3) are collinear if and only if $x+y+z=0$.

Proof. Note that by the "shoelace formula", the collinearity is equivalent to

$$0 = \det \begin{bmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{bmatrix}$$

But the determinant equals

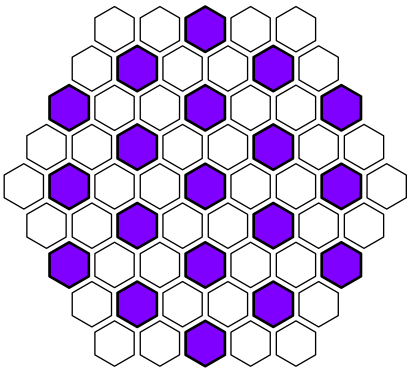
$$\sum_{\text{cyc}} x(y^3 - z^3) = (x-y)(y-z)(z-x)(x+y+z). \qquad \square$$

§4 USAMO 2014/4, proposed by Palmer Mebane

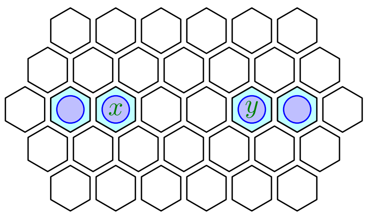
Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In her move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

The answer is $k = 6$.

Proof that A cannot win if $k = 6$. We give a strategy for B to prevent A's victory. Shade in every third cell, as shown in the figure below. Then A can never cover two shaded cells simultaneously on her turn. Now suppose B always removes a counter on a shaded cell (and otherwise does whatever he wants). Then he can prevent A from ever getting six consecutive counters, because any six consecutive cells contain two shaded cells.

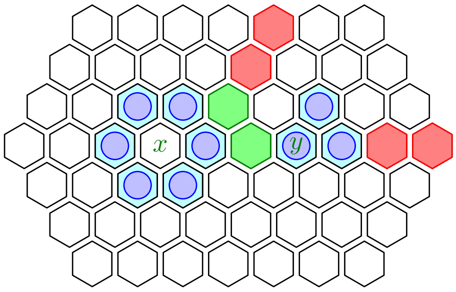


Example of a strategy for A when $k = 5$. We describe a winning strategy for A explicitly. Note that after B's first turn there is one counter, so then A may create an equilateral triangle, and hence after B's second turn there are two consecutive counters. Then, on her third turn, A places a pair of counters two spaces away on the same line. Label the two inner cells x and y as shown below.



Now it is B's turn to move; in order to avoid losing immediately, he must remove either x or y. Then on any subsequent turn, A can replace x or y (whichever was removed) and add one more adjacent counter. This continues until either x or y has all its neighbors filled (we ask A to do so in such a way that she avoids filling in the two central cells between x and y as long as possible).

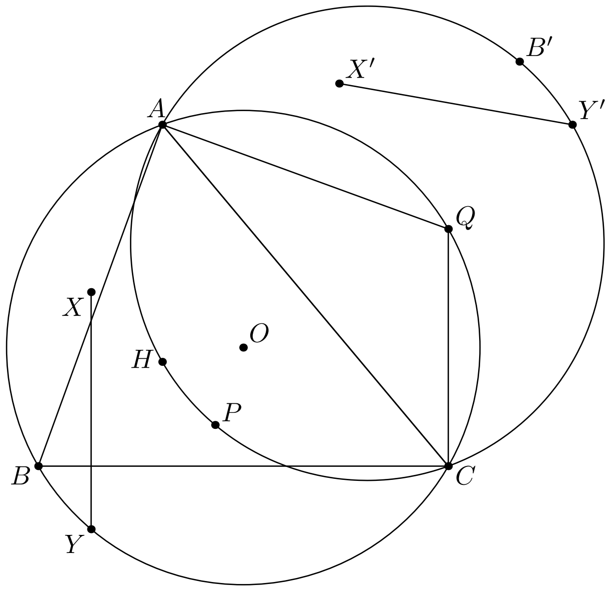
So, let's say without loss of generality (by symmetry) that x is completely surrounded by tokens. Again, B must choose to remove x (or A wins on her next turn). After x is removed by B, consider the following figure.



We let A play in the two marked green cells. Then, regardless of what move B plays, one of the two choices of moves marked in red lets A win. Thus, we have described a winning strategy when $k = 5$ for A.

§5 USAMO 2014/5, proposed by Titu Andreescu and Cosmin Pohoata

Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of $\angle BAC$. Let X be the circumcenter of triangle APB and let Y be the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.



We eliminate the floating orthocenter by reflecting P across $\overline{AC}$ to Q. Then Q lies on (ABC) and moreover $\angle QAC = \frac{1}{2}\angle BAC$. This motivates us to reflect B, X, Y to B', X', Y' and complex bash with respect to $\triangle AQC$. Obviously

$$y' = a + q + c.$$

Now we need to compute x'. You can get this using the formula

$$x' = a + \frac{(b'-a)(q-a)\left(\overline{q-a} - \overline{b'-a}\right)}{(b'-a)\overline{(q-a)} - \overline{(b'-a)}(q-a)}.$$

Using the angle condition we know $b = \frac{c^3}{q^2}$, and then that

$$b' = a + c - ac\overline{b} = a + c - \frac{aq^2}{c^2}.$$

Therefore

$$\begin{aligned} x' &= a + \frac{\left(c - \frac{aq^2}{c^2}\right)(q-a)\left(\frac{1}{q} - \frac{1}{a} - \frac{1}{c} + \frac{c^2}{aq^2}\right)}{\left(c - \frac{aq^2}{c^2}\right)\left(\frac{1}{q} - \frac{1}{a}\right) - \left(\frac{1}{c} - \frac{c^2}{aq^2}\right)(q-a)} \\ &= a + \frac{\frac{c^3-aq^2}{c^2}(q-a)\left(\frac{1}{q} - \frac{1}{a} - \frac{1}{c} + \frac{c^2}{aq^2}\right)}{-\frac{c^3-aq^2}{c^2}\frac{q-a}{qa} + \frac{c^3-aq^2}{aq^2c}(q-a)} \\ &= a + \frac{\frac{1}{q} - \frac{1}{a} - \frac{1}{c} + \frac{c^2}{aq^2}}{-\frac{1}{qa} + \frac{c}{aq^2}} \\ &= a + \frac{c^2 - q^2 + aq - \frac{aq^2}{c}}{c-q} \\ &= a + c + q + \frac{aq}{c} \end{aligned}$$

whence

$$\left|x' - y'\right| = \left|\frac{aq}{c}\right| = 1.$$

§6 USAMO 2014/6, proposed by Gabriel Dospinescu

Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > (cn)^{n/2}.$$

Let $N = n + 1$ and assume N is (very) large. We construct an $N \times N$ with cells (i, j) where $0 \le i, j \le n$ and in each cell place a prime p dividing $\gcd(a + i, b + j)$.

The central claim is at least 50% of the primes in this table exceed $0.001n^2$. We count the maximum number of squares they could occupy:

$$\sum_p \left\lceil \frac{N}{p} \right\rceil^2 \le \sum_p \left(\frac{N}{p} + 1 \right)^2 = N^2 \sum_p \frac{1}{p^2} + 2N \sum_p \frac{1}{p} + \sum_p 1.$$

Here the summation runs over primes $p \le 0.001n^2$.

Let $r = \pi(0.001n^2)$ denote the number of such primes. Now we consider the following three estimates. First,

$$\sum_p \frac{1}{p^2} < \frac{1}{2}$$

which follows by adding all the primes directly with some computation. Moreover,

$$\sum_p \frac{1}{p} < \sum_{k=1}^{r} \frac{1}{k} = O(\log r) < o(N)$$

using the harmonic series bound, and

$$\sum_p 1 < r \sim O\left(\frac{N^2}{\ln N} \right) < o(N^2)$$

via Prime Number Theorem. Hence the sum in question is certainly less than $\frac{1}{2}N^2$ for N large enough, establishing the central claim.

Hence some column $a + i$ has at least one half of its primes greater than $0.001n^2$. Because this is greater than n for large n, these primes must all be distinct, so $a + i$ exceeds their product, which is larger than

$$\left(0.001n^2\right)^{N/2} > c^n \cdot n^n$$

where c is some constant (better than the requested bound).

44th United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 28, 2015

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in a 1-point automatic deduction.

USAMO 1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

USAMO 2. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

USAMO 3. Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

44^{th} United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 29, 2015

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in a 1-point automatic deduction.

USAMO 4. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

USAMO 5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

USAMO 6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1,2,3\}$ and $\{2,1,3\}$ are equivalent, but $\{1,1,2,3\}$ and $\{1,2,3\}$ differ.)

44^{th} United States of America Mathematical Olympiad Solutions

Day I, II 12:30 PM – 5 PM EDT

April 28 - April 29, 2015

USAMO 1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Solution: Let $x + y = 3k$, with $k \in \mathbb{Z}$. Then $x^2 + x(3k - x) + (3k - x)^2 = (k+1)^3$, which reduces to

$$x^2 - (3k)x - (k^3 - 6k^2 + 3k + 1) = 0.$$

Its discriminant Δ is

$$9k^2 + 4(k^3 - 6k^2 + 3k + 1) = 4k^3 - 15k^2 + 12k + 4.$$

We notice the (double) root $k = 2$, so $\Delta = (4k+1)(k-2)^2$. It follows that $4k+1 = (2t+1)^2$ for some nonnegative integer t, hence $k = t^2 + t$ and

$$x = \frac{1}{2}(3(t^2 + t) \pm (2t+1)(t^2 + t - 2)).$$

We obtain $(x, y) = (t^3 + 3t^2 - 1, -t^3 + 3t + 1)$ and $(x, y) = (-t^3 + 3t + 1, t^3 + 3t^2 - 1)$, $t \in \{0, 1, 2, ...\}$.

OR

One can also try to simplify the original equation as much as possible. First with $k = \frac{x+y}{3} + 1$ we get

$$x^2 - 3xk + 3x = k^3 - 9k^2 + 18k - 9.$$

But then we recognize terms from the expansion of $(k-3)^3$ so we use $s = k - 3$ and obtain

$$x^2 - 3xs - 6x = s^3 - 9s - 9.$$

So again it becomes natural to use $x - 3 = u$. The equation becomes

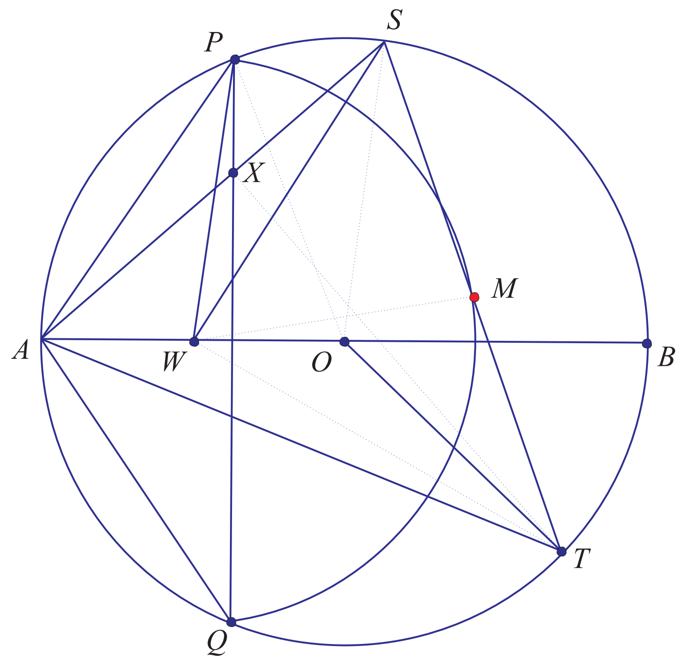
$$u^2 - 3su - s^3 = 0.$$

We view this as a quadratic in u, whose discriminant is $s^2(9 + 4s)$, and so $9 + 4s$ must be a perfect square, and because it is odd, it must be of the form $(2t+1)^2$. It follows that $s = t^2 + t - 2$, and so $k = t^2 + t + 1$. We obtain the same family of solutions.

USAMO 2. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Solution: Let O denote the center of ω, and let W denote the midpoint of segment $\overline{AO}$. Denote by Ω the circle centered at W with radius WP. We will show that $WM = WP$, which will imply that M always lies on Ω and so solve the problem.

We present two solutions. The first solution is more computational (in particular, with extensive applications of the formula for a median of a triangle); the second is more synthetic.



Set r to be the radius of circle ω. Applying the median formula in triangles APO, SWT, ASO, ATO gives

$$\begin{aligned} 4WP^2 &= 2AP^2 + 2OP^2 - AO^2 = 2AP^2 + r^2, \\ 4WM^2 &= 2WS^2 + 2WT^2 - ST^2, \\ 2WS^2 &= AS^2 + OS^2 - AO^2/2 = AS^2 + r^2/2, \\ 2WT^2 &= AT^2 + OT^2 - AO^2/2 = AT^2 + r^2/2. \end{aligned}$$

Adding the last three equations yields $4WM^2 = AS^2 + AT^2 - ST^2 + r^2$. It suffices to show that

$$4WP^2 = 4WM^2 \quad \text{or} \quad AS^2 + AT^2 - ST^2 = 2AP^2. \tag{1}$$

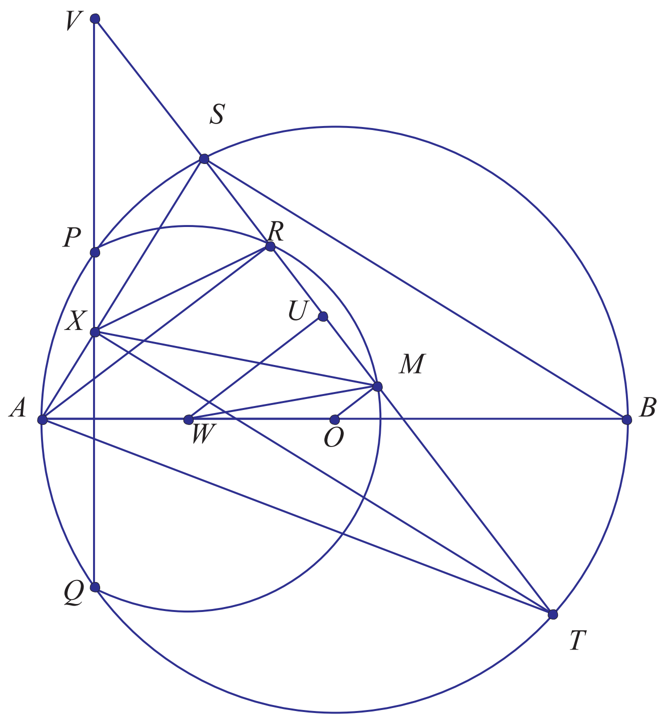
Because $\overline{XT} \perp \overline{AS}$,

$$AT^2 - ST^2 = (AX^2 + XT^2) - (SX^2 + XT^2)$$

$$\begin{aligned}
&= AX^2 - SX^2 \\
&= (AX + XS)(AX - XS) \\
&= AS(AX - XS).
\end{aligned}$$

It follows that $AS^2 + AT^2 - ST^2 = AS^2 + AS \cdot (AX - XS) = AS^2 + AS(2AX - AS) = 2AS{\cdot}AX$, and (1) reduces to $AP^2 = AS{\cdot}AX$, which is true because triangle APX is similar to triangle ASP (as $\angle PAX = \angle SAP$ and $\angle APX = \mathrm{arc}(AQ)/2 = \mathrm{arc}(AP)/2 = \angle ASP$).

OR



In the following solution, we use directed distances and directed angles in order to avoid issues with configuration (segments $\overline{ST}$ and $\overline{PQ}$ may intersect, or may not as depicted in the figure.)

Let R be the foot of the perpendicular from A to line ST. Note that $OM \perp ST$, and so $ARMO$ is a right trapezoid. Let U be the midpoint of segment $\overline{RM}$. Then $\overline{WU}$ is the midline of the trapezoid. In particular, $\overline{WU} \perp \overline{RM}$. Hence line WU is the perpendicular bisector of segment $\overline{RM}$. It is also clear that AW is the perpendicular bisector of segment $\overline{PQ}$. Therefore, W is the intersection of the perpendicular bisectors of segments $\overline{RM}$ and $\overline{PQ}$. It suffices to show that quadrilateral $PQMR$ is cyclic, since then W must be its circumcenter, and so $WP = WM$.

(To be precise, this argument fails when ST and PQ are parallel, because then $R = M$ and the perpendicular bisector of $\overline{RM}$ is not defined. However, it is easy to see that this can happen for only one position of X. Because the argument works for all other X, continuity then implies that M lies on Ω for this exceptional case as well.)

Let lines PQ and ST meet in V. By the converse of the power-of-a-point theorem, it suffices to show that $VP \cdot VQ = VR \cdot VM$. On the other hand, because $PQTS$ is cyclic, by the power-of-a-point theorem, we have $VP \cdot VQ = VS \cdot VT$. Therefore, we only need to show that

$$VS \cdot VT = VR \cdot VM. \tag{2}$$

Note that M is the midpoint of segment $\overline{ST}$. Then (2) is equivalent to

$$2VS \cdot VT = VR \cdot (2VM) = VR \cdot (VS + VT)$$

or

$$VS \cdot VT - VS \cdot VR = VT \cdot VR - VT \cdot VS$$

or equivalently

$$VS \cdot RT = VT \cdot SR \quad \text{or} \quad \frac{VS}{SR} = \frac{VT}{RT}. \tag{3}$$

We claim that XS bisects $\angle VXR$. Indeed, because AB is the symmetry line of the kite $APBQ$, $AB \perp PQ$, and so $\angle VXS = \angle QXA = 90° - \angle XAO = 90° - \angle SAO$. Because O is the circumcenter of triangle AST,

$$\angle VXS = 90° - \angle SAO = \angle ATS.$$

On the other hand, because $\angle AXT$ and $\angle ART$ are both right angles, quadrilateral $AXRT$ is cyclic, implying that $\angle SXR = \angle ATR = \angle ATS$. Our claim follows from the last two equations.

Combining our claim and the fact that $XS \perp XT$, we know that XS and XT are the interior and exterior bisectors of $\angle VXR$, from which (3) follows, by the angle-bisector theorem. We saw that (3) was equivalent to (2) and that this was enough to show that $PQMR$ is cyclic, which completes the solution, so we are done.

USAMO 3. Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

Solution: The answer is $3^n + 1$.

Specifically, the colorings we want are of the following forms: either there are no blue sets; or for each element $x \in S$ we define one of three types of restriction — either x must be in T, x can't be in T, or x is unrestricted — and the blue sets T are exactly the ones that satisfy every restriction. It's easy to check such a coloring meets the condition, using the formula

$$f(T) = \prod_{x \in T} a_x \prod_{x \notin T} b_x,$$

where $a_x = 2$ if x is unrestricted and 1 otherwise, and $b_x = 0$ if x is required to be present and 1 otherwise.

We want to show that if there's at least one blue set, then the class of blue sets is of this form.

If some element of S is in every blue set, take it out and induct. If some element of S is not in any blue set, take it out and induct. Otherwise, every element x has some blue set containing it and some blue set not containing it. In this case we'll show that all sets are blue (i.e. every element is unrestricted).

First show $\emptyset$ is blue. To show this, let T be a minimal blue set. If nonempty, take $x \in T$; by assumption there's blue T' not containing x. Then the condition is violated with T and T', since $f(T \cap T') = 0$. Next, show any singleton is blue. Otherwise, let U be a minimal blue set containing x, and let $T = \{x\}$ and $T' = U \setminus \{x\}$. We get $1 \cdot m = 1 \cdot (1 + m)$ (where $m = f(T')$), a contradiction. Finally, any set is blue. Otherwise, let U be a minimal non-blue set and x, y two different elements. Taking $T = U \setminus \{x\}, T' = U \setminus \{y\}$ gives a contradiction.

USAMO 4. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he is finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i, k), (i, l), (j, k), (j, l) for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Solution: We think of the pilings as assigning a positive integer to each square on the grid. Now, we restrict ourselves to the types of moves in which we take a lower left and upper right stone and move them to the upper left and lower right of our chosen rectangle. Call this a Type 1 stone move. We claim that we can perform a sequence of Type 1 stone moves on any piling to obtain an equivalent piling for which we cannot perform any Type 1 move, i.e. in which no square that has stones is above and to the right of any other square that has stones. We call such a piling a "down-right" piling.

To prove that any piling is equivalent to a down-right piling, first consider the squares in the leftmost column and topmost row of the grid. Let a be the entry (number of stones) in the upper left corner, and let b and c be the sum of the remaining entries in the leftmost column and topmost row respectively. If $b < c$, we can perform a sequence of Type 1 stone moves to remove all the stones from the leftmost column except for the top entry, and if $c < b$ we can similarly clear all squares in the top row except for the top left square. In the former case, we can now ignore the leftmost column and repeat the process on the second-to-leftmost column and the top row; similarly, in the latter case, we can ignore the top row and proceed as before. Since the corner square a cannot be part of any Type 1 move at each step in the process, it follows that we end up with a down-right piling.

We next show that down-right pilings in any size grid (not necessarily $n \times n$) are uniquely determined by their row-sums and column-sums, given that the row sums and column sums are nonnegative integers which sum to m both along the rows and the columns. Let the topmost row sum be R_1 and the leftmost column sum be C_1. Then the upper left square must contain $\min(R_1, C_1)$ stones, since otherwise there would be stones both in the first row and first column that are not in the upper left square. Whichever is smaller indicates that either the row or the column respectively is empty save for the upper left square; then we can remove this row or column and are reduced to a smaller grid in which we know all the row and column sums. Since one-row and one-column pilings are clearly uniquely determined by their column and row sums, it follows by induction that down-right pilings are determined uniquely by their row-sums and column sums.

Finally, notice that row sums and column sums are both invariant under stone moves. Therefore every piling is equivalent to a *unique* down-right piling. It therefore suffices to count the number of down-right pilings, which is also equivalent to counting the number of possibilities for the row-sums and column-sums. As stated above, the row sums and column sums can be the sums of any two n-tuples of nonnegative integers that each sum to m. The number of such tuples is $\binom{n+m-1}{m}$, and so the total number of non-equivalent pilings is the number of pairs of these tuples, i.e. $\left(\binom{n+m-1}{m}\right)^2$.

USAMO 5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Solution: We approach indirectly by assuming that $p = ac+bd$ is a prime. By symmetry, we may assume that $\max\{a, b, c, d\} = a$, then because $a^4 + b^4 = c^4 + d^4$, we infer that $\min\{a, b, c, d\} = b$. Note that $ac \equiv -bd \pmod{p}$, implying that $a^4c^4 \equiv b^4d^4 \pmod{p}$. Consequently, we have

$$b^4d^4 + b^4c^4 \equiv a^4c^4 + b^4c^4 = c^8 + c^4d^4 \pmod{p},$$

from which it follows that $(c^4 + d^4)(b^4 - c^4) \equiv 0 \pmod{p}$. Thus, p divides at least one of $b - c, b + c, b^2 + c^2, c^4 + d^4$. Because $p = ac + bd > c^2 + b^2$, and $-(b^2 + c^2) < b - c < 0$ (because b and c are distinct), p must divide $c^4 + d^4 = e^5$. Thus $p^5 = (ac + bd)^5$ divides $c^4 + d^4$, which is clearly impossible because it is evident that $(ac + bd)^5 > c^4 + d^4$.

OR

A stronger result is possible:

Claim. Suppose a, b, and e are positive integers such that $a^4 + b^4 = e^5$. Then a and b have a common prime factor.

Proof. Suppose on the contrary that $\gcd(a, b) = 1$. If e is even, then this forces a and b to both be odd, so $a^4 + b^4 \equiv 2 \pmod{8}$ and $e^5 \equiv 0 \pmod{8}$, a contradiction. Thus e is odd. Note for use below that 5 cannot divide both a and b, so we may assume without loss that 5 does not divide a (swapping the roles of a and b if necessary).

Factoring over the Gaussian integers we find $a^4+b^4=(a^2+ib^2)(a^2-ib^2)$ and $\gcd(a^2+ib^2,a^2-ib^2)=\gcd(a^2+ib^2,2a^2)$. But $\gcd(a,b)=1$ implies no prime factor of a can divide a^2+ib^2 and e odd implies no prime factor of 2 divides a^2+ib^2. Thus these factors are relatively prime, and hence both are a unit multiplied by a fifth power. Since every unit in the Gaussian integers is a fifth power, that means both factors are fifth powers, or

$$a^2+ib^2=(r+is)^5=r^5+5ir^4s-10r^3s^2-10ir^2s^3+5rs^4+is^5.$$

Thus

$$\begin{aligned} a^2 &= r(r^4-10r^2s^2+5s^4), \quad \text{and} \\ b^2 &= s(s^4-10r^2s^2+5r^4). \end{aligned}$$

Note that since $\gcd(a,b)=1$, $\gcd(r,s)=1$. Also since 5 does not divide a, it also does not divide r. Since

$$\gcd(r,r^4-10r^2s^2+5s^4)=\gcd(r,5s^4)=\gcd(r,5)=1,$$

r must be a perfect square and we have found an integer solution $(x,y,z)=(r,a/r,s)$ to

$$y^2=x^4-10x^2z^2+5z^4$$

with $\gcd(x,z)=1$. The following Lemma will then complete the proof of the claim.

Lemma. There are no nontrivial ($z\neq 0$) integer solutions to

$$y^2=x^4-10x^2z^2+5z^4.$$

Proof. Suppose (x,y,z) is a solution in the positive integers with minimal z. Note that this implies that x,y,z are pairwise relatively prime. (The only case that takes a little work is that if a prime p divides x and y, then p^2 divides $5z^4$, hence p also divides z. But then p^4 divides x^2 so p^2 divides x and $(x/p^2,y/p,z/p)$ is a smaller solution.) Rewrite this as

$$20z^4=(x^4-5z^2)^2-y^2=(x^2-5z^2+y)(x^2-5z^2-y).$$

The two factors on the right have the same parity and their product is even. Hence both are even. Any common factor p of $\frac{x^2-5z^2+y}{2}$ and $\frac{x^2-5z^2-y}{2}$ would have $p^2|5z^4$, hence $p|z$, and $p|\frac{x^2-5z^2+y}{2}-\frac{x^2-5z^2-y}{2}=y$, a contradiction. Thus these factors of $5z^4$ are relatively prime. Hence they must be $\pm v^4$ and $\pm 5w^4$ for some relatively prime v and w with $vw=z$. Then

$$x^2-5v^2w^2=x^2-5z^2=\frac{x^2-5z^2+y}{2}+\frac{x^2-5z^2-y}{2}=\pm v^4\pm 5w^4$$

or

$$x^2=\pm v^4+5v^2w^2\pm 5w^4.$$

If v and w both odd, then the right hand side is either $1+5+5\equiv 3 \pmod 8$ or $-1+5-5\equiv 7 \pmod 8$, neither of which is possible for a square like the left hand side. Hence one of v

and w is even, and in either case we get $x^2 \equiv \pm 1 \pmod 4$. Thus we must have the plus sign and

$$x^2 = v^4 + 5v^2w^2 + 5w^4.$$

This is not the equation we started with, so we repeat the argument above (with a few changes). Rewrite this new equation as

$$5w^4 = (2v^2 + 5w^2)^2 - 4x^2 = (2v^2 + 5w^2 + 2x)(2v^2 + 5w^2 - 2x).$$

There are two very similar cases depending on whether w is odd or even. These cases can be forced together, but we prefer to be more clear and keep them separate.

If w is odd, then the two factors on the right are both odd and any common (odd) prime factor p would have $p^2|5w^4$, hence $p|w$, and $p|(2v^2 + 5w^2 + 2x) - (2v^2 + 5w^2 - 2x) = 4x$, hence $p|x$. But then p also divides v and we get a contradiction. Thus these factors of $5w^4$ are relatively prime and so must be $\pm t^4$ and $\pm 5u^4$ for some relatively prime t and u with $tu = w$. Then

$$4v^2 + 10t^2u^2 = 4v^2 + 10w^2 = (2v^2 + 5w^2 + 2x) + (2v^2 + 5w^2 - 2x) = \pm(t^4 + 5u^4).$$

The left hand side is positive, so we must have the plus sign, and hence

$$(2v)^2 = t^4 - 10t^2u^2 + 5u^4.$$

Thus $(t, 2v, u)$ is a solution to the original equation. Since $u|w$ and $w|z$, we either have $u < z$ (contradicting the minimality of z) or $u = z$ and hence $t = v = 1$ (giving nonsense $4 = 1 - 10u^2 + 5u^4 \equiv 1 \pmod 5$). Thus this case cannot occur.

If w is even, then the two factors are even, congruent mod 4, and their product is divisible by 16. Hence both are multiples of 4. Any common prime factor p of $\frac{2v^2+5w^2+2x}{4}$ and $\frac{2v^2+5w^2-2x}{4}$ would have $p^2|5(w/2)^4$, hence $p|w$, and $p|\frac{2v^2+5w^2+2x}{4} - \frac{2v^2+5w^2-2x}{4} = x$. But this would mean $p|v$, a contradiction. Thus $\frac{2v^2+5w^2+2x}{4}$ and $\frac{2v^2+5w^2-2x}{4}$ must be $\pm t^4$ and $\pm 5u^4$ for some relatively prime t and u with $2tu = w$. Then

$$v^2 + 10t^2u^2 = v^2 + \frac{5}{2}w^2 = \frac{2v^2 + 5w^2 + 2x}{4} + \frac{2v^2 + 5w^2 - 2x}{4} = \pm(t^4 + 5u^4).$$

Again, the left hand side is positive, so we must have the plus sign, and hence

$$v^2 = t^4 - 10t^2u^2 + 5u^4.$$

Thus (t, v, u) is a solution to the original equation and since $2u|w$ and $w|z$, we have $u < z$. This contradicts the minimality of z and completes the proof of the lemma.

Remark. One can use essentially the same argument to show that any nontrivial integer solution to $x^2+y^4 = z^5$ has $\gcd(x, y) > 1$. In this case one cannot assume 5 does not divide r so there is a second case where $r = 5q^2$. Then $(x, y, z) = (s, a/(5q), q^2)$ is a solution to

$$y^2 = x^4 - 50x^2z^2 + 125z^4.$$

This Diophantine equation also has no nontrivial integer solutions and the proof is nearly identical to the proof of the Lemma above. This stronger result was (apparently) first proven by Nils Bruin (1999). This result is at least tangentially related to Beal's conjecture.

The more general solution is due to Richard Stong.

USAMO 6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1,2,3\}$ and $\{2,1,3\}$ are equivalent, but $\{1,1,2,3\}$ and $\{1,2,3\}$ differ.)

Solution: Set $b_n = |A_n|$, $a_n = n\lambda - A_n \geq 0$. There are $b_i - b_{i-1}$ elements equal to i. Therefore the sum of elements in A_n is

$$\sum_{i=1}^{n} i(b_i - b_{i-1}) = nb_n - \sum_{i=1}^{n} b_i.$$

Now $b_n = n\lambda - a_n$, so the sum of elements in A_n may be written as

$$\Sigma_n = \lambda \frac{n(n+1)}{2} - na_n + \sum_{i=1}^{n-1} a_i.$$

Assume, by way of contradiction, that for all $n \geq n_0$, the sum of elements in A_n is greater than $\lambda\frac{n(n+1)}{2}$. Then

$$na_n < a_{n-1} + a_{n-2} + \ldots + a_1,$$

so

$$a_n < \frac{a_{n-1} + a_{n-2} + \ldots + a_1}{n} \leq \frac{M_n \cdot (n-1)}{n} \tag{4}$$

where $M_n = \max\{a_1, a_2, \ldots, a_{n-1}\}$. We deduce that $a_n < \frac{(n-1)M_n}{n}$, so $M_{n+1} = M_n = M$, where $M = M_{n_0}$.

Let $\{x\}$ denote the fractional part of x; i.e., $\{x\} = x - \lfloor x \rfloor$. We note that $\{a_{k+1} - a_k\} = \lambda$, so $\{(M - a_k) - (M - a_{k+1})\} = \lambda$. We claim that

$$(M - a_k) + (M - a_{k+1}) \geq \min(\lambda, 1 - \lambda). \tag{5}$$

To see this, we first note that $M - a_k \geq 0$ and $M - a_{k+1} \geq 0$. If either $M - a_k \geq 1$ or $M - a_{k+1} \geq 1$, then we are done. Assume that $0 < M - a_k, M - a_{k+1} < 1$. Then $-1 < (M - a_k) - (M - a_{k+1}) < 1$, so either $(M - a_k) - (M - a_{k+1}) = \lambda - 1$ or $(M - a_k) - (M - a_{k+1}) = \lambda$. In the former case, we get $M - a_{k+1} > 1 - \lambda$, and in the latter case we get $M - a_k > \lambda$. In either case, (5) follows.

We deduce from (5) that $a_k + a_{k+1} \leq 2M - \mu$, where $\mu = \min(\lambda, 1 - \lambda)$. From this and (4), we see that

$$a_n \leq M - \frac{\mu}{2} \tag{6}$$

for $n \geq n_1 = n_0 + 1$.

Let $\delta = \mu/3$. We will use induction to prove that for any integer $k \geq 1$ and $n \geq n_k$,

$$a_n \leq M - k\delta. \tag{7}$$

We have already proved the base case. Assume that (7) is true for a given fixed k. Using (6), we see that $a_k + a_{k+1} \leq 2M - 2k\delta - \mu = 2M - (2k+3)\delta$. (Note that $\delta \leq 1/6$, so $\min(\delta, 1-\delta) = \delta$.) Now if we take $n > (2k+3)n_k$, we deduce that

$$a_n \leq \frac{n_k M + (n - n_k)(M - (k + \frac{3}{2})\delta)}{n} \leq M - (k+1)\delta.$$

Statement (7) then follows by induction. However, it then follows that $a_n < 0$ when $k > M/\delta$, and this is a contradiction.

USAMO 2015 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2015 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Solve in integers the equation
$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

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   As X varies on segment $\overline{PQ}$, show that M moves along a circle.

3. Let $S = \{1, 2, \dots, n\}$, where $n \ge 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

   Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,
$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

4. Steve is piling $m \ge 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k), (i,l), (j,k), (j,l)$ for some $1 \le i,j,k,l \le n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

   Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \le n\}$. Assume that for every $n \in \mathbb{N}$, the multiset A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

§1 USAMO 2015/1, proposed by Titu Andreescu

Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

We do the trick of setting $a = x + y$ and $b = x - y$. This rewrites the equation as

$$\frac{1}{4}\left((a+b)^2 + (a+b)(a-b) + (a-b)^2\right) = \left(\frac{a}{3} + 1\right)^3$$

where $a, b \in \mathbb{Z}$ have the same parity. This becomes

$$3a^2 + b^2 = 4\left(\frac{a}{3} + 1\right)^3$$

which is enough to imply $3 \mid a$, so let $a = 3c$. Miraculously, this becomes

$$b^2 = (c-2)^2(4c+1).$$

So a solution must have $4c + 1 = m^2$, with m odd. This gives

$$x = \frac{1}{8}\left(3(m^2-1) \pm (m^3 - 9m)\right) \quad \text{and} \quad y = \frac{1}{8}\left(3(m^2-1) \mp (m^3 - 9m)\right).$$

For mod 8 reasons, this always generates a valid integer solution, so this is the complete curve of solutions. Actually, putting $m = 2n + 1$ gives the much nicer curve

$$\boxed{x = n^3 + 3n^2 - 1 \quad \text{and} \quad y = -n^3 + 3n + 1}$$

and permutations.

For $n = 0, 1, 2, 3$ this gives the first few solutions are $(-1, 1)$, $(3, 3)$, $(19, -1)$, $(53, -17)$, (and permutations).

§2 USAMO 2015/2, proposed by Zuming Feng

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

We present three solutions, one by complex numbers, two more synthetic. (A fourth solution using median formulas is also possible.) Most solutions will prove that the center of the fixed circle is the midpoint of $\overline{AO}$ (with O the center of ω); this can be recovered empirically by letting

- X approach P (giving the midpoint of $\overline{BP}$)
- X approach Q (giving the point Q), and
- X at the midpoint of $\overline{PQ}$ (giving the midpoint of $\overline{BQ}$)

which determines the circle; this circle then passes through P by symmetry and we can find the center by taking the intersection of two perpendicular bisectors (which two?).

Complex solution (Evan Chen) Toss on the complex unit circle with $a = -1$, $b = 1$, $z = -\frac{1}{2}$. Let s and t be on the unit circle. We claim Z is the center.

It follows from standard formulas that

$$x = \frac{1}{2}\left(s + t - 1 + s/t\right)$$

thus

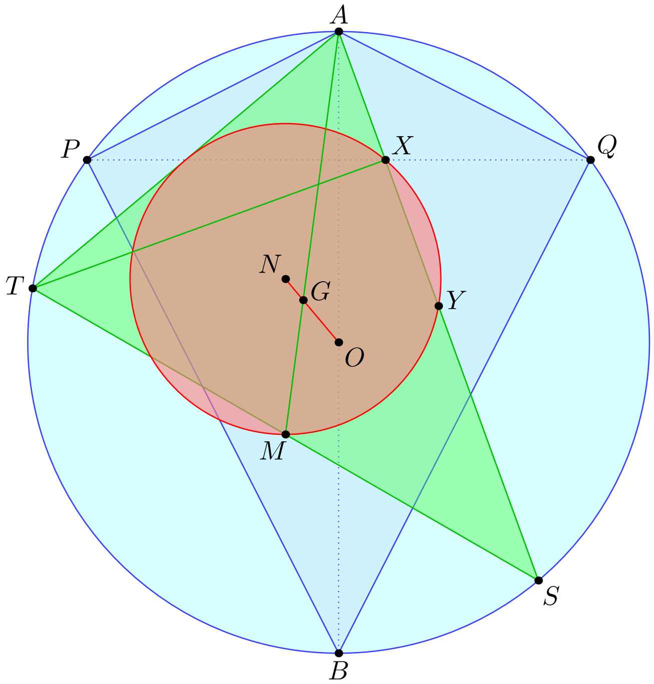
$$4 \operatorname{Re} x + 2 = s + t + \frac{1}{s} + \frac{1}{t} + \frac{s}{t} + \frac{t}{s}$$

which depends only on P and Q, and not on X. Thus

$$4\left|z - \frac{s+t}{2}\right|^2 = |s+t+1|^2 = 3 + (4 \operatorname{Re} x + 2)$$

does not depend on X, done.

Homothety solution (Alex Whatley) Let G, N, O denote the centroid, nine-point center, and circumcenter of triangle AST, respectively. Let Y denote the midpoint of $\overline{AS}$. Then the three points X, Y, M lie on the nine-point circle of triangle AST, which is centered at N and has radius $\frac{1}{2}AO$.



Let R denote the radius of ω. Note that the nine-point circle of $\triangle AST$ has radius equal to $\frac{1}{2}R$, and hence is independent of S and T. Then the power of A with respect to the nine-point circle equals

$$AN^2 - \left(\frac{1}{2}R\right)^2 = AX \cdot AY = \frac{1}{2}AX \cdot AS = \frac{1}{2}AQ^2$$

and hence

$$AN^2 = \left(\frac{1}{2}R\right)^2 + \frac{1}{2}AQ^2$$

which does not depend on the choice of X. So N moves along a circle centered at A.

Since the points O, G, N are collinear on the Euler line of $\triangle AST$ with

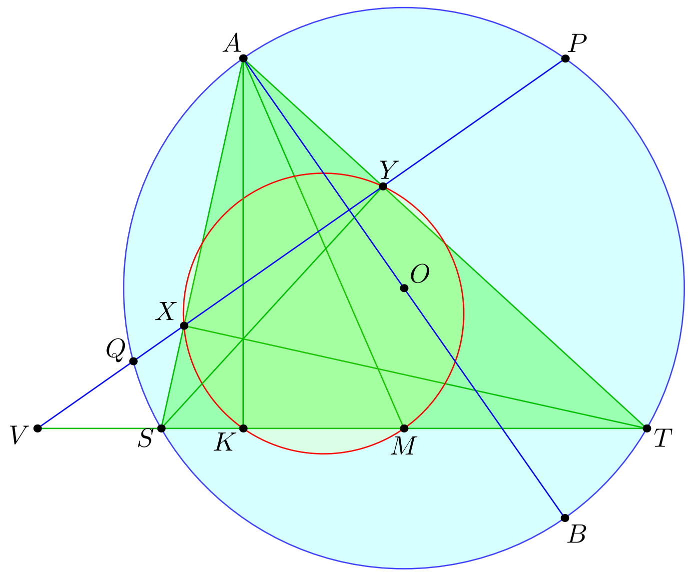
$$GO = \frac{2}{3}NO$$

it follows by homothety that G moves along a circle as well, whose center is situated one-third of the way from A to O. Finally, since A, G, M are collinear with

$$AM = \frac{3}{2}AG$$

it follows that M moves along a circle centered at the midpoint of $\overline{AO}$.

Power of a point solution (Zuming Feng, official solution) We complete the picture by letting $\triangle KYX$ be the orthic triangle of $\triangle AST$; in that case line XY meets the ω again at P and Q.



The main claim is:

Claim — Quadrilateral $PQKM$ is cyclic.

Proof. To see this, we use power of a point: let $V = \overline{QXYP} \cap \overline{SKMT}$. One approach is that since $(VK;ST) = -1$ we have $VQ \cdot VP = VS \cdot VT = VK \cdot VM$. A longer approach is more elementary:

$$VQ \cdot VP = VS \cdot VT = VX \cdot VY = VK \cdot VM$$

using the nine-point circle, and the circle with diameter $\overline{ST}$. □

But the circumcenter of $PQKM$, is the midpoint of $\overline{AO}$, since it lies on the perpendicular bisectors of $\overline{KM}$ and $\overline{PQ}$. So it is fixed, the end.

§3 USAMO 2015/3

Let $S = \{1, 2, \dots, n\}$, where $n \ge 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

For an n-coloring $\mathcal{C}$ (by which we mean a coloring of the subsets of $\{1, \dots, n\}$), define the **support** of $\mathcal{C}$ as

$$\operatorname{supp}(\mathcal{C}) = \{T \mid f(T) \neq 0\}.$$

Call a coloring **nontrivial** if $\operatorname{supp}(\mathcal{C}) \neq \varnothing$ (equivalently, the coloring is not all red). In that case, notice that

- the support is *closed under unions and intersections*: since if $f(T_1)f(T_2) \neq 0$ then necessarily both $f(T_1 \cap T_2)$ and $f(T_1 \cup T_2)$ are nonzero; and
- the support is obviously *upwards closed.*

Thus, the support must take the form

$$\operatorname{supp}(\mathcal{C}) = [X, S] \overset{\text{def}}{=} \{T \mid X \subseteq T \subseteq S\}$$

for some set X (for example by letting X be the minimal (by size) element of the support).

Say $\mathcal{C}$ has **full support** if $X = \varnothing$ (equivalently, $\varnothing$ is blue).

Lemma

For a given n and $B \subseteq \{1, \dots, n\}$, there is exactly one n-coloring with full support in which the singletons colored blue are exactly those in B. Therefore there are exactly 2^n n-colorings with full support.

Proof. To see there is at least one coloring, color only the subsets of B blue. In that case

$$f(T) = 2^{|T \cap B|}$$

which satisfies the condition by Inclusion-Exclusion. To see this extension is unique, note that $f(\{b\})$ is determined for each b and we can then determine $f(T)$ inductively on $|T|$; hence there is *at most* one way to complete a coloring of the singletons, which completes the proof. □

For a general nontrivial n-coloring $\mathcal{C}$, note that if $\operatorname{supp}(\mathcal{C}) = [X, S]$, then we can think of it as an $(n - |X|)$-coloring with full support. For $|X| = k$, there are $\binom{n}{k}$ possible choices of $X \subseteq S$. Adding back in the trivial coloring, the final answer is

$$1 + \sum_{k=0}^{n} \binom{n}{k} 2^k = \boxed{1 + 3^n}.$$

Remark. To be more explicit, the possible nontrivial colorings are exactly described by specifying two sets X and Y with $X \subseteq Y$, and coloring blue only the sets T with $X \subseteq T \subseteq Y$.

In particular, one deduces that in a working coloring, $f(T)$ is always either zero or a power of two. If one manages to notice this while working on the problem, it is quite helpful for motivating the solution, as it leads one to suspect that the working colorings have good structure.

§4 USAMO 2015/4

Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l),(j,k),(j,l)$ for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

The answer is $\binom{m+n-1}{n-1}^2$. The main observation is that the multi-set of column counts, and the multi-set of row counts, remains invariant. We call the pair (X,Y) of multisets the *signature* of the configuration.

We are far from done. This problem is a good test of mathematical maturity since the following steps are then necessary:

1. Check that signatures are invariant around moves (trivial)
2. Check conversely that two configurations are equivalent if they have the same signatures (the hard part of the problem), and
3. Show that each signature is realized by at least one configuration (not immediate, but pretty easy).

Most procedures to the second step are algorithmic in nature, but Ankan Bhattacharya gives the following far cleaner approach. Rather than having a grid of stones, we simply consider the multiset of ordered pairs (x,y). Then, the signatures correspond to the multisets of x and y coordinates, while *a stone move corresponds to switching two y-coordinates in different pairs*, say.

Then, the second part is completed just because transpositions generate any permutation. To be explicit, given two sets of stones, we can permute the labels so that the first set is (x_1,y_1), …, (x_m,y_m) and the second set of stones is (x_1,y'_1), …, (x_m,y'_m). Then we just induce the correct permutation on (y_i) to get (y'_i).

The third part is obvious since given two multisets $X = \{x_1,\dots,x_m\}$ and $Y = \{y_1,\dots,y_m\}$ we just put stones at (x_i,y_i) for $i = 1,\dots,m$.

In that sense, the entire grid is a huge red herring!

§5 USAMO 2015/5, proposed by Mohsen Jamali

Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Assume to the contrary that $p = ac + bd$, so that

$$\begin{aligned} ac &\equiv -bd \pmod p \\ \implies a^4c^4 &\equiv b^4d^4 \pmod p \\ \implies a^4(e^5 - d^4) &\equiv (e^5 - a^4)d^4 \pmod p \\ \implies a^4e^5 &\equiv d^4e^5 \pmod p \\ \implies e^5(a^4 - d^4) &\equiv 0 \pmod p \end{aligned}$$

and hence

$$p \mid e^5(a-d)(a+d)(a^2+d^2).$$

Claim — We should have $p > e$.

Proof. We have $e^5 = a^4 + b^4 \le a^5 + b^5 < (ac + bd)^5 = p^5$. □

Thus the above equation implies $p \le \max(a - d, a + d, a^2 + d^2) = a^2 + d^2$. Similarly, $p \le b^2 + c^2$. So

$$ac + bd = p \le \min\left\{a^2 + d^2, b^2 + c^2\right\}$$

or by subtraction

$$0 \le \min\left\{a(a-c) + d(d-b), b(b-d) + c(c-a)\right\}.$$

But since $a^4 + b^4 = c^4 + d^4$ the numbers $a - c$ and $d - b$ should have the same sign, and so this is an obvious contradiction.

§6 USAMO 2015/6

Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \le n\}$. Assume that for every $n \in \mathbb{N}$, the multiset A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

For brevity, $\#S$ denotes $|S|$. Let $x_n = n\lambda - \#A_n \ge 0$. We now proceed by contradiction by assuming the conclusion fails for n large enough; that is,

$$\begin{aligned} \frac{n(n+1)}{2}\lambda &< \sum_{a \in A_n} a \\ &= 1(\#A_1 - \#A_0) + 2(\#A_2 - \#A_1) + \dots + n(\#A_n - \#A_{n-1}) \\ &= n\#A_n - (\#A_1 + \dots + \#A_{n-1}) \\ &= n(n\lambda - x_n) - [(\lambda - x_1) + (2\lambda - x_2) + \dots + ((n-1)\lambda - x_{n-1})] \\ &= \frac{n(n+1)}{2}\lambda - nx_n + (x_1 + \dots + x_{n-1}). \end{aligned}$$

This means that for all sufficiently large n, we have

$$x_n < \frac{x_1 + \dots + x_{n-1}}{n} \qquad \forall n \gg 0.$$

Intuitively this means x_n should become close to each other, since they are also nonnegative.

Astonishingly, this intuition is false and (x_n) need not converge; Zhao Ting-Wei showed me that one can have a sequence which is zero "every so often" yet where the average is nonzero. However, we have a second condition we haven't used yet: the "integer" condition implies

$$|x_{n+1} - x_n| = |\lambda - \#\{n \in A\}| > \varepsilon$$

for some fixed $\varepsilon > 0$, namely $\varepsilon = \min\{\lambda, 1 - \lambda\}$. Using the fact that consecutive terms differ by some fixed ε, we will derive a contradiction.

Note that for some N_0 and M, we have

$$x_n < M \qquad \forall n > N_0.$$

Hence for $n > N_0$ we have $x_n + x_{n+1} < 2M - \varepsilon$, and so for large enough n the average must drop to just above $M - \frac{1}{2}\varepsilon$. Thus for some large $N_1 > N_0$, we will have

$$x_n < M - \frac{1}{3}\varepsilon \qquad \forall n > N_1.$$

If we repeat this argument then with a large $N_2 > N_1$, we obtain

$$x_n < M - \frac{2}{3}\varepsilon \qquad \forall n > N_2$$

and so on and so forth. This is a clear contradiction.

> **Remark.** Note that if $A = \{2, 2, 3, 4, 5, \dots\}$ and $\lambda = 1$ then contradiction. So the condition that $0 < \lambda < 1$ cannot be dropped, and (by scaling) neither can the condition that $A \subseteq \mathbb{Z}$.

45th United States of America Mathematical Olympiad

Day I 12:30PM — 5PM EDT

April 19, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct non-empty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

USAMO 2. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

USAMO 3. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

45th United States of America Mathematical Olympiad

Day II 12:30PM — 5PM EDT

April 20, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2 .$$

USAMO 5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

USAMO 6. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?

USAMO 2016 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2016 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

2. Prove that for any positive integer k,
$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$
is an integer.

3. Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

   Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

4. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,
$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

   Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

6. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

   We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

§1 USAMO 2016/1, proposed by Iurie Boreico

Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

Solution with Danielle Wang: the answer is that $|S| \geq 8$.

Proof of sufficiency Since we must have $2^{|S|} \geq 100$, we must have $|S| \geq 7$.

To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \dots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$.

Construction We will provide an inductive construction for a chain of subsets $X_1, X_2, \dots, X_{2^{n-1}+1}$ of $S = \{1, \dots, n\}$ satisfying $X_i \cap X_{i+1} = \varnothing$ and $X_i \cup X_{i+1} \neq S$ for each $n \geq 4$.

For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13 \ .$$

Now, given a chain of subsets of $\{1, 2, \dots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \dots, n+1\}$:

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;
2. glue the two copies together, joined by $\varnothing$ in between; and then
3. insert the element $n+1$ into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

$$\begin{array}{ccccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25 & \end{array}$$

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1}+1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \dots, 8\}$.

Remark. Here is the construction for $n = 8$ in its full glory.

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

§2 USAMO 2016/2, proposed by Kiran Kedlaya

Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \ge 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power q, we have

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor \ge \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor$$

Since both sides are integers, we show

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor.$$

If we denote by $\{x\}$ the fractional part of x, then $\lfloor x \rfloor = x - \{x\}$ so it's equivalent to

$$\left\{ \frac{k^2}{q} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} < 1 + \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\}.$$

However, the sum of remainders when $0, 1, \dots, k-1$ are taken modulo q is easily seen to be less than the sum of remainders when $k, k+1, \dots, 2k-1$ are taken modulo q. So

$$\sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} \le \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\}$$

follows, and we are done upon noting $\left\{ k^2/q \right\} < 1$.

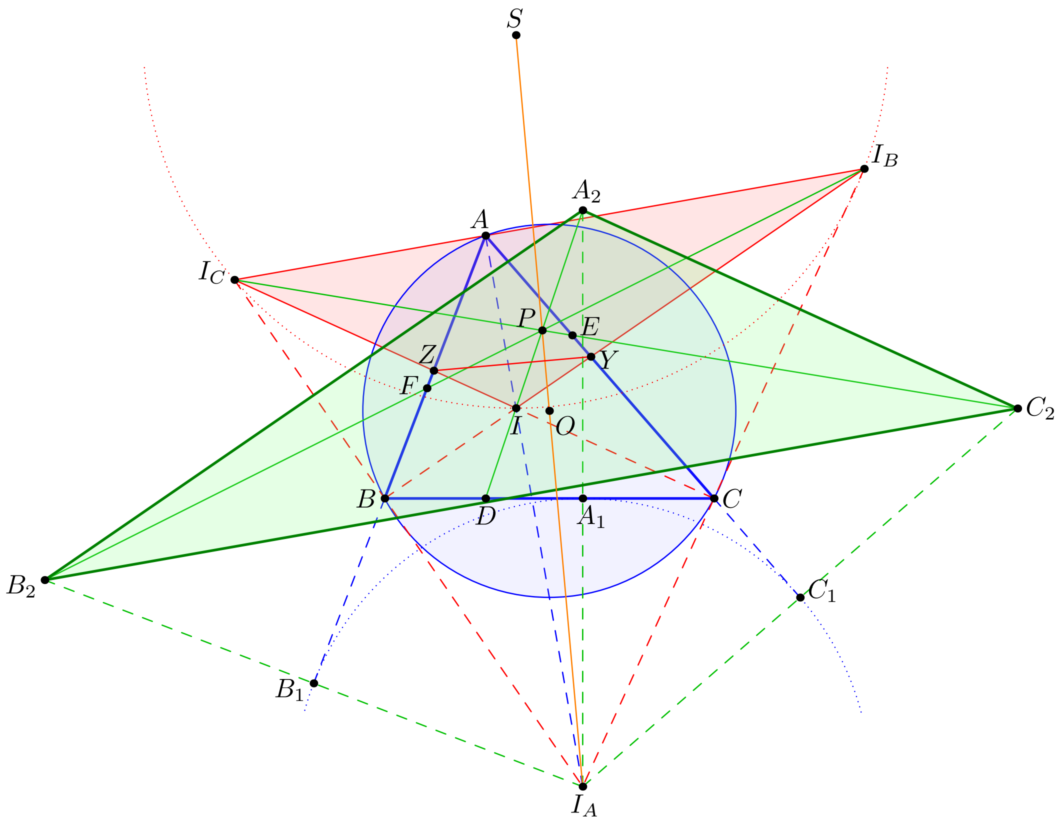
§3 USAMO 2016/3, proposed by Evan Chen and Telv Cohl

Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

We present two solutions.

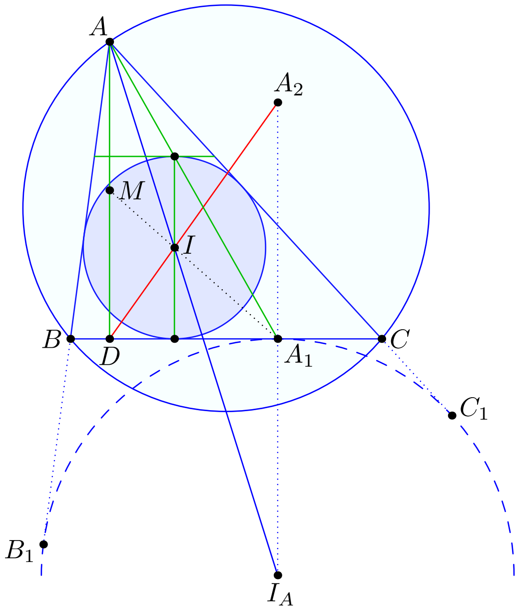
First solution Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle II_BI_C$.



We begin with the following observation:

Claim — Points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear.

Proof. This basically follows from the "midpoints of altitudes" lemma. To see D, I, A_2 are collinear, recall first that $\overline{IA_1}$ passes through the midpoint M of $\overline{AD}$.



Now since $\overline{AD} \parallel \overline{I_A A_2}$, and M and A_1 are the midpoints of $\overline{AD}$ and $\overline{I_A A_2}$, it follows from the collinearity of A, I, I_A that D, I, A_2 are collinear as well.

The other two claims follow in a dual fashion. For example, using the homothety taking the A to C-excircle, we find that $\overline{C_1 I_C}$ bisects the altitude $\overline{BE}$, and since I_C, B, I_A are collinear the same argument now gives I_C, E, C_2 are collinear. The fact that I_B, F, B_2 are collinear is symmetric. □

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

Second solution (barycentric, outline, Colin Tang) we are going to use barycentric coordinates to show that the line through O perpendicular to $\overline{YZ}$ is concurrent with $\overline{I_BF}$ and $\overline{I_CE}$.

The displacement vector $\overrightarrow{YZ}$ is proportional to $(a(b-c) : -b(a+c) : c(a+b))$, and so by strong perpendicularity criterion and doing a calculation gives the line

$$x(b-c)bc(a+b+c) + y(a+c)ac(a+b-c) + z(a+b)ab(-a+b-c) = 0.$$

On the other hand, line I_CE has equation

$$0 = \det \begin{bmatrix} a & b & -c \\ S_C & 0 & S_A \\ x & y & z \end{bmatrix} = bS_a \cdot x + (-cS_C - aS_A) \cdot y + (-bS_C) \cdot z$$

and similarly for I_BF. Consequently, concurrence of these lines is equivalent to

$$\det \begin{bmatrix} bS_A & -cS_C - aS_A & -bS_C \\ cS_A & -cS_B & -aS_A - bS_B \\ (b-c)bc(a+b+c) & (a+c)ac(a+b-c) & (a+b)ab(-a+b-c) \end{bmatrix} = 0$$

which is a computation.

Authorship comments I was intrigued by a Taiwan TST problem which implied that, in the configuration above, $\angle I_B D I_C$ was bisected by $\overline{DA}$. This motivated me to draw all three properties above where I_A and P were isogonal conjugates with respect to DEF. After playing around with this picture for a long time, I finally noticed that O was on line PI_A. (So the original was to show that I_BF, I_CE, DA_2 concurrent). Eventually I finally noticed in the picture that PI_A actually passed through the circumcenter of ABC as well. This took me many hours to prove.

The final restatement (which follows quickly from P, O, I_A collinear) was discovered by Telv Cohl when I showed him the problem.

§4 USAMO 2016/4, proposed by Titu Andreescu

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x)+xy) \cdot f(x-3y) + (f(y)+xy) \cdot f(3x-y) = (f(x+y))^2.$$

We claim that the only two functions satisfying the requirements are $f(x) \equiv 0$ and $f(x) \equiv x^2$. These work.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\bigstar)$$

for all x.

Now we claim

Claim — $f(z) = 0 \iff f(2z) = 0 \qquad (\spadesuit)$.

Proof. Let $(x, y) = (3t, t)$ in the given to get

$$\left(f(t) + 3t^2\right) f(8t) = f(4t)^2.$$

Now if $f(4t) \neq 0$ (in particular, $t \neq 0$), then $f(8t) \neq 0$. Thus we have $(\spadesuit)$ in the forwards direction.

Then $f(4t) \neq 0 \overset{(\bigstar)}{\Longrightarrow} f(t) = t^2 \neq 0 \overset{(\spadesuit)}{\Longrightarrow} f(2t) \neq 0$ implies the reverse direction, the last step being the forward direction $(\spadesuit)$. ☐

By putting together $(\bigstar)$ and $(\spadesuit)$ we finally get

$$\forall x \in \mathbb{R}: \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \geq 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c+b}{4}, \frac{c-b}{4}\right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x-y) = (f(x) + f(y) + 2xy)\, f(b)$$

Since $f(x) + f(y) + 2xy > 0$, if follows that $f(b) = 0$, as desired.

§5 USAMO 2016/5, proposed by Ivan Borsenco

An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

First solution (complex) In fact, we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OZ$, it follows that

$$m - a = i \cdot sz.$$

Similarly, $p - n = i \cdot sy$ and $a - q = i \cdot sx$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

Second solution (trig, Danielle Wang) Let δ and ϵ denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$\begin{aligned} BN &= (c - 1)\cos B + \cos\delta, \\ CP &= (b - 1)\cos C + \cos\epsilon, \text{ and} \\ a &= 1 + BN + CP \end{aligned}$$

from which it follows that

$$\cos\delta + \cos\epsilon = \cos B + \cos C - 1$$

Also, by the Law of Sines, we have $\frac{c-1}{\sin\delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin\epsilon - \sin\delta = \sin B - \sin C.$$

The sum-to-product formulas

$$\sin\epsilon - \sin\delta = 2\cos\left(\frac{\epsilon+\delta}{2}\right)\sin\left(\frac{\epsilon-\delta}{2}\right)$$
$$\cos\epsilon - \cos\delta = 2\cos\left(\frac{\epsilon+\delta}{2}\right)\cos\left(\frac{\epsilon-\delta}{2}\right)$$

give us

$$\tan\left(\frac{\epsilon-\delta}{2}\right) = \frac{\sin\epsilon - \sin\delta}{\cos\epsilon - \cos\delta} = \frac{\sin B - \sin C}{\cos B + \cos C - 1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi + \epsilon - \delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan \varphi = \frac{r - R\cos A}{\frac{1}{2}(b-c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r - R\cos A}{\frac{1}{2}(b-c)} = \frac{\cos B + \cos C + 1}{\sin B - \sin C}.$$

Using the fact that $b = 2R\sin B$, $c = 2R\sin C$, this reduces to the fact that $r/R + 1 = \cos A + \cos B + \cos C$, which is the so-called Carnot theorem.

§6 USAMO 2016/6, proposed by Gabriel Carroll

Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

The game is winnable if and only if $k < n$.

First, suppose $2 \leq k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. By taking the difference of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \dots, 2n-k$. If $k \leq n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.

46^{th} United States of America Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 19, 2017

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

USAMO 2. Let $m_1, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an *A-inversion* of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,
- $w_j > a_i \geq w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

USAMO 3. $(*)$ Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

46^{th} United States of America Mathematical Olympiad
Day 2. 12:30 PM – 5:00 PM EDT
April 20, 2017

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 4. Let $P_1, \ldots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$ other than $(1, 0)$. Each point is colored either red or blue, with exactly n of them red and n of them blue. Let $R_1, \ldots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 traveling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

USAMO 5. Let $\mathbf{Z}$ denote the set of all integers. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbf{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

USAMO 6. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4},$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

46^{th} United States of America Mathematical Olympiad Solutions

USAMO 1. (Proposed by Gregory Galperin)

Let n be an odd positive integer, and take $a = 2n - 1$, $b = 2n + 1$. Then $a^b + b^a \equiv 1 + 3 \equiv 0 \pmod 4$, and $a^b + b^a \equiv -1 + 1 \equiv 0 \pmod n$. Therefore $a + b = 4n$ divides $a^b + b^a$.

Alternate solution: Let $p > 5$ be a prime and let $p \not\equiv 1 \pmod 5$. For each such prime p we construct a pair of relatively prime numbers (a, b) that satisfy the conclusion of the problem. Thus, we will get infinitely many distinct pairs (a, b) as required.

Let $a = 3p + 2$, $b = 7p - 2$. Then $a + b = 10p$. We have $\varphi(10p) = 4(p - 1) = b - a$, where φ is Euler's function.

Obviously, a and b are odd and not divisible by p. They are not divisible by 5 because $p \not\equiv 1 \pmod 5$. Thus, a and b are relatively prime to $10p = a + b$, and therefore relatively prime to each other.

Therefore, using Euler's theorem,

$$a^b = a^{a+\varphi(10p)} = a^a \cdot a^{\varphi(10p)} \equiv a^a \pmod{10p} ,$$

and since $10p = a + b$,

$$a^b + b^a \equiv a^a + b^a \pmod{a + b} .$$

However, since a is odd, $a^a + b^a$ is divisible by $a + b$. Hence, $a^b + b^a$ is divisible by $a + b$.

USAMO 2. (Proposed by Maria Monks Gillespie)

It suffices to show the result for $B = (0, 0, \ldots, 0)$, since then any sequence is equivalent to any other sequence via B. We first show that the result holds for all sequences of the form $A = (a, a, \ldots, a)$ for some a.

For each positive integer i define the ith **lifting map** B_i on the permutations of $m_1, \ldots, m_n$ by $B_i(w_1, \ldots, w_n) = v_1, \ldots, v_n$ where $v_j = i$ if and only if $w_{n+1-j} = i$, and where the subsequence of v consisting of all entries not equal to i (taken in order) is equal to the subsequence of w consisting of all entries not equal to i.

Lemma 1. *Let $A_{i-1} = (i - 1, i - 1, \ldots, i - 1)$ and $A_i = (i, i, \ldots, i)$. Then the number of A_{i-1}-inversions of w equals the number of A_i-inversions of $B_i(w)$. Moreover, B_i is a bijection on the permutations of w, showing the result in this case.*

Proof. It is easy to see that B_i is a bijection for any i, since we can reverse the map.

Now, note that any A_{i-1}-inversions between entries not equal to i in w are still A_i-inversions in $B_i(w)$, and vice-versa. Notice also that there are no A_{i-1}-inversions in w having i as the left entry. Similarly there are no A_i-inversions having i as the *right* entry in $B_i(w)$.

On the other hand, in w, any non-i entry to the *left* of an i forms an A_{i-1}-inversion with that i. And in $B_i(w)$, any non-i entry to the *right* of an i forms an A_i-inversion with that i. Since the positions of the i's are reversed from w to $B_i(w)$, the number of inversions involving an i are equal in each case, and the result follows. □

For $j > i$, we denote $B_{i\to j} := B_j \circ B_{j-1} \circ \cdots \circ B_{i+2} \circ B_{i+1}$. Also, for $j > i$, we denote $B_{j\to i} := B_{i\to j}^{-1} = B_{i+1}^{-1} \circ B_{i+2}^{-1} \circ \cdots \circ B_j^{-1}$. And we let $B_{i\to i}$ be the identity permutation.

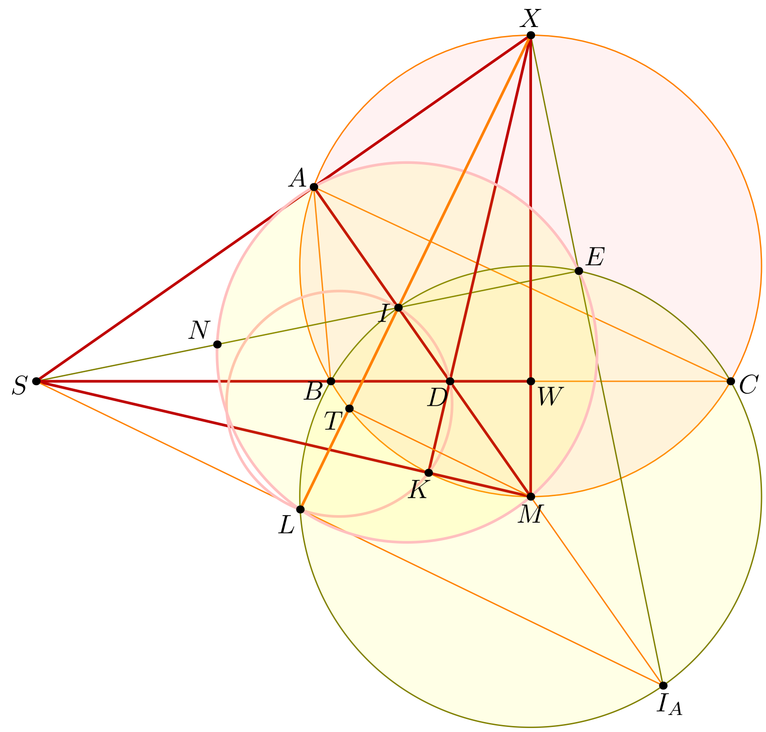
Additionally, for $A = (a_1, \ldots, a_n)$ and for a permutation w of $m_1, \ldots, m_n$ we define $\phi_A(w)$ as follows. Let $w^{(1)} = B_{0\to a_1}(w)$ and, inductively, for $i > 1$ let $w^{(i)}$ be the result of applying $B_{a_{i-1}\to a_i}$ to the last $n - i + 1$ terms of $w^{(i-1)}$ and leaving the first $i - 1$ terms unchanged. Finally let $\phi_A(w) = w^{(n)}$.

Lemma 2. *The number of A-inversions of $\phi_A(w)$ is equal to the number of B-inversions of w where $B = (0, 0, \ldots, 0)$.*

Proof. This is a consequence of the definition of ϕ_A: At any step $w^{(i)}$ in the process of computing $\phi_A(w)$, we consider the sequence $A^{(i)}$ formed by changing the last $n - i + 1$ terms of the previous sequence $A^{(i-1)}$ (starting at $A^{(0)} = (0, 0, \ldots, 0)$) from a_{i-1} to a_i. Then we have $A^{(n)} = A$, and at each step the number of $A^{(i)}$-inversions of $w^{(i)}$ is equal to the number of $A^{(i-1)}$-inversions of $w^{(i-1)}$ by Lemma 1. (More precisely, the lemma applies to the number of such inversions among the last $n - i + 1$ terms, but note that the number of inversions involving any of the first $i - 1$ terms is also unchanged at each step.) The result follows. □

And since ϕ_A is a bijection, being a composition of bijections, we are done.

USAMO 3. (Proposed by Evan Chen)



Let W be the midpoint of $\overline{BC}$, and let X be the point on Ω opposite M. Observe that line KD passes through X, and thus lines BC, MK, XA concur at the orthocenter of $\triangle DMX$, which is S. Denote by I_A the A-excenter of ABC.

Next, let E be the foot of the altitude from I to $\overline{XI_A}$; observe that E lies on the circle ω centered at M through I, B, C, I_A. Then, S is the radical center of ω, Ω, and the circle with diameter $\overline{IX}$; hence line SI passes through E; accordingly I is the orthocenter of $\triangle XSI_A$; denote by L the foot of the altitude from X to $\overline{I_AS}$.

We claim that this L lies on both the circumcircle of $\triangle KID$ and $\triangle MAN$. It lies on the circumcircle of $\triangle MAN$ since this circle is the nine-point circle of $\triangle XSI_A$. For the other, note that $\triangle MWI \sim \triangle MIX$, since they share the same angle at M and $MW \cdot MX = MB^2 = MI^2$. Consequently, $\angle IWM = \angle MIX = 180^\circ - \angle LIM = 180^\circ - \angle MLI$, enough to imply that quadrilateral $MWIL$ is cyclic. But lines IL, DK, and WM meet at X, so Power of a Point in cyclic quadrilaterals $DKMW$ and $MWIL$ gives $XD \cdot XK = XM \cdot XW = XI \cdot XL$, hence $KDIL$ is cyclic as needed.

All that remains to show is that the midpoint T of $\overline{IL}$ lies on Ω. But this follows from the fact that $\overline{TM} \parallel \overline{I_AL} \implies \angle MTX = 90^\circ$, thus the problem is solved.

Alternate Solution (by Titu Andreescu and Cosmin Pohoata): We refer to the same figure as in the first solution. Let X be the midpoint of arc BAC of Ω. A first key step in the problem is to note that D is the orthocenter of triangle XSM. This follows from the fact that $\overline{DK} \perp \overline{KM}$, which implies that line DK must pass through the antipode of M in Ω, which is precisely the point X. This together with the fact that $\overline{MX} \perp \overline{SW}$ implies the claim.

Next, it is essential to notice that I is also the orthocenter of triangle XSI_A, where I_A denotes the A-excenter of triangle ABC. This can be argued as follows: since D is the orthocenter of $\triangle XSM$, we have by Power of a Point that $AX \cdot AS = AD \cdot AM$ (we are implicitly using the fact that the reflection of D across line XS lies on the circumcircle of triangle XSM). However, the 4-tuple (A, I, D, I_A) is a harmonic division and M is the midpoint of $\overline{II_A}$, which easily implies that $AD \cdot AM = AI \cdot AI_A$. By Power of a Point once again, this yields that the reflection of I across line XS lies on the circumcircle of triangle XSI_A, so I must indeed be the orthocenter of triangle XSI_A. This is crucial, since then the circumcircle of triangle MAN is nothing but the nine-point circle of $\triangle XSI_A$, so the foot of altitude L from X on $\overline{SI_A}$ becomes a good candidate for L_1 or L_2. If T denotes the midpoint of segment $\overline{IL}$, then $\overline{TM}$ is a midline in triangle ILI_A, so $\overline{TM} \perp \overline{TX}$; therefore T is on the circle of diameter $\overline{MX}$, which is precisely Ω. It remains to show that L also lies on the circumcircle of triangle KID, but this is clear: $ASKD$ is cyclic, so $XA \cdot XS = XD \cdot XK$; also, $ASLI$ is cyclic, so $XA \cdot XS = XI \cdot XL$; hence $XD \cdot XK = XI \cdot XL$, which by Power of a Point means that $ILKD$ is cyclic, thus completing the proof.

USAMO 4. (Proposed by Maria Monks Gillespie)

We may assume the points have been labeled as $P_1, P_2, \ldots, P_{2n}$ in order, going counterclockwise from $(1, 0)$. Now, write out the color of each point in order, and replace each R with a $+1$ and each B with a -1, to get a list $p_1, \ldots, p_{2n}$ of $+1$'s and -1's. Consider the partial sums $p_1 + \cdots + p_k$ of this sequence, and choose the index k such that the kth partial sum has as small a value as possible; if several partial sums are tied for smallest, let k be the lowest index among them. Now, rotate the circle clockwise so that points $P_1, \ldots, P_k$ are moved past $(1, 0)$; the resulting sequence of $+1$'s and -1's from the new orientation now has all nonnegative partial sums, and the total sum is 0.

Consider any red point in the rotated diagram and label it R_1. The arc $R_1 \to B_1$ does not cross $(1, 0)$, for otherwise the sequence ends with a string of $+1$'s and the partial sums before those $+1$'s would be negative. Furthermore, the sequence of entries from R_1 to B_1 looks like $+1, +1, +1, \ldots, +1, -1$, and so removing R_1 and B_1 is equivalent to removing a consecutive pair of a $+1$ and -1, so the partial sums remain all nonnegative. It follows that the next pairing also doesn't cross $(1, 0)$, and so on, so no matter which way we pick the ordering of the red points in the rotated circle, there are no counterclockwise arcs $R_i \to B_i$ containing $(1, 0)$.

Finally, note that in any ordering of the red points, the blue points among $P_1, \ldots, P_k$ are all paired with red points, and those red points among $P_1, \ldots, P_k$ are paired with blue points in this same subsequence since there are no crossings in the rotated picture. Let m be the difference between the number of blue and red points among $P_1, \ldots, P_k$. Then it follows that exactly m blue points in $P_1, \ldots, P_k$ were matched with red points from $P_{k+1}, \ldots, P_{2n}$. Therefore, when we rotate the circle back to its original position, there are exactly m crossings, no matter which ordering we pick for the red points. Since m is independent of the ordering, the proof is complete.

USAMO 5. (Proposed by Ricky Liu)

The answer is $c < \sqrt{2}$.

First suppose $c < \sqrt{2}$. We can partition $\mathbf{Z}^2$ into two subsets

$$L_1 = \{(x,y) \mid x+y \text{ is odd}\} \quad \text{and} \quad L_1' = \{(x,y) \mid x+y \text{ is even}\}.$$

Both L_1 and L_1' are square lattices with unit length $\sqrt{2}$ (that is, they are similar to $\mathbf{Z}^2$ with a scaling factor of $\sqrt{2}$). Hence we can similarly partition L_1' into two square lattices L_2 and L_2' with unit length $\sqrt{2}^2$, then partition L_2' into two square lattices L_3 and L_3' with unit length $\sqrt{2}^3$, and so forth. Hence for any $N \geq 1$, $\mathbf{Z}^2$ can be partitioned into $N+1$ square lattices $L_1, L_2, \ldots, L_N, L_N'$ with unit lengths $\sqrt{2}, \sqrt{2}^2, \ldots, \sqrt{2}^N, \sqrt{2}^N$, respectively.

Since $\frac{\sqrt{2}}{c} > 1$, there exists a positive integer N such that $(\frac{\sqrt{2}}{c})^{N+1} \geq \sqrt{2}$, or equivalently, $c^{N+1} \leq \sqrt{2}^N$. For $i = 1, \ldots, N$, label all points in L_i by i, and then label all points in L_N' by $N+1$. Any two points in L_i lie at least $\sqrt{2}^i > c^i$ apart, while any two points in L_N' lie at least $\sqrt{2}^N \geq c^{N+1}$ apart, so this is a valid labeling.

Suppose instead that $c \geq \sqrt{2}$. For a nonnegative integer m, define

$$R_m = \{(x,y) \mid 1 \leq x \leq 2^a,\, 1 \leq y \leq 2^b\} \subseteq \mathbf{Z}^2, \text{ where } (a,b) = \begin{cases} (\frac{m}{2}, \frac{m}{2}) & \text{if } m \text{ is even,} \\ (\frac{m-1}{2}, \frac{m+1}{2}) & \text{if } m \text{ is odd.} \end{cases}$$

We will show by induction that R_m does not have a valid labeling using only labels at most m, which will prove that $\mathbf{Z}^2$ has no valid labeling. The case $m = 0$ is trivial.

Suppose $m > 0$ is odd and that R_{m-1} does not have a valid labeling using only $1, \ldots, m-1$ (the inductive hypothesis), but that R_m does have a valid labeling using only $1, \ldots, m$. Consider this labeling of R_m. Since $R_m \supseteq R_{m-1}$, some point (x_0, y_0) with $y_0 \leq 2^{(m-1)/2}$ must be labeled m. But then (x_0, y_0) lies directly below a translate R' of R_{m-1} inside R_m. The distance between (x_0, y_0) and any point in R' is at most

$$\sqrt{(2^{\frac{m-1}{2}} - 1)^2 + (2^{\frac{m-1}{2}})^2} < \sqrt{2}^m \leq c^m,$$

so no points in R' can be labeled m. But by the inductive hypothesis, R' has no valid labeling using only $1, \ldots, m-1$, which is a contradiction.

Now suppose $m > 0$ is even and that R_{m-1} does not have a valid labeling using only $1, \ldots, m-1$ (the inductive hypothesis), but R_m does have a valid labeling using only $1, \ldots, m$. By the inductive hypothesis, some point (x_0, y_0) with $\frac{1}{4} \cdot 2^{m/2} < y_0 \leq \frac{3}{4} \cdot 2^{m/2}$ must be labeled m (since the corresponding rows of R_m form a rotated copy of R_{m-1}). But then (x_0, y_0) lies either directly to the left or to the right of a translate R' of R_{m-1} inside R_m. The distance between (x_0, y_0) and any point of R' is less than

$$\sqrt{(\tfrac{3}{4} \cdot 2^{\frac{m}{2}})^2 + (2^{\frac{m-2}{2}})^2} = \frac{\sqrt{13}}{4} \cdot \sqrt{2}^m < \sqrt{2}^m \leq c^m,$$

so no points in R' can be labeled m. But by the inductive hypothesis, R' has no valid labeling using only $1, \ldots, m-1$, which is a contradiction. This completes the proof.

USAMO 6. (Proposed by Titu Andreescu)

We will show that the minimum is $\frac{2}{3}$. (In particular, the value $\frac{4}{5}$, obtained by making the natural guess $a = b = c = d = 1$, is not the right answer.)

We have

$$\frac{4a}{b^3+4} = a - \frac{ab^3}{b^3+4} \geq a - \frac{ab}{3},$$

since

$$b^3 + 4 = \frac{b^3}{2} + \frac{b^3}{2} + 4 \geq 3b^2,$$

by the Arithmetic Mean-Geometric Mean Inequality.

Then

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4} \geq \frac{a+b+c+d}{4} - \frac{ab+bc+cd+da}{12}.$$

But $a + b + c + d = 4$ and

$$4(ab + bc + cd + da) = 4(a + c)(b + d) \leq (a + b + c + d)^2 = 16.$$

Hence

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4} \geq 1 - \frac{4}{12} = \frac{2}{3}.$$

The minimum is realized when, for example, $a = b = 2$ and $c = d = 0$.

USAMO 2017 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2017 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

2. Let $m_1, m_2, \dots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \dots, a_n)$ and any permutation $w = w_1, \dots, w_n$ of $m_1, \dots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \ge w_i > w_j$,
- $w_j > a_i \ge w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \dots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \dots, m_n$ having exactly k B-inversions.

3. Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

4. Let $P_1, P_2, \dots, P_{2n}$ be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let $R_1, R_2, \dots, R_n$ be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \dots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \dots, R_n$ of the red points.

5. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points in $\mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

6. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

§1 USAMO 2017/1, proposed by Gregory Galperin

Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

One construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d+2^d}{d+2}$. Then set

$$a = \frac{x+d}{2}, \qquad b = \frac{x-d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$\begin{aligned} a+b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\ &\iff b^{a-b} \equiv 1 \pmod{a+b} \\ &\iff b^d \equiv 1 \pmod{d+2b} \\ &\iff (-2)^d \equiv d^d \pmod{d+2b} \\ &\iff d+2b \mid d^d + 2^d. \end{aligned}$$

So it would be enough that

$$d + 2b = \frac{d^d + 2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d+2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$.

> **Remark.** Ryan Kim points out that in fact, $(a, b) = (2n - 1, 2n + 1)$ is always a solution.

§2 USAMO 2017/2, proposed by Maria Monks

Let m_1, m_2, …, m_n be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \dots, a_n)$ and any permutation $w = w_1, \dots, w_n$ of $m_1, \dots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \ge w_i > w_j$,
- $w_j > a_i \ge w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \dots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \dots, m_n$ having exactly k B-inversions.

The following solution was posted by Michael Ren, and I think it is the most natural one (since it captures all the combinatorial ideas using a q-generating function that is easier to think about, and thus makes the problem essentially a long computation).

Denote by M our multiset of n positive integers. Define an *inversion* of a permutation to be pair $i < j$ with $w_i < w_j$ (which is a $(0, \dots, 0)$-inversion in the problem statement); this is the usual definition (see https://en.wikipedia.org/wiki/Inversion_(discrete_mathematics)). So we want to show the number of A-inversions is equal to the number of usual inversions. In what follows we count permutations on M with multiplicity: so $M = \{1, 1, 2\}$ still has $3! = 6$ permutations.

We are going to do what is essentially recursion, but using generating functions in a variable q to do our book-keeping. (Motivation: there's no good closed form for the number of inversions, but there's a great generating function known — which is even better for us, since we're only trying to show two numbers are equal!) First, we prove two claims.

Claim — For any positive integer n, the generating function for the number of permutations of $(1, 2, \dots, n)$ with exactly k inversions is

$$n!_q \overset{\text{def}}{=} 1 \cdot (1+q) \cdot (1+q+q^2) \cdot \ldots (1+q+\dots+q^{n-1}).$$

Here we mean that the coefficient of q^s above gives the number of permutations with exactly s inversions.

Proof. This is an induction on n, with $n = 1$ being trivial. Suppose we choose the first element to be i, with $1 \le i \le n$. Then there will always be exactly $i - 1$ inversions using the first element, so this contributes $q^i \cdot (n-1)!_q$. Summing $1 \le i \le n$ gives the result. □

Unfortunately, the main difficulty of the problem is that there are repeated elements, which makes our notation much more horrific.

Let us define the following. We take our given multiset M of n positive integers, we suppose the distinct numbers are $\theta_1 < \theta_2 < \dots < \theta_m$. We let e_i be the number of times θ_i appears. Therefore the multiplicities e_i should have sums

$$e_1 + \dots + e_m = n$$

and m denotes the number of distinct elements. Finally, we let

$$F(e_1, \dots, e_m) = \sum_{\text{permutations } \sigma} q^{\text{number inversions of } \sigma}$$

be the associated generating function for the number of inversions. For example, the first claim we proved says that $F(1, \dots, 1) = n!_q$.

Claim — We have the explicit formula

$$F(e_1, \dots, e_m) = n!_q \cdot \prod_{i=1}^{m} \frac{e_i!}{e_i!_q}.$$

Proof. First suppose we perturb all the elements slightly, so that they are no longer equal. Then the generating function would just be $n!_q$.

Then, we undo the perturbations for each group, one at a time, and claim that we get the above $e_i!_q$ factor each time. Indeed, put the permutations into classes of $e_1!$ each where permutations in the same classes differ only in the order of the perturbed θ_1's (with the other $n - e_1$ elements being fixed). Then there is a factor of $e_1!_q$ from each class, owing to the slightly perturbed inversions we added within each class. So we remove that factor and add $e_1! \cdot q^0$ instead. This accounts for the first term of the product.

Repeating this now with each term of the product implies the claim. □

Thus we have the formula for the number of inversions in general. We wish to show this also equals the generating function the number of A-inversions, for any fixed choice of A. This will be an induction by n, with the base case being immediate.

For the inductive step, fix A, and assume the first element satisfies $\theta_k \le a_1 < \theta_{k+1}$ (so $0 \le k \le m$; we for convenience set $\theta_0 = -\infty$ and $\theta_m = +\infty$). We count the permutations based on what the first element θ_i of the permutation is. Then:

- Consider permutations starting with $\theta_i \in \{\theta_1, \dots, \theta_k\}$. Then the number of inversions which will use this first term is $(e_1 + \dots + e_{i-1}) + (e_{k+1} + \dots + e_m)$. Also, there are e_i ways to pick which θ_i gets used as the first term. So we get a contribution of

$$q^{e_1 + \dots + e_{i-1} + (e_{k+1} + \dots + e_m)} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)$$

  in this case (with inductive hypothesis to get the last F-term).

- Now suppose $\theta_i \in \{\theta_{k+1}, \dots, \theta_m\}$. Then the number of inversions which will use this first term is $e_{k+1} + \dots + e_{i-1}$. Thus by a similar argument the contribution is

$$q^{e_{k+1} + \dots + e_{i-1}} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m).$$

Therefore, to complete the problem it suffices to prove

$$\begin{aligned} &\sum_{i=1}^{k} q^{(e_1 + \dots + e_{i-1}) + (e_{k+1} + \dots + e_m)} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m) \\ &+ \sum_{i=k+1}^{m} q^{e_{k+1} + \dots + e_{i-1}} \cdot e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m) \\ &= F(e_1, \dots, e_m). \end{aligned}$$

Now, we see that

$$\frac{e_i \cdot F(e_1, \dots, e_i - 1, \dots, e_m)}{F(e_1, \dots, e_m)} = \frac{1 + \dots + q^{e_i - 1}}{1 + q + \dots + q^{n-1}} = \frac{1 - q^{e_i}}{1 - q^n}$$

so it's equivalent to show

$$1 - q^n = q^{e_{k+1} + \dots + e_m} \sum_{i=1}^{k} q^{e_1 + \dots + e_{i-1}} (1 - q^{e_i}) + \sum_{i=k+1}^{m} q^{e_{k+1} + \dots + e_{i-1}} (1 - q^{e_i})$$

which is clear, since the left summand telescopes to $q^{e_{k+1} + \dots + e_m} - q^n$ and the right summand telescopes to $1 - q^{e_{k+1} + \dots + e_m}$.

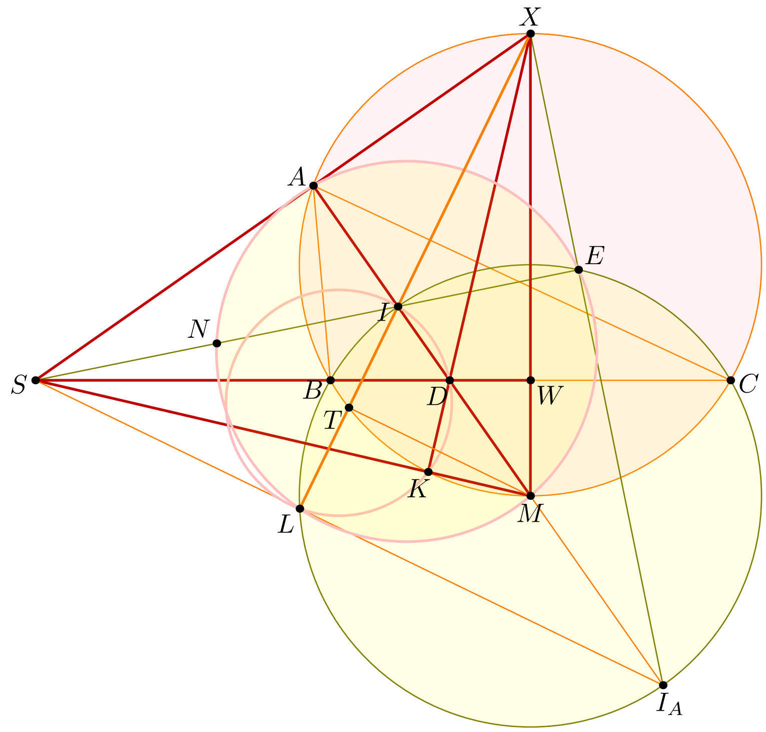
Remark. Technically, we could have skipped straight to the induction, without proving the first two claims. However I think the solution reads more naturally this way.

§3 USAMO 2017/3, proposed by Evan Chen

Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Let W be the midpoint of $\overline{BC}$, let X be the point on Ω opposite M. Observe that $\overline{KD}$ passes through X, and thus lines BC, MK, XA concur at the orthocenter of $\triangle DMX$, which we call S. Denote by I_A the A-excenter of ABC.

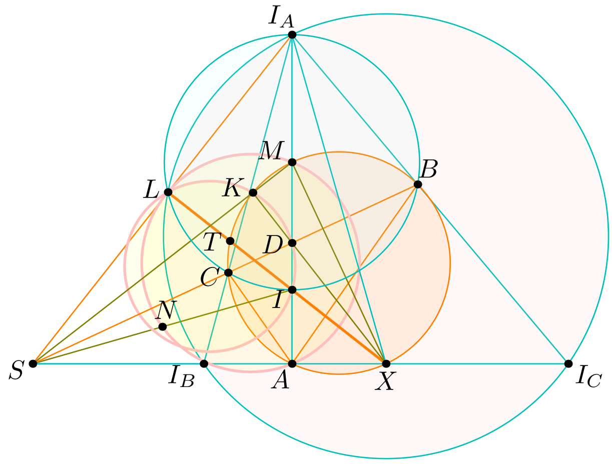
Next, let E be the foot of the altitude from I to $\overline{XI_A}$; observe that E lies on the circle centered at M through I, B, C, I_A. Then, S is the radical center of Ω and the circles with diameter $\overline{IX}$ and $\overline{II_A}$; hence line SI passes through E; accordingly I is the orthocenter of $\triangle XSI_A$; denote by L the foot from X to $\overline{SI_A}$.



We claim that this L lies on both the circumcircle of $\triangle KID$ and $\triangle MAN$. It lies on the circumcircle of $\triangle MAN$ since this circle is the nine-point circle of $\triangle XSI_A$. Also, $XD \cdot XK = XW \cdot XM = XA \cdot XS = XI \cdot XL$, so $KDIL$ are concyclic.

All that remains to show is that the midpoint T of $\overline{IL}$ lies on Ω. But this follows from the fact that $\overline{TM} \parallel \overline{LI_A} \implies \angle MTX = 90^\circ$, thus the problem is solved.

> **Remark.** Some additional facts about this picture: the point T is the contact point of the A-mixtilinear incircle (since it is collinear with X and I), while the point K is such that $\overline{AK}$ is an A-symmedian (since $\overline{KD}$ and $\overline{AD}$ bisect $\angle A$ and $\angle K$, say).



Remark. In fact, the point L is the Miquel point of cyclic quadrilateral I_BI_CBC (inscribed in the circle with diameter $\overline{I_BI_C}$). This implies many of the properties that L has above. For example, it directly implies that L lies on the circumcircles of triangles $I_AI_BI_C$ and BCI_A, and that the point L lies on $\overline{SI_A}$ (since $S = \overline{BC} \cap \overline{I_BI_C}$). For this reason, many students found it easier to think about the problem in terms of $\triangle I_AI_BI_C$ rather than $\triangle ABC$.

§4 USAMO 2017/4, proposed by Maria Monks

Let P_1, P_2, ..., P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, ..., R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, ..., B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, ..., R_n of the red points.

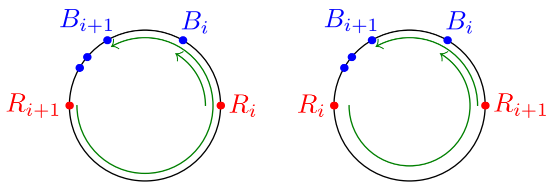
We present two solutions, one based on swapping and one based on an invariant.

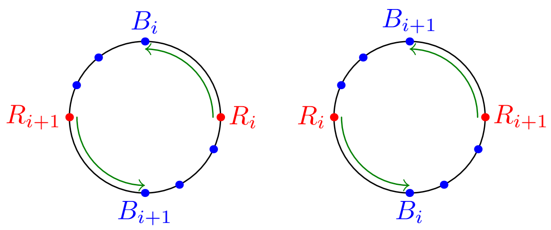
First "local" solution by swapping two points Let $1 \le i < n$ be any index and consider the two red points R_i and R_{i+1}. There are two blue points B_i and B_{i+1} associated with them.

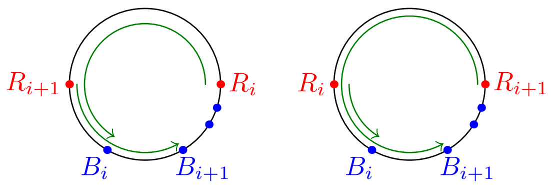
Claim — If we swap the locations of points R_i and R_{i+1} then the new arcs $R_i \to B_i$ and $R_{i+1} \to B_{i+1}$ will cover the same points.

Proof. Delete all the points R_1, ..., R_{i-1} and B_1, ..., B_{i-1}; instead focus on the positions of R_i and R_{i+1}.

The two blue points can then be located in three possible ways: either 0, 1, or 2 of them lie on the arc $R_i \to R_{i+1}$. For each of the cases below, we illustrate on the left the locations of B_i and B_{i+1} and the corresponding arcs in green; then on the right we show the modified picture where R_i and R_{i+1} have swapped. (Note that by hypothesis there are no other blue points in the green arcs).

Case 1 

Case 2 

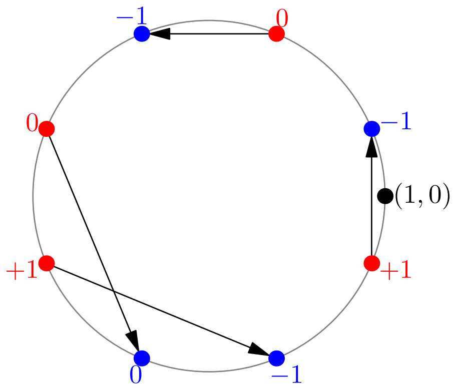
Case 3 

Observe that in all cases, the number of arcs covering any given point on the circumference is not changed. Consequently, this proves the claim. □

Finally, it is enough to recall that any permutation of the red points can be achieved by swapping consecutive points (put another way: $(i\ i+1)$ generates the permutation group S_n). This solves the problem.

> **Remark.** This proof does *not* work if one tries to swap R_i and R_j if $|i-j| \neq 1$. For example if we swapped R_i and R_{i+2} then there are some issues caused by the possible presence of the blue point B_{i+1} in the green arc $R_{i+2} \to B_{i+2}$.

Second longer solution using an invariant Visually, if we draw all the segments $R_i \to B_i$ then we obtain a set of n chords. Say a chord is *inverted* if satisfies the problem condition, and *stable* otherwise. The problem contends that the number of stable/inverted chords depends only on the layout of the points and not on the choice of chords.



In fact we'll describe the number of inverted chords explicitly. Starting from $(1,0)$ we keep a running tally of $R-B$; in other words we start the counter at 0 and decrement by 1 at each blue point and increment by 1 at each red point. Let $x \le 0$ be the lowest number ever recorded. Then:

> **Claim** — The number of inverted chords is $-x$ (and hence independent of the choice of chords).

This is by induction on n. I think the easiest thing is to delete chord R_1B_1; note that the arc cut out by this chord contains no blue points. So if the chord was stable certainly no change to x. On the other hand, if the chord is inverted, then in particular the last point before $(1,0)$ was red, and so $x<0$. In this situation one sees that deleting the chord changes x to $x+1$, as desired.

§5 USAMO 2017/5, proposed by Ricky Liu

Find all real numbers $c > 0$ such that there exists a labeling of the lattice points in $\mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

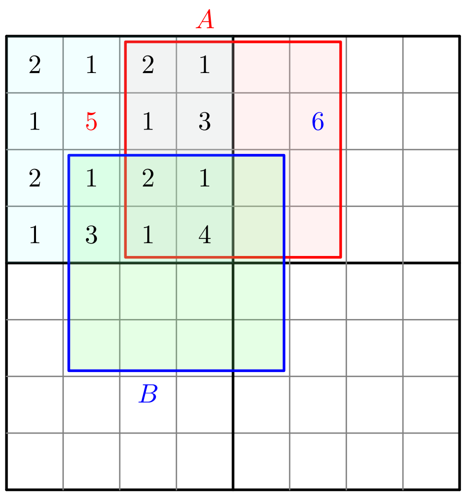
The answer is $c < \sqrt{2}$. Here is a solution with Calvin Deng.

The construction for any $c < \sqrt{2}$ can be done as follows. Checkerboard color the lattice points and label the black ones with 1. The white points then form a copy of $\mathbb{Z}^2$ again scaled up by $\sqrt{2}$ so we can repeat the procedure with 2 on half the resulting points. Continue this dyadic construction until a large N for which $c^N < 2^{\frac{1}{2}(N-1)}$, at which point we can just label all the points with N.

I'll now prove that $c = \sqrt{2}$ (and hence $c \ge \sqrt{2}$) can't be done.

Claim — It is impossible to fill a $2^n \times 2^n$ square with labels not exceeding $2n$.

The case $n = 1$ is clear. So now assume it's true up to $n - 1$; and assume for contradiction a $2^n \times 2^n$ square S only contains labels up to $2n$. (Of course every $2^{n-1} \times 2^{n-1}$ square contains an instance of a label at least $2n - 1$.)



Now, we contend there are fewer than four copies of $2n$:

Lemma

In a unit square, among any four points, two of these points have distance ≤ 1 apart.

Proof. Look at the four rays emanating from the origin and note that two of them have included angle $\le 90°$. □

So WLOG the northwest quadrant has no $2n$'s. Take a $2n - 1$ in the northwest and draw a square of size $2^{n-1} \times 2^{n-1}$ directly right of it (with its top edge coinciding with the top of S). Then A can't contain $2n - 1$, so it must contain a $2n$ label; that $2n$ label must be in the northeast quadrant.

Then we define a square B of size $2^{n-1} \times 2^{n-1}$ as follows. If $2n - 1$ is at least as high $2n$, let B be a $2^{n-1} \times 2^{n-1}$ square which touches $2n - 1$ north and is bounded east by $2n$. Otherwise let B be the square that touches $2n - 1$ west and is bounded north by $2n$. We then observe B can neither have $2n - 1$ nor $2n$ in it, contradiction.

> **Remark.** To my knowledge, essentially all density arguments fail because of hexagonal lattice packing.

§6 USAMO 2017/6, proposed by Titu Andreescu

Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

The minimum $\frac{2}{3}$ is achieved at $(a,b,c,d)=(2,2,0,0)$ and cyclic permutations.

The problem is an application of the tangent line trick: we observe the miraculous identity

$$\frac{1}{b^3+4} \ge \frac{1}{4} - \frac{b}{12}$$

since $12-(3-b)(b^3+4) = b(b+1)(b-2)^2 \ge 0$. Moreover,

$$ab+bc+cd+da = (a+c)(b+d) \le \left(\frac{(a+c)+(b+d)}{2}\right)^2 = 4.$$

Thus

$$\sum_{\text{cyc}} \frac{a}{b^3+4} \ge \frac{a+b+c+d}{4} - \frac{ab+bc+cd+da}{12} \ge 1 - \frac{1}{3} = \frac{2}{3}.$$

Remark. The main interesting bit is the equality at $(a,b,c,d)=(2,2,0,0)$. This is the main motivation for trying tangent line trick, since a lower bound of the form $\sum a(1-\lambda b)$ preserves the unusual equality case above. Thus one takes the tangent at $b=2$ which miraculously passes through the point $(0,1/4)$ as well.

47^{th} United States of America Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAMO 2. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

USAMO 3. For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

47^{th} United States of America Mathematical Olympiad
Day 2. 12:30 PM – 5:00 PM EDT
April 19, 2018

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 4. Let p be a prime, and let $a_1, a_2, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \dots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

USAMO 5. ($*$) In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in this order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

USAMO 6. Let a_n be the number of permutations $(x_1, x_2, \dots, x_n)$ of the numbers $(1, 2, \dots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

2018 U.S.A. Mathematical Olympiad Solutions

USAMO 1.

First solution. Assume without loss of generality that $c = \min(a, b, c)$. By the AM-GM inequality and the given condition, we have

$$\begin{aligned} 4c(a+b+c) + 4ab &\geq 2\sqrt{16 \cdot abc(a+b+c)} \\ &= 2\sqrt{16\left(\frac{a+b+c}{4}\right)^3 (a+b+c)} \\ &= (a+b+c)^2. \end{aligned}$$

Subtracting $2(ab + bc + ca)$ from both sides, this gives

$$2(ab + bc + ca) + 4c^2 \geq a^2 + b^2 + c^2,$$

as desired.

Remark. The equality in the AM-GM step occurs if and only if $c(a + b + c) = ab$. Solving for $a+b+c$ and substituting into the condition $a+b+c = 4\sqrt[3]{abc}$, this implies $8c^2 = ab$. Substituting this back into the equation $c(a + b + c) = ab$, we conclude that

$$c(a+b+c) = 8c^2 \implies a + b = 7c.$$

We then have

$$a - b = \pm\sqrt{(a+b)^2 - 4ab} = \pm\sqrt{49c^2 - 32c^2} = \pm\sqrt{17}c.$$

It follows that $\{2a, 2b\} = \{(7 - \sqrt{17})c, (7 + \sqrt{17})c\}$. Hence, equality holds if and only if (a, b, c) is a permutation of

$$\left((7 - \sqrt{17})r, (7 + \sqrt{17})r, 2r\right)$$

for some positive real number r.

Second solution. Suppose, as above, that $c = \min(a, b, c)$, and write $A = a/c$, $B = b/c$, and $D = A+B$. The given condition becomes $A+B+1 = 4\sqrt[3]{AB}$, or equivalently, $AB = (D+1)^3/64$. In terms of A and B, the problem asks us to prove that

$$2(AB + A + B) + 4 \geq A^2 + B^2 + 1,$$

which can be rearranged as

$$2(A + B) + 3 - (A + B)^2 + 4AB \geq 0.$$

After substituting in D, this inequality becomes

$$2D + 3 - D^2 + (D + 1)^3/16 \geq 0.$$

Since the left-hand side factors as $(D+1)(D-7)^2/16$, the inequality always holds.

Third solution: Assuming that $c = \min(a, b, c)$ and by adding $2(ab + bc + ca)$ to both sides, our inequality becomes

$$4c(a+b+c) + 4ab \geq (a+b+c)^2.$$

Since both the given condition and the desired claim are homogeneous, we may assume without loss of generality that $a + b + c = 8$, so our task is to prove that if $ab = 8/c$, then $32c + 4ab \geq 64$. This clearly holds, since for any positive real number c we have $32\left(c + \frac{1}{c}\right) \geq 64$.

USAMO 2.

For any $u, v, w \in (0, 1)$ satisfying $u + v + w = 1$, we may set $x = \frac{u}{v}$, $y = \frac{v}{w}$, and $z = \frac{w}{u}$ to obtain

$$f\left(\frac{u+v}{w}\right) + f\left(\frac{v+w}{u}\right) + f\left(\frac{w+u}{v}\right) = 1,$$

and thus

$$f\left(\frac{1}{w} - 1\right) + f\left(\frac{1}{u} - 1\right) + f\left(\frac{1}{v} - 1\right) = 1.$$

First, let $g : (0, 1) \to (0, \infty)$ be given by $g(x) = f\left(\frac{1}{x} - 1\right)$, so that the above equation reads

$$g(u) + g(v) + g(w) = 1 \text{ for all } u, v, w \in (0, 1) \text{ with } u + v + w = 1.$$

Note that this condition implies actually $g(x) < 1$ for all x.

Next, consider the function $h : (-1/3, 2/3) \to (-1/3, 2/3)$ given by $h(x) = g(x + 1/3) - 1/3$. Then, we have for all $x, y, z \in (-1/3, 2/3)$ with $x + y + z = 0$ that

$$h(x) + h(y) + h(z) = 0. \tag{1}$$

We now establish the key properties of h in a series of claims.

Claim 1. *We have $h(0) = 0$ and for all $x \in (-1/3, 1/3)$, we have $h(-x) = -h(x)$.*

Proof. Setting $x = y = z = 0$ in (1) gives $h(0) = 0$. Then, setting $z = 0$ and $y = -x$ yields $h(-x) = -h(x)$, as long as $x \in (-1/3, 1/3)$. □

Claim 2. *For all $x, y \in (0, 2/3)$ with $x + y < 2/3$, we have $h(x + y) = h(x) + h(y)$.*

Proof. In the case where $x, y < 1/3$, we immediately have from Claim 1 and (1) that

$$h(x) + h(y) = -h(-x) - h(-y) = h(x + y).$$

This allows us to deduce the same property for all x and y satisfying the specified conditions. Indeed, we have

$$h(x+y) = h\left(\frac{x+y}{2}\right) + h\left(\frac{x+y}{2}\right) = 2h\left(\frac{x}{2}\right) + 2h\left(\frac{y}{2}\right) = h(x) + h(y),$$

where we have used the fact that $x + y < 2/3$ implies $x/2, y/2, (x + y)/2$ are all less than $1/3$. □

Claim 3. *For all $x \in (-1/3, 2/3)$, we have $h(x) = 3h(1/3)x$.*

Proof. Note that by repeated applications of Claim 2, we have $h(nx) = nh(x)$ for all real numbers x and positive integers n satisfying $nx \in (0, 2/3)$. Thus, for any positive integers p and q, we have

$$h\left(\frac{p}{q}\right) = 3ph\left(\frac{1}{3q}\right) = \frac{3p}{q}h(1/3),$$

which proves the claim when x is positive and rational.

Next, suppose for sake of contradiction that for some $x \in (0, 2/3)$, we have $|h(x) - 3h(1/3)x| = \delta > 0$. Consider any positive rational $r < x$. Then, we have by Claim 2 that

$$h(x-r) = h(x) - h(r) = h(x) - 3h(1/3)r = h(x) - 3h(1/3)x + 3h(1/3)(x-r).$$

Thus, by taking r sufficiently close to x, we can ensure that

$$x - r < \frac{1}{3 \cdot \lceil 1/\delta \rceil} \text{ and } |h(x-r)| > \frac{\delta}{2}.$$

However, this implies (again by repeated applications of Claim 2)

$$|h(2 \cdot \lceil 1/\delta \rceil \cdot (x-r))| = 2 \cdot \lceil 1/\delta \rceil \cdot |h(x-r)| > 1,$$

which is a contradiction, since h must take values in $(-1/3, 2/3)$.

Thus, we have proved the claim for all positive x in the domain of h. Applying Claim 1, the result extends also to negative x, completing the proof. □

By Claim 3, we conclude that h must take the form $h(x) = cx$, where c is a constant. Moreover, since h maps $(-1/3, 2/3)$ to itself, we must have $c \in [-1/2, 1]$. In terms of f, this means we must have

$$f(x) = g(1/(x+1)) = \frac{1}{3} + c \cdot \left(\frac{1}{x+1} - \frac{1}{3}\right)$$

for some constant $-1/2 \le c \le 1$. And we can readily check that all functions of this form do indeed work, by plugging this expression into the original equation, and choosing u, v, w such that $x = \frac{u}{v}, y = \frac{v}{w}, z = \frac{w}{u}$ as at the beginning of this solution (which can be done whenever $xyz = 1$).

USAMO 3.

The integer m in the statement of the problem is $\varphi(n)$, where φ is the Euler totient function. Throughout our proof we write $p^s \,||\, m$, if s is the greatest power of p that divides m.

We begin with the following lemma:

Lemma 1. If p is a prime and p^s divides n for some positive integer s, then $1^k + 2^k + \cdots + n^k$ is divisible by p^{s-1} for any integer $k \ge 1$.

Proof. Let $\{a_1, a_2, \ldots, a_m\}$ be a complete reduced residue set modulo p^s and $m = p^{s-1}(p-1)$. First we prove by induction on s that for any positive integer k, $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by p^{s-1}. The base case $s = 1$ is true. Suppose the statement holds for some value of s. Consider the statement for $s+1$. Note that

$$\{a_1, \ldots, a_m, p^s + a_1, \ldots, p^s + a_m, \ldots, p^s(p-1) + a_1, \ldots, p^s(p-1) + a_m\}$$

is a complete reduced residue set modulo p^{s+1}. Therefore, the desired sum of k-th powers is equal to

$$a_1^k + \cdots + a_m^k + \cdots + (p^s(p-1) + a_1)^k + \cdots + (p^s(p-1) + a_m)^k \equiv p(a_1^k + \cdots + a_m^k) \equiv 0 \pmod{p^s},$$

where we have used the induction hypothesis for the second congruence. This gives the induction step.

Now we are ready to prove the lemma. Because numbers from 1 to n can be split into blocks of consecutive numbers of length p^s, it is enough to show that $1^k + 2^k + \cdots + (p^s)^k$ is divisible by p^{s-1} for any positive integer k. We use induction on s. The statement is true for $s = 1$. Assume the statement is true for $s - 1$. The sum

$$1^k + 2^k + \cdots + (p^s)^k = a_1^k + a_2^k + \cdots + a_m^k + p^k \left(1^k + 2^k + \cdots + (p^{s-1})^k\right)$$

is divisible by p^{s-1}, because $p^{s-1} \mid a_1^k + \cdots + a_m^k$ and by the induction hypothesis $p^{s-2} \mid 1^k + \cdots + (p^{s-1})^k$. □

Now we proceed to prove a second lemma, from which the statement of the problem will immediately follow:

Lemma 2. Suppose p is a prime dividing n. Let $\{a_1, \ldots, a_m\}$ be a complete reduced residue set mod n, and define s by $p^s \mid\mid m$. Then p^s divides $a_1^k + \cdots + a_m^k$ for any integer $k \geq 1$.

Proof. We fix p, and use induction on the number of prime factors of n (counted by multiplicity) that are different from p. If there are no prime factors other than p, then $n = p^{s+1}$, $m = p^s(p-1)$, and we proved in Lemma 1 that $a_1^k + \cdots + a_m^k$ is divisible by p^s. Now suppose the statement is true for n. We show that it is true for nq, where q is a prime not equal to p.

Case 1. q divides n. We have $p^s \mid\mid \varphi(n)$ and $p^s \mid\mid \varphi(nq)$, because $\varphi(nq) = q\varphi(n)$. If $\{a_1, a_2, \ldots, a_m\}$ is a complete reduced residue set modulo n, then

$$\{a_1, \ldots, a_m, n + a_1, \ldots, n + a_m, \ldots, n(q-1) + a_1, \ldots, n(q-1) + a_m\}$$

is a complete reduced residue set modulo nq. The new sum of k-th powers is equal to

$$a_1^k + \cdots + a_m^k + \cdots + (n(q-1) + a_1)^k + \cdots + (n(q-1) + a_m)^k = mn^k \left(1^k + \cdots + (q-1)^k\right) +$$

$$\binom{k}{1} n^{k-1}\left(1^{k-1}+\cdots+(q-1)^{k-1}\right)(a_1+\cdots+a_m)+\cdots+q(a_1^k+\cdots+a_m^k).$$

This sum is divisible by p^s because $p^s \,||\, m$ and $p^s \mid a_1^j + a_2^j + \cdots + a_m^j$ for any positive integer j.

Case 2. q doesn't divide n. Suppose $p^b \,||\, q-1$, where $b \geq 0$. Note that $\varphi(nq) = \varphi(n)(q-1)$, so $p^s \,||\, \varphi(n)$ and $p^{s+b} \,||\, \varphi(nq)$. Let $\{a_1, \ldots, a_m\}$ be a complete reduced residue set modulo n. The complete reduced residue set modulo nq consists of the mq numbers

$$\{a_1, \ldots, a_m, n+a_1, \ldots, n+a_m, \ldots, n(q-1)+a_1, \ldots, n(q-1)+a_m\}$$

with the m elements $\{qa_1, qa_2, \ldots, qa_m\}$ removed.

The new sum of k-th powers is equal to

$$a_1^k+\cdots+a_m^k+\cdots+(n(q-1)+a_1)^k+\cdots+(n(q-1)+a_m)^k - q^k(a_1^k+\cdots+a_m^k) =$$

$$mn^k\left(1^k+\cdots+(q-1)^k\right)+\binom{k}{1}n^{k-1}\left(1^{k-1}+\cdots+(q-1)^{k-1}\right)(a_1+\cdots+a_m)+\cdots$$

$$\cdots+\binom{k}{k-1}n\left(1+\cdots+(q-1)\right)(a_1^{k-1}+\cdots+a_m^{k-1})+q(a_1^k+\cdots+a_m^k)-q^k(a_1^k+\cdots+a_m^k).$$

Each term

$$\binom{k}{j}n^{k-j}\left(1^{k-j}+\cdots+(q-1)^{k-j}\right)(a_1^j+\cdots+a_m^j),$$

for $0 \leq j \leq k-1$, is divisible by p^{s+b} because $p \mid n^{k-j}$, $p^s \mid a_1^j + \cdots + a_m^j$, and $p^{b-1} \mid 1^{k-j} + \cdots + (q-1)^{k-j}$ by Lemma 1.

Also $(q^k - q)(a_1^k + \cdots + a_m^k)$ is divisible by p^{s+b} because $p^b \mid q-1 \mid q^k - q$ and $p^s \mid a_1^k + \cdots + a_m^k$. Thus p^{s+b} divides our sum and our proof is complete. □

Remark. In fact, one can also show the converse statement: if $\{a_1, a_2, \ldots, a_m\}$ is as defined in the problem and $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k, then every prime that divides m also divides n.

USAMO 4.

The statement is trivial for $p = 2$, so assume $p = 2q+1$ is odd. Create a $p \times p$ table of numbers, as follows:

$$\begin{array}{cccc}
a_1 + 1\cdot 0 & a_2 + 2\cdot 0 & \cdots & a_p + p\cdot 0 \\
a_1 + 1\cdot 1 & a_2 + 2\cdot 1 & \cdots & a_p + p\cdot 1 \\
\vdots & \vdots & \ddots & \vdots \\
a_1 + 1\cdot(p-1) & a_2 + 2\cdot(p-1) & \cdots & a_p + p\cdot(p-1)
\end{array}$$

Interpret all the numbers above modulo p. Examine two different columns, say columns i and j. We claim they agree (modulo p) in exactly one row. Indeed, $a_i + ik \equiv a_j + jk \pmod{p}$ holds if and only if $(i-j)k \equiv a_j - a_i \pmod{p}$. Since p is prime and $i \not\equiv j \pmod{p}$, this condition holds for a unique value of k (namely, $k \equiv (a_j - a_i)(i-j)^{-1} \pmod{p}$).

Thus, there are $\binom{p}{2} = \frac{p(p-1)}{2} = pq$ pairs of integers that are congruent modulo p and lie in the same row of the table. Since there are only p rows, some row, say $\{a_n + nk\}_n$, must contain at most q such pairs.

We claim that this k satisfies our requirement. Indeed, if we read the p entries in this row one by one, each entry either is distinct from all the previous ones, or is congruent to at least one previous entry and thereby completes a pair. Since the latter case happens at most q times, there must be at least $p - q = (p+1)/2$ distinct entries (modulo p), completing the proof.

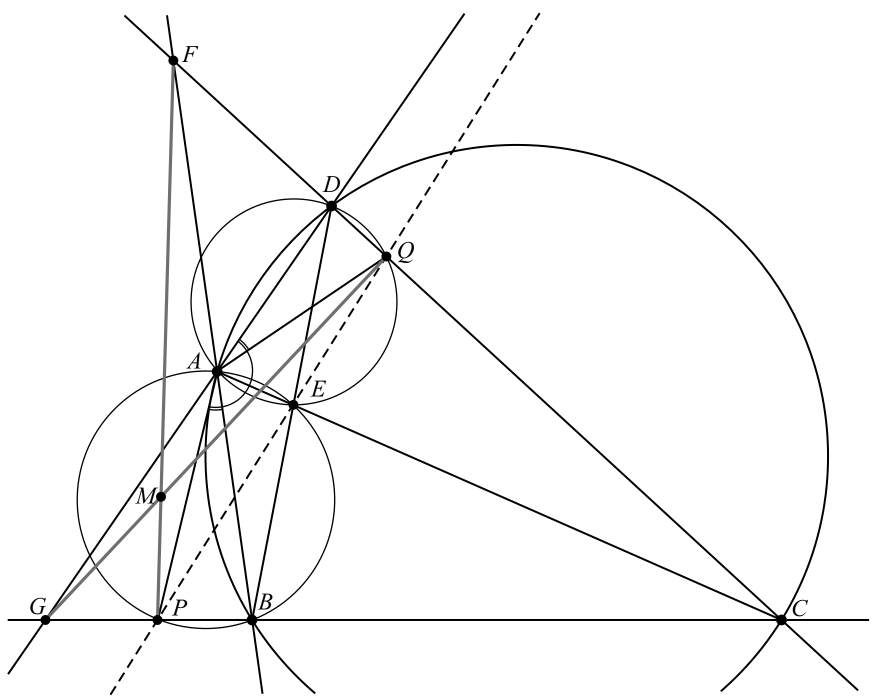
USAMO 5.

First solution. In this particular configuration, we have

$$\angle BAE = \angle BAC = \angle BDC = \angle EDQ = \angle EAQ,$$

$$\angle PAE = 180^\circ - \angle PBE = \angle CBD = \angle CAD = \angle EAD,$$

hence line AC is the internal angle bisector of angles BAQ and PAD. If we could prove that $\angle GAM = \angle MAP$, then line AM would prove to be the external angle bisector of $\angle BAQ$ and hence perpendicular to AC.



Since $\triangle PAF$ and $\triangle QAG$ are related by $\angle PAF = \angle QAG$, it now suffices to prove that

$$\frac{\sin \angle GAM}{\sin \angle MAQ} = \frac{\sin \angle PAM}{\sin \angle MAF}, \tag{1}$$

which is but a repeated application of the Law of Sines. Using the Ratio Lemma in $\triangle PAF$ and $\triangle QAG$, (1) is equivalent to

$$\frac{GM}{MQ}\bigg/\frac{AG}{AQ} = \frac{PM}{MF}\bigg/\frac{AP}{AF}, \qquad \text{i.e.} \qquad \frac{GM}{MP}\cdot\frac{FM}{MQ} = \frac{AF\cdot AG}{AP\cdot AQ}. \tag{2}$$

We now calculate

$$\frac{GM}{MP}\cdot\frac{FM}{MQ} = \frac{\sin \angle GPF}{\sin \angle CGQ}\cdot\frac{\sin \angle GQF}{\sin \angle PFC}$$

$$= \frac{GF\cdot\frac{\sin \angle CGF}{FP}}{CQ\cdot\frac{\sin \angle GCF}{GQ}}\cdot\frac{GF\cdot\frac{\sin \angle GFC}{GQ}}{PC\cdot\frac{\sin \angle GCF}{FP}} = \frac{GF^2}{\sin^2 \angle GCF}\cdot\frac{\sin \angle CGF\cdot\sin \angle GFC}{CQ\cdot CP} = \frac{CF\cdot CG}{CP\cdot CQ}. \tag{3}$$

However, from $\triangle CAP \sim \triangle CBE$ and $\triangle CAQ \sim \triangle CDE$, we have $\frac{CP}{AP} = \frac{CE}{BE}$ and $\frac{CQ}{AQ} = \frac{CE}{DE}$. Hence

$$\frac{CP\cdot CQ}{AP\cdot AQ} = \frac{EC^2}{EB\cdot ED} = \frac{EC^2}{EA\cdot EC}. \tag{4}$$

Further computations give

$$\frac{CF}{AF}\cdot\frac{CG}{AG} = \frac{\sin \angle BAC}{\sin \angle ACD}\cdot\frac{\sin \angle CAD}{\sin \angle ACB} = \frac{\sin \angle BAC}{\sin \angle ACB}\cdot\frac{\sin \angle CAD}{\sin \angle ACD} = \frac{\sin \angle CDB}{\sin \angle BDA}\cdot\frac{CD}{DA} = \frac{EC}{EA}.$$

Combining this with (3) and (4), we finally have

$$\frac{GM}{MP}\cdot\frac{FM}{MQ} = \frac{CF\cdot CG}{CP\cdot CQ} = \frac{CF\cdot CG}{AP\cdot AQ}\cdot\frac{EA}{EC} = \frac{AF\cdot AG}{AP\cdot AQ},$$

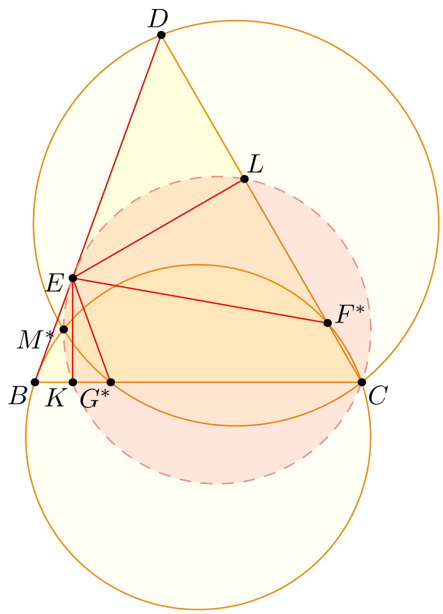
which gives us (2) and therefore (1). This completes the proof.

Second solution. Note by Power of a Point that $CE\cdot CA = CP\cdot CB = CQ\cdot CD$. Thus we can perform an inversion at C swapping these pairs of points. The point G is mapped to a point G^* on ray $\overrightarrow{CB}$ for which QEG^*C is cyclic, but then (using directed angles modulo 180°) we have

$$\angle CG^*E = \angle CQE = \angle CQP = \angle DBC = \angle EBC$$

and so we conclude $EB = EG^*$. Similarly, $ED = EF^*$.

Now, M^*, the image of M, is the intersection (distinct from C) of the circumcircles of $\triangle CG^*D$ and $\triangle CF^*B$; and we wish to show that $\angle EM^*C = 90°$.



Note that triangles M^*BG^* and M^*F^*D are similar, because (again with directed angles)

$$\angle M^*BG^* = \angle M^*BC = \angle M^*F^*C = \angle M^*F^*D$$

and

$$\angle M^*G^*B = \angle M^*G^*C = \angle M^*DC = \angle M^*DF^*.$$

Then, the same spiral similarity that sends $\triangle M^*BG^*$ to $\triangle M^*F^*D$ also maps the midpoint K of $\overline{BG^*}$ to the midpoint L of $\overline{F^*D}$. Consequently, $\angle KM^*L = \angle BM^*F^* = \angle BCF^* = \angle KCL$, which means that M^* lies on the circumcircle of triangle KLC as well. In other words, $ELCKM^*$ is a cyclic pentagon with circumdiameter $\overline{CE}$, implying that $\angle EM^*C = 90°$, as desired.

Third solution. Similarly to the first solution, we begin by noting that

$$\angle GAC = 180° - \angle DAC = 180° - \angle DBC = \angle PBE = 180° - \angle PAE.$$

Thus, AC is the external bisector of $\angle GAP$. By symmetry, AC is also the external bisector of $\angle FAQ$.

Now, for a small $\epsilon > 0$, consider a homothety of factor $1 - \epsilon$ centered at C taking A, G, and Q to A', G', and Q', respectively. Let

$$X = AP \cap A'G', \quad Y = AF \cap A'Q', \quad M' = PF \cap G'Q'.$$

Note that $A'G'Q'$ and APF are perspective from the point C. Thus, by Desargues' theorem, we know that X, Y, and M' are collinear.

Moreover, since AC externally bisects $\angle GAP$ and $G'A' || GA$, it follows that $\triangle AXA'$ is isosceles, and X lies on the perpendicular bisector of $\overline{AA'}$. Similarly, Y also lies on this perpendicular bisector, so the line through M', X, and Y is perpendicular to AC.

Now, taking $\epsilon \to 0$, we see that $M' \to M$ while $X \to A$ and $Y \to A$. It follows that MA is perpendicular to AC, as desired.

USAMO 6.

For any permutation $x = (x_1, x_2, \dots, x_n)$ there is an inverse permutation $y = (y_1, y_2, \dots, y_n)$ where we define $y_j = k$ if and only if $x_k = j$. Then the ratios for the permutation y are $\frac{y_j}{j} = \frac{k}{x_k}$, hence the reciprocals of those for the permutation x. Thus we see that y has distinct ratios if and only if x does. In particular, modulo 2, a_n is the same as the number of permutations x which are equal to their own inverse and have distinct ratios.

A permutation x is its own inverse if and only if it can be formed by breaking the numbers $1, 2, \dots, n$ into singletons and pairs and defining $x_k = k$ if k is a singleton and $x_j = k$, $x_k = j$ if $\{j, k\}$ is a pair. Any singleton gives a ratio of 1, so the distinct ratio condition forces there to be at most one singleton (and hence, there is one singleton if n is odd and none if n is even). Thus we see that $a_n \equiv b_n \pmod 2$, where b_n is the number of ways to form $\lfloor n/2 \rfloor$ disjoint pairs of elements of $\{1, 2, \dots, n\}$ such that no pair forms the same ratio as any other pair. (To avoid ambiguity, interpret "the ratio of a pair" to mean the ratio of its larger to its smaller element.)

Note that for any set of $\lfloor n/2 \rfloor$ disjoint pairs of elements of $\{1, 2, \dots, n\}$, if we have two pairs with the same ratio, say $\{a, b\}$ and $\{c, d\}$ with $a/b = c/d$ (or equivalently $ad = bc$), then replacing $\{a, b\}$ and $\{c, d\}$ with $\{a, c\}$ and $\{b, d\}$ gives another such pairing. Accordingly, refer to a pair of pairs $\{\{a, b\}, \{c, d\}\}$ satisfying $a/b = c/d$ as a *potential swap*. Notice that this move is reversible: we can apply it to potential swap $\{\{a, b\}, \{c, d\}\}$ to get to potential swap $\{\{a, c\}, \{b, d\}\}$, and vice versa.

Now build a graph whose vertices are sets of $\lfloor n/2 \rfloor$ disjoint pairs of elements from $\{1, 2, \dots, n\}$, and where two such pairings are connected by an edge if they differ by simultaneously applying the move above to some non-empty collection of (disjoint) potential swaps. This graph G has $(2\lfloor (n-1)/2 \rfloor + 1)!!$ vertices, hence an odd number of vertices. (The notation $k!!$ means $1 \cdot 3 \cdot 5 \cdots k$, where k is odd. To see why this formula holds, note that for even n, we have $n-1$ possible partners for the element 1 and then $(n-3)!!$ ways to pair up the remaining elements by induction. Then, for odd n, we have n choices for the singleton and $(n-2)!!$ ways to pair up the remaining elements.)

Moreover, b_n is the number of isolated vertices of G, since all pairs in a given pairing have different ratios if and only if there are no potential swaps.

Whenever we are given a set of $m \geq 2$ pairs all with the same ratio, then we can form k disjoint potential swaps from among these m pairs in $\binom{m}{2k}(2k-1)!!$ ways. (For $k = 0$, we define $(-1)!! = 1$.) Hence, the total number of ways to choose disjoint potential swaps from these m is

$$d_m = \sum_k \binom{m}{2k}(2k-1)!! \equiv \sum_k \binom{m}{2k} = 2^{m-1} \pmod 2.$$

Thus the number of choices (including the empty choice of no potential swaps) is even. More generally, if we are given a set of pairs, for which at least two of them (but not necessarily all) have the same ratio, then the number of ways to form disjoint potential swaps from them is again even: we can arrange the pairs into groups of pairs having the same ratio, and the desired number is just the product of d_m, as m ranges over the sizes of the various groups. Thus, for any collection of

$\lfloor n/2 \rfloor$ disjoint pairs from $\{1, 2, \ldots, n\}$, if the pairs do not all have distinct ratios, then the number of ways of constructing zero or more disjoint potential swaps among these pairs is even. Excluding the empty choice, we see that every non-isolated vertex of G has odd degree. Thus, b_n can also be described as the number of vertices of G of even degree.

However, by the handshake lemma, any finite graph G has an even number of vertices of odd degree. Thus, G, having an odd number of vertices, also has an odd number of vertices of even degree. That is, b_n is odd and hence so is a_n.

USAMO 2018 Solution Notes

Compiled by Evan Chen

April 17, 2020

This is an compilation of solutions for the 2018 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\ge a^2+b^2+c^2.$$

2. Find all functions $f\colon (0,\infty)\to(0,\infty)$ such that

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

for all $x,y,z>0$ with $xyz=1$.

3. Let $n\ge 2$ be an integer, and let $\{a_1,\dots,a_m\}$ denote the $m=\varphi(n)$ integers less than n and relatively prime to n. Assume that every prime divisor of m also divides n. Prove that m divides $a_1^k+\dots+a_m^k$ for every positive integer k.

4. Let p be a prime, and let a_1, …, a_p be integers. Show that there exists an integer k such that the numbers

$$a_1+k,\ a_2+2k,\ \dots,\ a_p+pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

5. Let $ABCD$ be a convex cyclic quadrilateral with $E=\overline{AC}\cap\overline{BD}$, $F=\overline{AB}\cap\overline{CD}$, $G=\overline{DA}\cap\overline{BC}$. The circumcircle of $\triangle ABE$ intersects line CB at B and P, and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q. Assume C, B, P, G and C, Q, D, F are collinear in that order. Let $M=\overline{FP}\cap\overline{GQ}$. Prove that $\angle MAC=90^\circ$.

6. Let a_n be the number of permutations $(x_1,\dots,x_n)$ of $(1,\dots,n)$ such that the ratios x_k/k are all distinct. Prove that a_n is odd for all $n\ge 1$.

§1 USAMO 2018/1, proposed by Titu Andreescu

Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2) \ge a^2+b^2+c^2.$$

WLOG let $c=\min(a,b,c)=1$ by scaling. The given inequality becomes equivalent to

$$4ab+2a+2b+3 \ge (a+b)^2 \qquad \forall a+b=4(ab)^{1/3}-1.$$

Now, let $t=(ab)^{1/3}$ and eliminate $a+b$ using the condition, to get

$$4t^3+2(4t-1)+3 \ge (4t-1)^2 \iff 0 \le 4t^3-16t^2+16t = 4t(t-2)^2$$

which solves the problem.

Equality occurs only if $t=2$, meaning $ab=8$ and $a+b=7$, which gives

$$\{a,b\} = \left\{ \frac{7 \pm \sqrt{17}}{2} \right\}$$

with the assumption $c=1$. Scaling gives the curve of equality cases.

§2 USAMO 2018/2, proposed by Titu Andreescu and Nikolai Nikolov

Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

for all $x, y, z > 0$ with $xyz = 1$.

The main part of the problem is to show all solutions are linear. As always, let $x = b/c$, $y = c/a$, $z = a/b$ (classical inequality trick). Then the problem becomes

$$\sum_{\text{cyc}} f\left(\frac{b+c}{a}\right) = 1.$$

Let $f(t) = g(\frac{1}{t+1})$, equivalently $g(s) = f(1/s - 1)$. Thus $g\colon (0,1) \to (0,1)$ which satisfies $\sum_{\text{cyc}} g\left(\frac{a}{a+b+c}\right) = 1$, or equivalently

$$\boxed{g(a)+g(b)+g(c)=1} \qquad \forall a+b+c=1.$$

The rest of the solution is dedicated to solving this equivalent functional equation in g. It is a lot of technical details and I will only outline them (with apologies to the contestants who didn't have that luxury).

Claim — The function g is linear.

Proof. This takes several steps, all of which are technical. We begin by proving g is linear over $[1/8, 3/8]$.

- First, whenever $a + b \le 1$ we have

  $$1 - g(1-(a+b)) = g(a) + g(b) = 2g\left(\frac{a+b}{2}\right).$$

  Hence g obeys Jensen's functional equation over $(0, 1/2)$.

- Define $h\colon [0,1] \to \mathbb{R}$ by $h(t) = g(\frac{2t+1}{8}) - (1-t)\cdot g(1/8) - t\cdot g(3/8)$, then h satisfies Jensen's functional equation too over $[0,1]$. We have also arranged that $h(0) = h(1) = 0$, hence $h(1/2) = 0$ as well.

- Since

  $$h(t) = h(t) + h(1/2) = 2h(t/2 + 1/4) = h(t + 1/2) + h(0) = h(t + 1/2)$$

  for any $t < 1/2$, we find h is periodic modulo $1/2$. It follows one can extend $\widetilde{h}$ by

  $$\widetilde{h}\colon \mathbb{R} \to \mathbb{R} \qquad \text{by} \qquad \widetilde{h}(t) = h(t - \lfloor t \rfloor)$$

  and still satisfy Jensen's functional equation. Because $\widetilde{h}(0) = 0$, it's well-known this implies $\widetilde{h}$ is additive (because $\widetilde{h}(x+y) = 2\widetilde{h}\left((x+y)/2\right) = \widetilde{h}(x) + \widetilde{h}(y)$ for any real numbers x to y).

But $\widetilde{h}$ is bounded below on $[0,1]$ since $g \geq 0$, and since $\widetilde{h}$ is also additive, it follows (well-known) that $\widetilde{h}$ is linear. Thus h is the zero function. So, the function g is linear over $[1/8, 3/8]$; thus we may write $g(x) = kx + \ell$, valid for $1/8 \leq x \leq 3/8$.

Since $3g(1/3) = 1$, it follows $k + 3\ell = 1$.

For $0 < x < 1/8$ we have $g(x) = 2g(0.15) - g(0.3 - x) = 2(0.15k + \ell) - (k(0.3 - x) + \ell) = kx + \ell$, so g is linear over $(0, 3/8)$ as well. Finally, for $3/8 < x < 1$, we use the given equation

$$1 = g\left(\frac{1-x}{2}\right) + g\left(\frac{1-x}{2}\right) + g(x) \implies g(x) = 1 - 2\left(k \cdot \frac{1-x}{2} + \ell\right) = kx + \ell$$

since $\frac{1-x}{2} < \frac{5}{16} < \frac{3}{8}$. Thus g is linear over all. □

Putting this back in, we deduce that $g(x) = kx + \frac{1-k}{3}$ for some $k \in [-1/2, 1]$, and so

$$f(x) = \frac{k}{x+1} + \frac{1-k}{3}$$

for some $k \in [-1/2, 1]$. All such functions work.

§3 USAMO 2018/3, proposed by Ivan Borsenco

Let $n \ge 2$ be an integer, and let $\{a_1, \dots, a_m\}$ denote the $m = \varphi(n)$ integers less than n and relatively prime to n. Assume that every prime divisor of m also divides n. Prove that m divides $a_1^k + \dots + a_m^k$ for every positive integer k.

For brevity, given any n, we let $A(n) = \{1 \le x \le n, \gcd(x, n) = 1\}$ (thus $|A(n)| = \varphi(n)$). Also, let $S(n, k) = \sum_{a \in A(n)} a^k$.

We will prove the stronger statement (which eliminates the hypothesis on n).

Claim — Let $n \ge 2$ be arbitrary (and $k \ge 0$). If $p \mid n$, then

$$\nu_p(\varphi(n)) \le \nu_p(S(n, k)).$$

We start with the special case where n is a prime power.

Lemma

Let p be prime, $e \ge 1$, $k \ge 0$. We always have

$$S(p^e, k) = \sum_{x \in A(p^e)} x^k \equiv 0 \pmod{p^{e-1}}.$$

Proof. For p odd, this follows by taking a primitive root modulo p^{e-1}. In the annoying case $p = 2$, the proof is broken into two cases: for k odd it follows by pairing x with $2^e - x$ and when k is even one can take 5 as a generator of all the quadratic residues as in the $p > 2$ case. □

Corollary

We have $\nu_p(1^k + \dots + t^k) \ge \nu_p(t) - 1$ for any k, t, p.

Proof. Assume $p \mid t$. Handle the terms in that sum divisible by p (by induction) and apply the lemma a bunch of times. □

Now the idea is to add primes q one at a time to n, starting from the base case $n = p^e$. So, formally we proceed by induction on the number of prime divisors of n. We'll also assume $k \ge 1$ in what follows since the base case $k = 0$ is easy.

- First, suppose we want to go from n to nq where $q \nmid n$. In that case $\varphi(nq)$ gained $\nu_p(q-1)$ factors of p and then we need to show $\nu_p(S(nq, k)) \ge \nu_p(\varphi(n)) + \nu_p(q-1)$. The trick is to write

$$A(nq) = \{a + nh \mid a \in A(n) \text{ and } h = 0, \dots, q-1\} \setminus qA(n)$$

and then expand using binomial theorem:

$$\begin{aligned} S(nq,k) &= \sum_{a\in A(n)} \sum_{h=0}^{q-1} (a+nh)^k - \sum_{a\in A(n)} (qa)^k \\ &= -q^k S(n,k) + \sum_{a\in A(n)} \sum_{h=0}^{q-1} \sum_{j=0}^{k} \left[\binom{k}{j} a^{k-j} n^j h^j \right] \\ &= -q^k S(n,k) + \sum_{j=0}^{k} \left[\binom{k}{j} n^j \left(\sum_{a\in A(n)} a^{k-j} \right) \left(\sum_{h=0}^{q-1} h^j \right) \right] \\ &= -q^k S(n,k) + \sum_{j=0}^{k} \left[\binom{k}{j} n^j S(n,k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right] \\ &= (q-q^k) S(n,k) + \sum_{j=1}^{k} \left[\binom{k}{j} n^j S(n,k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right]. \end{aligned}$$

We claim every term here has enough powers of p. For the first term, $S(n,k)$ has at least $\nu_p(\varphi(n))$ factors of p; and we have the $q-q^k$ multiplier out there. For the other terms, we apply induction to $S(n,k-j)$; moreover $\sum_{h=1}^{q-1} h^j$ has at least $\nu_p(q-1)-1$ factors of p by corollary, and we get one more factor of p (at least) from n^j.

- On the other hand, if q already divides n, then this time

$$A(nq) = \{a+nh \mid a \in A(n) \text{ and } h = 0, \dots, q-1\}.$$

and we have no additional burden of p to deal with; the same calculation gives

$$S(nq,k) = qS(n,k) + \sum_{j=1}^{k} \left[\binom{k}{j} n^j S(n,k-j) \left(\sum_{h=1}^{q-1} h^j \right) \right].$$

which certainly has enough factors of p already.

Remark. A curious bit about the problem is that $\nu_p(\varphi(n))$ can exceed $\nu_p(n)$, and so it is not true that the residues of $A(n)$ are well-behaved modulo $\varphi(n)$. For example, the official solutions give the following examples:

- Let $n = 7 \cdot 13$, so $\varphi(n) = 72$. Then $A(91)$ contains nine elements which are $0 \pmod 9$, and only seven elements congruent to $7 \pmod 9$.
- Let $n = 3 \cdot 7 \cdot 13 = 273$, so $\varphi(n) = 144$. Then $A(273)$ contains 26 elements congruent to $1 \pmod 9$ and only 23 elements congruent to $4 \pmod 9$.

Note also $n = 2 \cdot 3 \cdot 7 \cdot 13$ is an example where $\operatorname{rad} \varphi(n) \mid n$.

Remark. The converse of the problem is true too (but asking both parts would make this too long for exam).

§4 USAMO 2018/4, proposed by Ankan Bhattacharya

Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, \; a_2 + 2k, \; \dots, \; a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For each $k = 0, \dots, p-1$ let G_k be the graph on on $\{1, \dots, p\}$ where we join $\{i, j\}$ if and only if

$$a_i + ik \equiv a_j + jk \pmod p \iff k \equiv -\frac{a_i - a_j}{i - j} \pmod p.$$

So we want a graph G_k with at least $\frac{1}{2}p$ connected components.

However, each $\{i, j\}$ appears in exactly one graph G_k, so some graph has at most $\frac{1}{p}\binom{p}{2} = \frac{1}{2}(p-1)$ edges (by "pigeonhole"). This graph has at least $\frac{1}{2}(p+1)$ connected components, as desired.

Remark. Here is an example for $p = 5$ showing equality can occur:

$$\begin{bmatrix} 0 & 0 & 3 & 4 & 3 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}.$$

Ankan Bhattacharya points out more generally that $a_i = i^2$ is sharp in general.

§5 USAMO 2018/5, proposed by Kada Williams

Let $ABCD$ be a convex cyclic quadrilateral with $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AB} \cap \overline{CD}$, $G = \overline{DA} \cap \overline{BC}$. The circumcircle of $\triangle ABE$ intersects line CB at B and P, and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q. Assume C, B, P, G and C, Q, D, F are collinear in that order. Let $M = \overline{FP} \cap \overline{GQ}$. Prove that $\angle MAC = 90°$.

We present three general routes. (The second route, using the fact that $\overline{AC}$ is an angle bisector, has many possible variations.)

First solution (Miquel points) This is indeed a Miquel point problem, but the main idea is to focus on the self-intersecting cyclic quadrilateral $PBQD$ as the key player, rather than on the given $ABCD$.

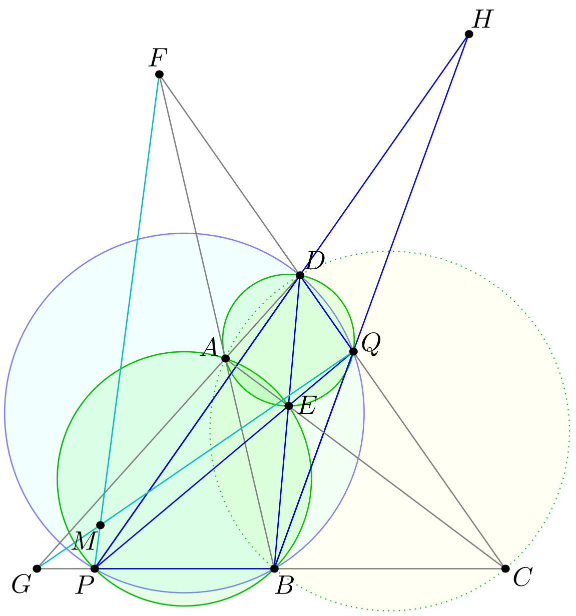
Indeed, we will prove that A is its Miquel point; this follows from the following two claims.

Claim — The self-intersecting quadrilateral $PQDB$ is cyclic.

Proof. By power of a point from C: $CQ \cdot CD = CA \cdot CE = CB \cdot CP$. □

Claim — Point E lies on line PQ.

Proof. $\measuredangle AEP = \measuredangle ABP = \measuredangle ABC = \measuredangle ADC = \measuredangle ADQ = \measuredangle AEQ$. □



To finish, let $H = \overline{PD} \cap \overline{BQ}$. By properties of the Miquel point, we have A is the foot from H to $\overline{CE}$. But also, points M, A, H are collinear by Pappus theorem on $\overline{BPG}$ and $\overline{DQF}$, as desired.

Second solution (projective) We start with a synthetic observation.

Claim — The line $\overline{AC}$ bisects $\angle PAD$ and $\angle BAQ$.

Proof. Angle chase: $\measuredangle PAC = \measuredangle PAE = \measuredangle PBE = \measuredangle CBD = \measuredangle CAD$. $\square$

There are three ways to finish from here:

- (Michael Kural) Suppose the external bisector of $\angle PAD$ and $\angle BAQ$ meet lines BC and DC at X and Y. Then
$$-1 = (GP; XC) = (FD; YC)$$
which is enough to imply that $\overline{XY}$, $\overline{GQ}$, $\overline{PF}$ are concurrent (by so-called prism lemma).

- (Daniel Liu) Alternatively, apply the dual Desargues involution theorem to complete quadrilateral $GQFPCM$, through the point A. This gives that an involutive pairing of
$$(AC, AM)\ (AP, AQ)\ (AG, AF).$$
This is easier to see if we project it onto the line ℓ through C perpendicular to $\overline{AC}$; if we let P', Q', G', F' be the images of the last four lines, we find the involution coincides with negative inversion through C with power $\sqrt{CB' \cdot CQ'}$ which implies that $\overline{AM} \cap \ell$ is an infinity point, as desired.

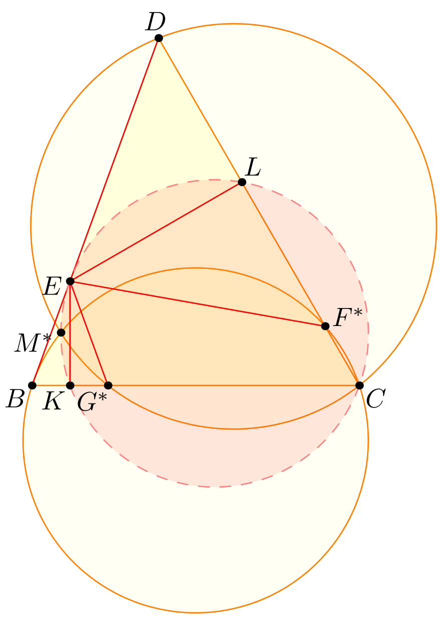
- (Kada Williams) The official solution instead shows the external angle bisector by a long trig calculation.

Third solution (inversion, Andrew Wu) Noting that $CE \cdot CA = CP \cdot CB = CQ \cdot CD$, we perform an inversion at C swapping these pairs of points. The point G is mapped to a point G^* ray CB for which QEG^*C is cyclic, but then

$$\measuredangle CG^*E = \measuredangle CQE = \measuredangle CQP = \measuredangle DBC = \measuredangle CBE$$

and so we conclude $EB = EG^*$. Similarly, $ED = EF^*$.

Finally, $M^* = (CG^*D) \cap (CF^*B) \neq C$, and we wish to show that $\angle EM^*C = 90^\circ$.



Note that M^* is the center of the spiral similarity sending $\overline{BG^*}$ to $\overline{F^*E}$. Hence it also maps the midpoint K of BG^* to the midpoint L of $\overline{F^*E}$. Consequently, M^* lies on the circumcircle KLC as well. In other words, $ELCKM^*$ is a cyclic pentagon with circumdiameter $\overline{CE}$, as desired.

§6 USAMO 2018/6, proposed by Richard Stong

Let a_n be the number of permutations $(x_1, \dots, x_n)$ of $(1, \dots, n)$ such that the ratios x_k/k are all distinct. Prove that a_n is odd for all $n \geq 1$.

This is the official solution; the proof has two main insights.

The first idea:

Lemma

If a permutation x works, so does the inverse permutation.

Thus it suffices to consider permutations x in which all cycles have length at most 2. Of course, there can be at most one fixed point (since that gives the ratio 1), and hence exactly one if n is odd, none if n is even.

We consider the graph K_n such that the edge $\{i, j\}$ is labeled with i/j (for $i < j$). The permutations we're considering are then equivalent to maximal matchings of this K_n. We call such a matching *fantastic* if it has an all of distinct edge labels.

Now the second insight is that if edges ab and cd have the same label for $a < b$ and $c < d$, then so do edges ac and bd. Thus:

Definition. Given a matching $\mathcal{M}$ as above we say the *neighbors* of $\mathcal{M}$ are those other matchings obtained as follows: for each label ℓ, we take some disjoint pairs of edges (possibly none) with label ℓ and apply the above switching operation (in which we replace ab and cd with ac and bd).

This neighborship relation is reflexive, and most importantly it is *symmetric* (because one can simply reverse the moves). But it is not transitive.

The second observation is that:

Claim — The matching $\mathcal{M}$ has an odd number of neighbors (including itself) if and only if it is not fantastic.

Proof. Consider the label ℓ, and assume it appears $n_\ell \geq 1$ times.

If we pick k disjoint pairs and swap them, the number of ways to do this is $\binom{n_\ell}{2k}(2k-1)!!$, and so the total number of ways to perform operations on the edges labeled ℓ is

$$\sum_k \binom{n_\ell}{2k}(2k-1)!! \equiv \sum_k \binom{n_\ell}{2k} = 2^{n_\ell - 1} \pmod 2.$$

This is even if and only if $n_\ell > 1$.

Finally, note that the number of neighbors of $\mathcal{M}$ is the product across all ℓ of the above. So it is odd if and only if each factor is odd, if and only if $n_\ell = 1$ for every ℓ. $\square$

To finish, consider a huge simple graph Γ on all the maximal matchings, with edge relations given by neighbor relation (we don't consider vertices to be connected to themselves). Observe that:

- Fantastic matchings correspond to isolated vertices (of degree zero, with no other neighbors) of Γ.
- The rest of the vertices of Γ have odd degrees (one less than the neighbor count)

- The graph Γ has an even number of vertices of odd degree (this is true for any simple graph, see "handshake lemma").
- The number of vertices of Γ is odd, namely $(2\lceil n/2\rceil - 1)!!$.

This concludes the proof.

2019 USAMO Problems

Contents

Day 1

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

Let $\mathbb{N}$ be the set of positive integers. A function $f:\mathbb{N}\to\mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots))=\frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

Solution

Problem 2

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2+BC^2=AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD=\angle BPC$. Show that line PE bisects $\overline{CD}$.

Solution

Problem 3

Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n)\in K$ whenever $n\in K$.

Solution

Day 2

Problem 4

Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j}\subseteq\{1,2,\ldots,2n\}$, for integers i,j with $0\leq i,j\leq n$, such that: for all $0\leq i,j\leq n$, the set $S_{i,j}$ has $i+j$ elements; and $S_{i,j}\subseteq S_{k,l}$ whenever $0\leq i\leq k\leq n$ and $0\leq j\leq l\leq n$.

Solution

Problem 5

Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Solution

Problem 6

Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

Solution

2019 USAJMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=176&year=2019))	
Preceded by **2018 USAMO**	Followed by **2020 USAMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAJMO Problems and Solutions	

2019 U.S.A. Mathematical Olympiad Solutions

USAMO 1. Answer: $f(1000)$ may be any even positive integer.

To prove this, first, two bits of terminology: we say that f *fixes* the positive integer n if $f(n) = n$; and we write f^k for the function given by iterating f k times.

Now, note that as long as f fixes all odd numbers and f^2 fixes all even numbers (which in particular implies $f(n)$ is even whenever n is), the function f satisfies the equation. Thus, for any even m, we may take $f(1000) = m$, $f(m) = 1000$, and $f(n) = n$ for all other n, and the condition is satisfied.

To see that $f(1000)$ cannot be odd, we show the following two claims.

Claim 1. *f is injective.*

Proof. If $f(a) = f(b)$, then $a^2 = f^{f(a)}(a)f(f(a)) = f^{f(b)}(b)f(f(b)) = b^2$, so $a = b$. □

Claim 2. *f fixes every odd number.*

Proof. We prove this by induction on odd $n \geq 1$.

Assume f fixes each element of $S = \{1, 3, \ldots, n-2\}$ now (allowing $S = \varnothing$ for the base case $n = 1$). Notice that if $f(m) \in S$, then $f(m) = f(f(m))$, implying $m = f(m) \in S$ by injectivity. Applying this repeatedly, we see that if $f^k(m) \in S$ for any $k \geq 1$ then $m \in S$.

Now, we contend $f(f(n)) = n$. Indeed, suppose $f(f(n)) \neq n$. The two numbers $f^{f(n)}(n)$ or $f(f(n))$ have product n^2 and aren't both equal to n, so one of them must be less than n, and also odd, therefore in S. However, by the result of the previous paragraph, this implies $n \in S$, which is a contradiction.

Hence $f(f(n)) = n$. Let $y = f(n)$, so $f(y) = n$. Then we now have

$$y^2 = f^n(y) \cdot y = ny$$

where the step $f^n(y) = n$ used the fact that n is odd. We conclude $n = y$, as desired. □

Now, if $f(n)$ is odd, then $f(n) = f(f(n))$ implying $n = f(n)$. In particular, $f(n)$ cannot be odd for any even n. This completes the proof.

Remark. An argument similar to the one for the second claim shows that in fact f^2 fixes every even number, so the functions identified in the beginning of the solution are actually the only solutions to the equation.

This problem was proposed by Evan Chen.

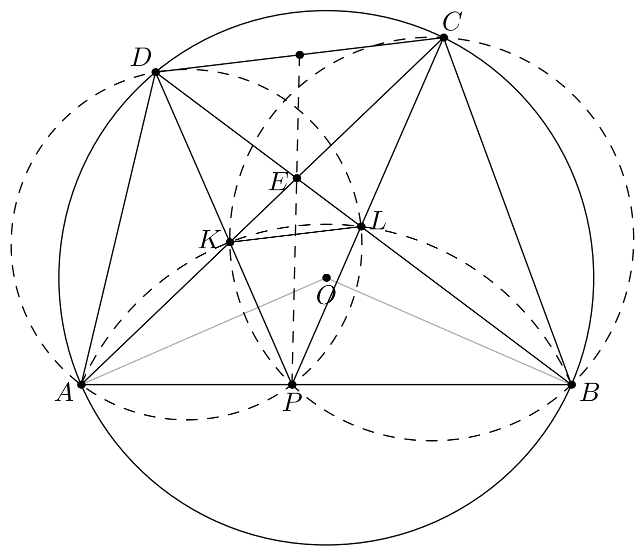
USAMO 2. Note that there can only be one point P on $\overline{AB}$ satisfying the given angle condition, since as P moves from A to B, $\angle APD$ decreases while $\angle BPC$ increases. Consequently, if we can show that there is a single point P on $\overline{AB}$ such that $\angle APD = \angle BPC$ and line PE bisects $\overline{CD}$, then it must coincide with the point in the problem statement, and we will be done. We construct such a point as follows.

Since $AD^2 + BC^2 = AB^2$, there exists a point P on $\overline{AB}$ satisfying

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

Thus $AP/AD = AD/AB$ and $BP/BC = BC/BA$. We then have similar triangles, $\triangle APD \sim \triangle ADB$ and $\triangle BPC \sim \triangle BCA$, from which $\angle APD = \angle ADB = \angle ACB = \angle BPC$.

Now we show that line PE bisects $\overline{CD}$. Define $K = \overline{AC} \cap \overline{PD}$ and $L = \overline{BD} \cap \overline{PC}$.



The quadrilaterals $APLD$ and $BPKC$ are cyclic, because

$$\measuredangle ADL = \measuredangle ACB = \measuredangle BPC = \measuredangle APL$$

and similarly $\measuredangle KCB = \measuredangle KPB$. (The notation $\measuredangle$ here refers to directed angles taken modulo 180°.)

Now the quadrilateral $AKLB$ is also cyclic, because

$$\measuredangle AKB = \measuredangle CKB = \measuredangle CPB$$

and similarly $\measuredangle ALB = \measuredangle APD$, and these are equal.

Now the cyclic quadrilaterals imply $\measuredangle KCD = \measuredangle ABD = \measuredangle ABL = \measuredangle AKL = \measuredangle CKL$, from which we conclude $\overline{CD} \parallel \overline{KL}$. Thus $CDKL$ is a trapezoid whose legs intersect at P and whose diagonals intersect at E. As is well-known (and can be quickly shown using Ceva's theorem), this implies that line PE bisects the bases $\overline{CD}$ and $\overline{KL}$, as desired.

This problem was proposed by Ankan Bhattacharya.

USAMO 3. For an integer x, let $l(x)$ be the length of its base-10 representation. We will show that the only solutions are

- $f(X) = c$, with $c \in K$;
- $f(X) = ax$, with a a power of 10; and
- $f(X) = aX + b$ with a a power of 10, $b \in K$ and $l(b) < l(a)$.

Clearly all of these work. The following lemma is crucial to show that there are no other possibilities:

Lemma 1. *The only $x \in K$ such that $xy \in K$ for all $y \in K$ are the powers of 10.*

Proof. Assume x has the property and is not a power of 10. By induction we get $x^n \in K$ for any n. But, as is well-known, we can find a power of x that starts with any desired finite sequence of digits (in particular, we can find one that starts with 7), which gives a contradiction. For completeness, we give a proof of this fact in the next paragraph.

In general, suppose N is the number representing the desired sequence of digits. Assume that $N+1$ is not a power of 10 (if it is, just replace N by $10N$). Then the claim is that there exist integers $j, k \geq 0$ such that $N \cdot 10^k < x^j < (N+1) \cdot 10^k$. Taking $\log_{10}$ of both sides, this is equivalent to $k + \log_{10}(N) < j \log_{10}(x) < k + \log_{10}(N+1)$. Thus, what we need is

$$\{\log_{10}(N)\} < \{j \log_{10}(x)\} < \{\log_{10}(N+1)\}$$

where $\{\cdots\}$ denotes the fractional part. To see that there is such a j, let M be large enough such that $1/M < \log_{10}(N+1) - \log_{10}(N)$. Divide the unit interval into M equal-sized subintervals. Consider the values of $\{t \log_{10}(x)\}$ for $t = 1, 2, \ldots, M+1$. By the pigeonhole principle, some two of them fall in the same subinterval, and these two cannot be equal since $\log_{10}(x)$ is irrational. Hence, by subtracting, $0 < \{(t'-t)\log_{10}(x)\} \leq 1/M$ for some t', t. If $t' > t$, then consider the multiples $r \cdot \{(t'-t)\log_{10}(x)\}$ (for $r = 1, 2, 3, \ldots$); one of them must eventually lie between $\{\log_{10}(N)\}$ and $\{\log_{10}(N+1)\}$, and then $j = r(t'-t)$ is our desired value. If $t' < t$, then similarly some multiple $r \cdot \{(t'-t)\log_{10}(x)\}$ must lie between $1 - \{\log_{10}(N+1)\}$ and $1 - \{\log_{10}(N)\}$, and the corresponding value $j = r(t-t')$ does the trick. □

Next, write $f(X) = a_d X^d + \ldots + a_1 X + a_0$. First let us prove that $a_i \in K \cup \{0\}$ for all i. By assumption

$$f(10^n) = \sum_{j=0}^{d} a_j 10^{jn} \in K.$$

Choosing $n > \max_j l(a_j)$, the base-10 representation of $f(10^n)$ will consist only of the digits in base 10 of the a_j's and zeroes, hence all nonzero a_j belong to K. A similar argument will yield the crucial:

Lemma 2. *For $0 \leq r \leq s \leq d$, with a_s nonzero, and any $k \in K$, we have $a_s k^{s-r}\binom{s}{r} \in K$.*

Proof. Fix $k \in K$ and pick n large enough. The binomial formula yields

$$f(10^n + k) = \sum_{j=0}^{d} a_j (10^n + k)^j = \sum_{j=0}^{d} a_j \sum_{i=0}^{j} 10^{ni} k^{j-i} \binom{j}{i} = \sum_{r=0}^{d} 10^{nr} \sum_{s=r}^{d} a_s k^{s-r} \binom{s}{r}.$$

Picking $n > \max_{0 \le r \le d} l\left(\sum_{s=r}^{d} a_s k^{s-r}\binom{s}{r}\right)$, we conclude as above that $\sum_{s=r}^{d} a_s k^{s-r}\binom{s}{r} \in K$. Since k was arbitrary, we can replace k by $10^p k$ and so also obtain $\sum_{s=r}^{d} a_s 10^{(s-r)p} k^{s-r}\binom{s}{r} \in K$ for any $k \in K$ and $p \ge 1$. Fixing k and choosing p large enough yields the result, by the same argument. $\square$

Suppose now that $d \ge 2$. Thanks to the lemma (pick $s = d$ and $r = d-1, d-2$) we obtain $da_d k \in K$ and $\binom{d}{2} a_d k^2 \in K$ for all $k \in K$. For $k \in K$ and p large enough we also have $\binom{d}{2} a_d (10^p + k)^2 \in K$ and arguing as above yields $2\binom{d}{2} a_d k \in K$. Applying the first lemma, we deduce that da_d and $2\binom{d}{2} a_d$ are powers of 10, thus their ratio $d-1$ is also a power of 10 and so $d = 2$. Since $da_d = 2a_d$ is a power of 10 and $a_d k^2 = a_d k^2 \binom{d}{2} \in K$ for $k \in K$, we obtain $5k^2 \in K$ for all $k \in K$. Taking $k = 12$ yields a contradiction, since $5 \cdot 12^2 = 720$. This contradiction shows that $d \le 1$.

Consider the case $d = 1$ (the case $d = 0$ being trivial). If $a_0 = 0$, then $a_1 x \in K$ whenever $x \in K$, so the first lemma implies a_1 is a power of 10. Otherwise, the above discussion shows that $a_0, a_1 \in K$ and a_1 is again a power of 10. We claim that the only extra restriction is that $l(a_0) < l(a_1)$. This condition is clearly sufficient. On the other hand, suppose that $l(a_0) \ge l(a_1)$ and let $a_1 = 10^f$, $a_0 = g \cdot 10^e +$ (lower powers). If $g < 7$ picking $x = (7-g) \cdot 10^{f-e} \in K$ yields $a_1 x + a_0 = 7 \cdot 10^e +$ (lower powers), and this is not in K, a contradiction. If $g > 7$, picking $x = (17-g) \cdot 10^{f-e}$, provides the desired contradiction.

This problem was proposed by Titu Andreescu, Vlad Matei, and Cosmin Pohoata.

USAMO 4. The answer is $(2n)! \cdot 2^{n^2}$. It may be helpful to view the sets $S_{i,j}$ as being placed in a grid, as shown in Figure 1. We say a choice of sets $S_{i,j}$ is *valid* if it satisfies the two conditions in the problem. In a slight abuse of terminology, we also apply this definition at times when only some of the $(n+1)^2$ total sets are chosen, with the rest left undetermined (in this case, the conditions are ignored when one or more of the sets involved is undetermined).

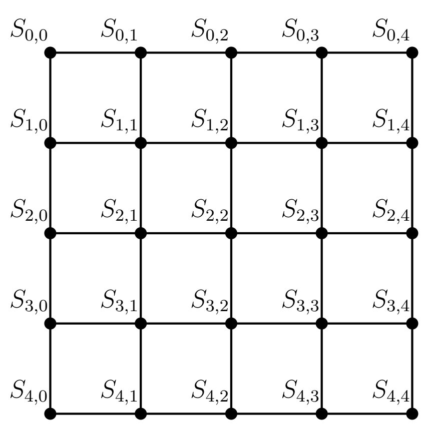


Figure 1: The $S_{i,j}$ arranged in a grid.

Let us define an *initial configuration* to be a valid choice of the sets corresponding to the top row and rightmost column (i.e. sets of the form $S_{0,j}$ and $S_{i,n}$). We first count the number of initial configurations. Since we must have

$$\emptyset = S_{0,0} \subseteq S_{0,1} \subseteq S_{0,2} \subseteq \cdots \subseteq S_{0,n} \subseteq S_{1,n} \subseteq S_{2,n} \subseteq \cdots \subseteq S_{n,n} = \{1, 2, \ldots, 2n\}$$

and recalling that $|S_{i,j}| = i + j$, it follows that the above sequence of sets is obtained by adding different elements of $\{1, 2, \ldots, 2n\}$ one at a time. We may add these $2n$ elements in any order, so the number of initial configurations is $(2n)!$.

Next, for any $0 \le i, j < n$, consider the sets $S_{i,j}$, $S_{i+1,j}$, $S_{i,j+1}$, and $S_{i+1,j+1}$. If they are part of a valid choice, we must have

$$S_{i,j} \subseteq S_{i+1,j+1} \quad \text{and} \quad |S_{i+1,j+1}| = i + j + 2 = |S_{i,j}| + 2,$$

which implies $S_{i+1,j+1} \setminus S_{i,j} = \{x, y\}$ for some distinct $x, y \in \{1, 2, \ldots 2n\}$. Then, $S_{i+1,j}$ and $S_{i,j+1}$ are each either $S_{i,j} \cup \{x\}$ or $S_{i,j} \cup \{y\}$. Let us say the ordered pair (i, j) is *hot* if $S_{i+1,j}$ and $S_{i,j+1}$ are different and *cold* if they are the same. We define a *hot-cold configuration* to consist of a designation of "hot" or "cold" for each of the n^2 ordered pairs (i, j). Clearly, there are 2^{n^2} hot-cold configurations.

Finally, we claim that given any initial configuration and any hot-cold configuration, there is a unique valid choice of sets $S_{i,j}$ for $0 \le i, j \le n$ that agrees with both the initial configuration and the hot-cold configuration. Indeed, we start with the initial configuration of $2n + 1$ sets and choose the remaining sets one by one. We choose them in the following order:

$$\begin{array}{cccc} S_{1,n-1}, & S_{1,n-2}, & \ldots, & S_{1,0}, \\ S_{2,n-1}, & S_{2,n-2}, & \ldots, & S_{2,0}, \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,n-1}, & S_{n,n-2}, & \ldots, & S_{n,0}, \end{array}$$

and we will make sure our choice of sets remains valid at each step. In terms of the grid in Figure 1, this corresponds to going row by row, going right to left in each row.

The above ordering ensures that when we are choosing $S_{i,j}$, the sets $S_{i-1,j}$, $S_{i-1,j+1}$, and $S_{i,j+1}$ have all been chosen already. Based on whether $(i-1, j)$ is required to be hot or cold, we are forced to set $S_{i,j}$ to be $S_{i-1,j} \cup (S_{i,j+1} \setminus S_{i-1,j+1})$ or $S_{i-1,j+1}$, respectively. Moreover, it is straightforward to check that the resulting choice of sets indeed remains valid, because we have ensured that $S_{i-1,j} \subseteq S_{i,j} \subseteq S_{i,j+1}$.

Thus, at the end of the procedure, we arrive at a unique valid choice of all $(n+1)^2$ of the $S_{i,j}$, establishing the claim. It follows that there are $(2n)! \cdot 2^{n^2}$ valid choices in total, as desired.

This problem was proposed by Ricky Liu.

USAMO 5. The answer is all (m, n) such that $m + n$ is a power of 2.

First, if $p \mid m + n$ for some prime $p > 2$, we show that any number $\frac{a}{b}$ written on the board will always have $p \mid a + b$. Indeed, if $p \mid s + t$ and $p \mid u + v$, then the arithmetic mean of $\frac{s}{t}$ and $\frac{u}{v}$ is $\frac{sv+tu}{2tv}$, and we note that

$$sv + tu + 2tv \equiv sv + tu + tv + su \equiv (s + t)(u + v) \equiv 0 \pmod{p}.$$

Since neither t nor v (nor 2) is divisible by p, we see that p still divides the sum of the numerator and denominator after the fraction has been reduced. Similarly, the harmonic mean $\frac{2su}{sv+tu}$ also satisfies the condition.

However, $1 = \frac{1}{1}$, and no prime $p > 2$ divides $1+1$, so no such prime can divide $m+n$ if Evan is to ever be able to write 1 on the board. So we need $m+n$ to be a power of 2.

We now show that Evan can fulfill his goal whenever $m+n$ is a power of 2. In fact, he can do this by only using the arithmetic mean. To show this, first notice that since $m+n$ is a power of 2, if he started with the numbers 0 and $m+n$ on the board, by repeatedly taking arithmetic means, he could eventually produce any integer between 0 and $m+n$; in particular, he could obtain the value m. But if $f(x) = cx+d$ is any linear function, the arithmetic mean of $f(x)$ and $f(y)$ is $f\left(\frac{x+y}{2}\right)$, so by replicating the same sequence of steps that gets to m starting from 0 and $m+n$, he can also get to $f(m)$ starting from $f(0)$ and $f(m+n)$. In particular, by taking $c = \frac{n-m}{mn}$ and $d = \frac{m}{n}$, we have $f(0) = \frac{m}{n}$, $f(m+n) = \frac{n}{m}$, and $f(m) = 1$, so by starting from $\frac{m}{n}$ and $\frac{n}{m}$, Evan can eventually reach 1, as needed.

(Note that the harmonic mean operation is never needed.)

This problem was proposed by Yannick Yao.

USAMO 6. We will first prove that $P(x) = c(x^2+3)$ is a solution for any real number c. This reduces to checking that

$$x(x^2+3) + y(y^2+3) + z(z^2+3) = xyz((x-y)^2 + (y-z)^2 + (z-x)^2 + 9)$$

whenever $2xyz = x+y+z$. Using the factorization of $a^3+b^3+c^3-3abc$ and the relation $x+y+z = 2xyz$, the left-hand side equals

$$\begin{aligned}(x^3+y^3+z^3) + 3(x+y+z) &= 3xyz + (x+y+z)(x^2+y^2+z^2-xy-yz-zx) + 3(x+y+z) \\ &= xyz(9 + (x-y)^2 + (y-z)^2 + (z-x)^2),\end{aligned}$$

as desired.

Next, we prove that these are all solutions of the problem. If $P(x) = c$ is constant, then the left-hand side of the original equation equals $\frac{c(x+y+z)}{xyz} = 2c$, while the right-hand side equals $3c$. This is only possible if $c = 0$. Therefore, if $P(x)$ is a nonzero solution, it is not constant.

If $x \neq 0$, then $y = \frac{1}{x}$ and $z = x + \frac{1}{x}$ satisfy $2xyz = x+y+z$, so

$$xP(x) + \frac{1}{x}P\left(\frac{1}{x}\right) + \left(x+\frac{1}{x}\right)P\left(x+\frac{1}{x}\right) = \left(x+\frac{1}{x}\right)\left(P\left(x-\frac{1}{x}\right) + P(-x) + P\left(\frac{1}{x}\right)\right). \quad (1)$$

Note that the left-hand side is symmetric with respect to $x \to \frac{1}{x}$, thus so must be the right-hand side. It follows that

$$P\left(x-\frac{1}{x}\right) + P(-x) + P\left(\frac{1}{x}\right) = P\left(\frac{1}{x}-x\right) + P(x) + P\left(-\frac{1}{x}\right).$$

This can be rewritten as $Q\left(x-\frac{1}{x}\right)=Q(x)+Q\left(-\frac{1}{x}\right)$, where $Q(X)=P(X)-P(-X)$. We also know $Q(0)=P(0)-P(0)=0$. Hence, as $x\to\infty$,

$$Q(x)-Q\left(x-\frac{1}{x}\right)=-Q\left(-\frac{1}{x}\right)\to 0.$$

Now, if $Q(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ with $n\geq 2$, then the left-hand side of the above equation is of the form $na_nx^{n-2}+$ (lower-order terms), which fails to go to 0 as $x\to\infty$. Thus, Q has degree at most 1, and since $Q(0)=0$, then $Q(x)=2ax$ for some real number a.

Using $P(x)-P(-x)=2ax$, we conclude that the odd part of $P(x)$ is ax, so that $P(x)=ax+f(x^2)$ for a polynomial f with real coefficients. Replacing $P(x)=ax+f(x^2)$ in relation (1) yields

$$ax^2+xf(x^2)+\frac{a}{x^2}+\frac{1}{x}f\left(\frac{1}{x^2}\right)+a\left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)f\left(x^2+2+\frac{1}{x^2}\right)$$

$$=\left(x+\frac{1}{x}\right)\left(f\left(x^2-2+\frac{1}{x^2}\right)+f(x^2)+f\left(\frac{1}{x^2}\right)\right).$$

Multiplying by x, we deduce that $2ax\left(x^2+1+\frac{1}{x^2}\right)$ is a function of x^2, which implies that $a=0$. Letting $t=x^2$, the previous relation becomes

$$f(t)+tf\left(\frac{1}{t}\right)=(t+1)\left(f\left(t+2+\frac{1}{t}\right)-f\left(t-2+\frac{1}{t}\right)\right).$$

Write $f(t)=b_nt^n+...+b_0$ with $b_n\neq 0$ and suppose that $n>1$. The largest term on the left-hand side is b_nt^n. However, the largest term on the right-hand side is the same as the largest term of

$$t(f(t+2)-f(t-2)),$$

which is $4b_nt^n$. This contradicts $b_n\neq 0$, which means $f(t)$ must be linear. We may check, if $f(t)=cx+d$ in the last formula, that $d=3c$. Therefore, $f(x)=c(x+3)$, so $P(x)=f(x^2)=c(x^2+3)$.

This problem was proposed by Titu Andreescu and Gabriel Dospinescu.

USAMO 2019 Solution Notes

COMPILED BY EVAN CHEN

April 17, 2020

This is an compilation of solutions for the 2019 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies
$$\underbrace{f(f(\dots f}_{f(n)\text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$
for all positive integers n. What are all possible values of $f(1000)$?

2. Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

3. Let K be the set of positive integers not containing the decimal digit 7. Determine all polynomials $f(x)$ with nonnegative coefficients such that $f(x) \in K$ for all $x \in K$.

4. Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that
   - $|S_{ij}| = i + j$, and
   - $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

5. Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac12(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m, n) can Evan write 1 on the board in finitely many steps?

6. Find all polynomials P with real coefficients such that
$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$
for all nonzero real numbers x, y, z obeying $2xyz = x + y + z$.

§1 USAMO 2019/1, proposed by Evan Chen

A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\underbrace{f(f(\dots f}_{f(n) \text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. What are all possible values of $f(1000)$?

Actually, we classify all such functions: f can be any function which fixes odd integers and acts as an involution on the even integers. In particular, $f(1000)$ may be any even integer.

It's easy to check that these all work, so now we check they are the only solutions.

Claim — f is injective.

Proof. If $f(a) = f(b)$, then $a^2 = f^{f(a)}(a)f(f(a)) = f^{f(b)}(b)f(f(b)) = b^2$, so $a = b$. □

Claim — f fixes the odd integers.

Proof. We prove this by induction on odd $n \ge 1$.

Assume f fixes $S = \{1, 3, \dots, n-2\}$ now (allowing $S = \varnothing$ for $n = 1$). Now we have that

$$f^{f(n)}(n) \cdot f^2(n) = n^2.$$

However, neither of the two factors on the left-hand side can be in S since f was injective. Therefore they must both be n, and we have $f^2(n) = n$.

Now let $y = f(n)$, so $f(y) = n$. Substituting y into the given yields

$$y^2 = f^n(y) \cdot y = f^{n+1}(n) \cdot y = ny$$

since $n + 1$ is even. We conclude $n = y$, as desired. □

Thus, f maps even integers to even integers. In light of this, we may let $g = f(f(n))$ (which is also injective), so we conclude that

$$g^{f(n)/2}(n)g(n) = n^2 \qquad \text{for } n = 2, 4, \dots.$$

Claim — The function g is the identity function.

Proof. The proof is similar to the earlier proof of the claim. Note that g fixes the odd integers already. We proceed by induction to show g fixes the even integers; so assume g fixes the set $S = \{1, 2, \dots, n-1\}$, for some even integer $n \ge 2$. In the equation

$$g^{f(n)/2}(n) \cdot g(n) = n^2$$

neither of the two factors may be less than n. So they must both be n. □

These three claims imply that the solutions we claimed earlier are the only ones.

> **Remark.** The last claim is not necessary to solve the problem; after realizing f and injective fixes the odd integers, this answers the question about the values of $f(1000)$. However, we chose to present the "full" solution anyways.

> **Remark.** After noting f is injective, another approach is outlined below. Starting from any n, consider the sequence
> $$n,\ f(n),\ f(f(n)),$$
> and so on. We may let m be the smallest term of the sequence; then $m^2 = f(f(m)) \cdot f^{f(m)}(m)$ which forces $f(f(m)) = f^{f(m)}(m) = m$ by minimality. Thus the sequence is 2-periodic. Therefore, $f(f(n)) = n$ always holds, which is enough to finish.

Authorship comments I will tell you a great story about this problem. Two days before the start of grading of USAMO 2017, I had a dream that I was grading a functional equation. When I woke up, I wrote it down, and it was

$$f^{f(n)}(n) = \frac{n^2}{f(f(n))}.$$

You can guess the rest of the story (and imagine how surprised I was the solution set was interesting). I guess some dreams do come true, huh?

§2 USAMO 2019/2, proposed by Ankan Bhattacharya

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Here are three solutions. The first two are similar although the first one makes use of symmedians. The last solution by inversion is more advanced.

First solution using symmedians We define point P to obey

$$\frac{AP}{BP} = \frac{AD^2}{BC^2} = \frac{AE^2}{BE^2}$$

so that $\overline{PE}$ is the E-symmedian of $\triangle EAB$, therefore the E-median of $\triangle ECD$.

Now, note that

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

This implies $\triangle APD \sim \triangle ABD$ and $\triangle BPC \sim \triangle BDP$. Thus

$$\measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP$$

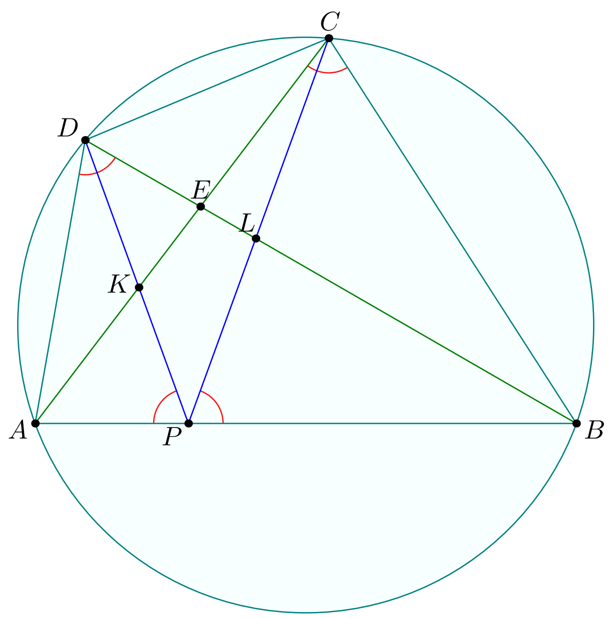
and so P satisfies the condition as in the statement (and is the unique point to do so), as needed.

Second solution using only angle chasing (by proposer) We again re-define P to obey $AD^2 = AP \cdot AB$ and $BC^2 = BP \cdot BA$. As before, this gives $\triangle APD \sim \triangle ABD$ and $\triangle BPC \sim \triangle BDP$ and so we let

$$\theta \stackrel{\text{def}}{=} \measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP.$$

Our goal is to now show $\overline{PE}$ bisects $\overline{CD}$.

Let $K = \overline{AC} \cap \overline{PD}$ and $L = \overline{AD} \cap \overline{PC}$. Since $\measuredangle KPA = \theta = \measuredangle ACB$, quadrilateral $BPKC$ is cyclic. Similarly, so is $APLD$.



Finally $AKLB$ is cyclic since

$$\measuredangle BKA = \measuredangle BKC = \measuredangle BPC = \theta = \measuredangle DPA = \measuredangle DLA = \measuredangle BLA.$$

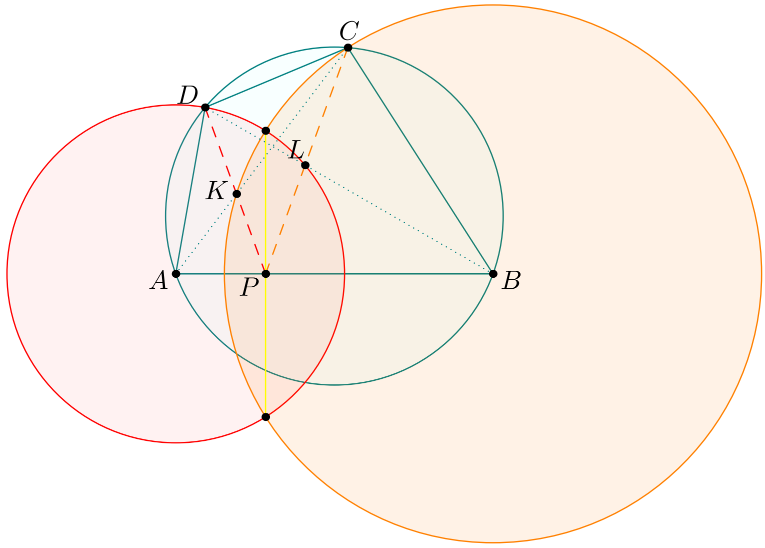
This implies $\measuredangle CKL = \measuredangle LBA = \measuredangle DCK$, so $\overline{KL} \parallel \overline{BC}$. Then PE bisects $\overline{BC}$ by Ceva's theorem on $\triangle PCD$.

Third solution (using inversion) By hypothesis, the circle ω_a centered at A with radius AD is orthogonal to the circle ω_b centered at B with radius BC. For brevity, we let $\mathbf{I}_a$ and $\mathbf{I}_b$ denote inversion with respect to ω_a and ω_b.

We let P denote the intersection of $\overline{AB}$ with the radical axis of ω_a and ω_b; hence $P = \mathbf{I}_a(B) = \mathbf{I}_b(A)$. This already implies that

$$\measuredangle DPA \overset{\mathbf{I}_a}{=} \measuredangle ADB = \measuredangle ACB \overset{\mathbf{I}_b}{=} \measuredangle BPC$$

so P satisfies the angle condition.



Claim — The point $K = \mathbf{I}_a(C)$ lies on ω_b and $\overline{DP}$. Similarly $L = \mathbf{I}_b(D)$ lies on ω_a and $\overline{CP}$.

Proof. The first assertion follows from the fact that ω_b is orthogonal to ω_a. For the other, since (BCD) passes through A, it follows $P = \mathbf{I}_a(B)$, $K = \mathbf{I}_a(C)$, and $D = \mathbf{I}_a(D)$ are collinear. $\square$

Finally, since C, L, P are collinear, we get A is concyclic with $K = \mathbf{I}_a(C)$, $L = \mathbf{I}_a(L)$, $B = \mathbf{I}_a(B)$, i.e. that $AKLB$ is cyclic. So $\overline{KL} \parallel \overline{CD}$ by Reim's theorem, and hence $\overline{PE}$ bisects $\overline{CD}$ by Ceva's theorem.

§3 USAMO 2019/3, proposed by Titu Andreescu, Vlad Matei, and Cosmin Pohoata

Let K be the set of positive integers not containing the decimal digit 7. Determine all polynomials $f(x)$ with nonnegative coefficients such that $f(x) \in K$ for all $x \in K$.

The answer is only the obvious ones: $f(x) = 10^e x$, $f(x) = k$, and $f(x) = 10^e x + k$, for any choice of $k \in K$ and $e > \log_{10} k$ (with $e \geq 0$).

Now assume f satisfies $f(K) \subseteq K$; such polynomials will be called *stable*. We first prove the following claim which reduces the problem to the study of monomials.

Lemma (Reduction to monomials)
If $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ is stable, then each monomial a_0, $a_1 x$, $a_2 x^2$, … is stable.

Proof. For any $x \in K$, plug in $f(10^e x)$ for large enough e: the decimal representation of f will contain a_0, $a_1 x$, $a_2 x^2$ with some zeros padded in between. □

Let's tackle the linear case next. Here is an ugly but economical proof.

Claim (Linear classification) — If $f(x) = cx$ is stable, then $c = 10^e$ for some nonnegative integer e.

Proof. We will show when $c \neq 10^e$ then we can find $x \in K$ such that cx starts with the digit 7. This can actually be done with the following explicit cases in terms of how c starts in decimal notation:

- For $9 \cdot 10^e \leq c < 10 \cdot 10^e$, pick $x = 8$.
- For $8 \cdot 10^e \leq c < 9 \cdot 10^e$, pick $x = 88$.
- For $7 \cdot 10^e \leq c < 8 \cdot 10^e$, pick $x = 1$.
- For $4.4 \cdot 10^e \leq c < 7 \cdot 10^e$, pick $11 \leq x \leq 16$.
- For $2.7 \cdot 10^e \leq c < 4.4 \cdot 10^e$, pick $18 \leq x \leq 26$.
- For $2 \cdot 10^e \leq c < 2.7 \cdot 10^e$, pick $28 \leq x \leq 36$.
- For $1.6 \cdot 10^e \leq c < 2 \cdot 10^e$, pick $38 \leq x \leq 46$.
- For $1.3 \cdot 10^e \leq c < 1.6 \cdot 10^e$, pick $48 \leq x \leq 56$.
- For $1.1 \cdot 10^e \leq c < 1.3 \cdot 10^e$, pick $58 \leq x \leq 66$.
- For $1 \cdot 10^e \leq c < 1.1 \cdot 10^e$, pick $x = 699 \dots 9$ for suitably many 9's. □

The hardest part of the problem is the case where $\deg f > 1$. We claim that no solutions exist then:

Claim (Higher-degree classification) — No monomial of the form $f(x) = cx^d$ is stable for any $d > 1$.

Proof. Note that $f(10x+3)$ is stable too. Thus

$$f(10x+3) = 3^d + 10d \cdot 3^{d-1}x + 100\binom{d}{2} \cdot 3^{d-1}x^2 + \dots$$

is stable. By applying the lemma the linear monomial $10d \cdot 3^{d-1}x$ is stable, so $10d \cdot 3^{d-1}$ is a power of 10, which can only happen if $d = 1$. $\square$

Thus the only nonconstant stable polynomials with nonnegative coefficients must be of the form $f(x) = 10^e x + k$ for $e \geq 0$. It is straightforward to show we then need $k < 10^e$ and this finishes the proof.

Remark. The official solution replaces the proof for $f(x) = cx$ with Kronecker density. From $f(1) = c \in K$, we get $f(c) = c^2 \in K$, et cetera and hence $c^n \in K$. But it is known that when c is not a power of 10, some power of c starts with any specified prefix.

§4 USAMO 2019/4, proposed by Ricky Liu

Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that

- $|S_{ij}| = i + j$, and
- $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$. First, we note that $\varnothing = S_{00} \subsetneq S_{01} \subsetneq \dots \subsetneq S_{nn} = \{1, \dots, 2n\}$ and thus multiplying by $(2n)!$ we may as well assume $S_{0i} = \{1, \dots, i\}$ and $S_{in} = \{1, \dots, n+i\}$. We illustrate this situation by placing the sets in a grid, as below for $n = 4$; our goal is to fill in the rest of the grid.

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & & & & \\ 12 & & & & \\ 1 & & & & \\ \varnothing & & & & \end{bmatrix}$$

We claim the number of ways to do so is 2^{n^2}. In fact, more strongly even the partial fillings are given exactly by powers of 2.

Claim — Fix a choice T of cells we wish to fill in, such that whenever a cell is in T, so are all the cells above and left of it. (In other words, T is a Young tableau.) The number of ways to fill in these cells with sets satisfying the inclusion conditions is $2^{|T|}$.

An example is shown below, with an indeterminate set marked in red (and the rest of T marked in blue).

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & 1234 & 12346 & 123467 & \\ 12 & 124 & 1234 \text{ or } 1246 & & \\ 1 & 12 & & & \\ \varnothing & 2 & & & \end{bmatrix}$$

Proof. The proof is by induction on $|T|$, with $|T| = 0$ being vacuous.

Now suppose we have a corner $\begin{bmatrix} B & C \\ A & S \end{bmatrix}$ where A, B, C are fixed and S is to be chosen. Then we may write $B = A \cup \{x\}$ and $C = A \cup \{x, y\}$ for $x, y \notin A$. Then the two choices of S are $A \cup \{x\}$ (i.e. B) and $A \cup \{y\}$, and both of them are seen to be valid.

In this way, we gain a factor of 2 any time we add one cell as above to T. Since we can achieve any Young tableau in this way, the induction is complete. □

§5 USAMO 2019/5, proposed by Yannick Yao

Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m,n) can Evan write 1 on the board in finitely many steps?

We claim this is possible if and only $m+n$ is a power of 2. Let $q = m/n$, so the numbers on the board are q and $1/q$.

Impossibility: The main idea is the following.

Claim — Suppose p is an odd prime. Then if the initial numbers on the board are $-1 \pmod p$, then all numbers on the board are $-1 \pmod p$.

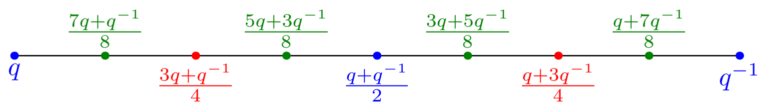
Proof. Let $a \equiv b \equiv -1 \pmod p$. Note that $2 \not\equiv 0 \pmod p$ and $a+b \equiv -2 \not\equiv 0 \pmod p$. Thus $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$ both make sense modulo p and are equal to $-1 \pmod p$. □

Thus if there exists *any* odd prime divisor p of $m+n$ (implying $p \nmid mn$), then

$$q \equiv \frac{1}{q} \equiv -1 \pmod p.$$

and hence all numbers will be $-1 \pmod p$ forever. This implies that it's impossible to write 1, whenever $m+n$ is divisible by some odd prime.

Construction: Conversely, suppose $m+n$ is a power of 2. We will actually construct 1 without even using the harmonic mean.



Note that

$$\frac{n}{m+n} \cdot q + \frac{m}{m+n} \cdot \frac{1}{q} = 1$$

and obviously by taking appropriate midpoints (in a binary fashion) we can achieve this using arithmetic mean alone.

§6 USAMO 2019/6, proposed by Titu Andreescu and Gabriel Dospinescu

Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

for all nonzero real numbers x, y, z obeying $2xyz = x + y + z$.

The given can be rewritten as saying that

$$\begin{aligned} Q(x,y,z) \overset{\text{def}}{=} & \; xP(x) + yP(y) + zP(z) \\ & - xyz\left(P(x-y) + P(y-z) + P(z-x)\right) \end{aligned}$$

is a polynomial vanishing whenever $xyz \neq 0$ and $2xyz = x + y + z$, for real numbers x, y, z.

Claim — This means $Q(x,y,z)$ vanishes also for any complex numbers x, y, z obeying $2xyz = x + y + z$.

Proof. Indeed, this means that the rational function

$$R(x,y) \overset{\text{def}}{=} Q\left(x, y, \frac{x+y}{2xy-1}\right)$$

vanishes for any real numbers x and y such that $xy \neq \frac{1}{2}$, $x \neq 0$, $y \neq 0$, $x + y \neq 0$. This can only occur if R is identically zero as a rational function with real coefficients. If we then regard R as having complex coefficients, the conclusion then follows. □

Remark (Algebraic geometry digression on real dimension). Note here we use in an essential way that z can be solved for in terms of x and y. If $s(x,y,z) = 2xyz - (x+y+z)$ is replaced with some general condition, the result may become false; e.g. we would certainly not expect the result to hold when $s(x,y,z) = x^2 + y^2 + z^2 - (xy + yz + zx)$ since for real numbers $s = 0$ only when $x = y = z$!

The general condition we need here is that $s(x,y,z) = 0$ should have "real dimension two". Here is a proof using this language, in our situation.

Let $M \subset \mathbb{R}^3$ be the surface $s = 0$. We first contend M is two-dimensional manifold. Indeed, the gradient $\nabla s = \langle 2yz - 1, 2zx - 1, 2xy - 1 \rangle$ vanishes only at the points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ where the $\pm$ signs are all taken to be the same. These points do not lie on M, so the result follows by the *regular value theorem*. In particular the topological closure of points on M with $xyz \neq 0$ is all of M itself; so Q vanishes on all of M.

If we now identify M with the semi-algebraic set consisting of maximal ideals $(x-a, y-b, z-c)$ in $\operatorname{Spec} \mathbb{R}[x,y,z]$ satisfying $2abc = a + b + c$, then we have real dimension two, and thus the Zariski closure of M is a two-dimensional closed subset of $\operatorname{Spec} \mathbb{R}[x,y,z]$. Thus it must be $Z = \mathcal{V}(2xyz - (x+y+z))$, since this Z is an irreducible two-dimensional closed subset (say, by *Krull's principal ideal theorem*) containing M. Now Q is a global section vanishing on all of Z, therefore Q is contained in the (radical, principal) ideal $(2xyz - (x+y+z))$ as needed. So it is actually divisible by $2xyz - (x+y+z)$ as desired.

Now we regard P and Q as complex polynomials instead. First, note that substituting $(x,y,z) = (t, -t, 0)$ implies P is even. We then substitute

$$(x,y,z) = \left(x, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right)$$

to get

$$\begin{aligned} &xP(x) + \frac{i}{\sqrt{2}} \left(P\left(\frac{i}{\sqrt{2}}\right) - P\left(\frac{-i}{\sqrt{2}}\right) \right) \\ &= \frac{1}{2}x \left(P(x - i/\sqrt{2}) + P(x + i/\sqrt{2}) + P(\sqrt{2}i) \right) \end{aligned}$$

which in particular implies that

$$P\left(x + \frac{i}{\sqrt{2}}\right) + P\left(x - \frac{i}{\sqrt{2}}\right) - 2P(x) \equiv P(\sqrt{2}i)$$

identically in x. The left-hand side is a second-order finite difference in x (up to scaling the argument), and the right-hand side is constant, so this implies $\deg P \le 2$.

Since P is even and $\deg P \le 2$, we must have $P(x) = cx^2 + d$ for some real numbers c and d. A quick check now gives the answer $P(x) = c(x^2 + 3)$ which all work.

2020 USAMO Problems

Contents

Day 1

Problem 1

Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Solution

Problem 2

An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

● The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)

● No two beams have intersecting interiors.

● The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

Solution

Problem 3

Let p be an odd prime. An integer x is called a *quadratic non-residue* if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \leq a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

Solution

Day 2

Problem 4

Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Solution

Problem 5

A finite set S of points in the coordinate plane is called *overdetermined* if $|S| \geq 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S| - 2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \geq 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is not overdetermined, but has k overdetermined subsets.

Solution

Problem 6

Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be $2n$ real numbers such that

$$0 = x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$$
$$\text{and } 1 = x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

Solution

2020 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=2020))	
Preceded by **2019 USAMO**	Followed by **2021 USAMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAMO Problems and Solutions	

USAMO 2020 Solution Notes

Compiled by Evan Chen

January 1, 2021

This is an compilation of solutions for the 2020 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

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What is the smallest positive number of beams that can be placed to satisfy these conditions?

3. Let p be an odd prime. An integer x is called a *quadratic non-residue* if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

4. Suppose that (a_1, b_1), (a_2, b_2), $\dots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

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For each integer $n \ge 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is *not* overdetermined, but has k overdetermined subsets.

6. Let $n \ge 2$ be an integer. Let $x_1 \ge x_2 \ge \cdots \ge x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n$ be $2n$ real numbers such that

$$\begin{aligned} 0 &= x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n, \\ \text{and} \quad 1 &= x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2. \end{aligned}$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \ge \frac{2}{\sqrt{n-1}}.$$

§1 USAMO 2020/1, proposed by Zuming Feng

Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

We prove $[OO_1O_2] \ge \frac{1}{4}[ABC]$, with equality if and only if $\overline{CX} \perp \overline{AB}$.

First approach (Bobby Shen) We use two simultaneous inequalities:

- Let M and N be the midpoints of CX and DX. Then MN equals the length of the O-altitude of $\triangle OO_1O_2$, since $\overline{O_1O_2}$ and $\overline{DX}$ meet at N at a right angle. Moreover, we have
$$MN = \frac{1}{2}CD \ge \frac{1}{2}h_a$$
where h_a denotes the A-altitude.
- The projection of O_1O_2 onto line AB has length exactly $AB/2$. Thus
$$O_1O_2 \ge \frac{1}{2}AB.$$

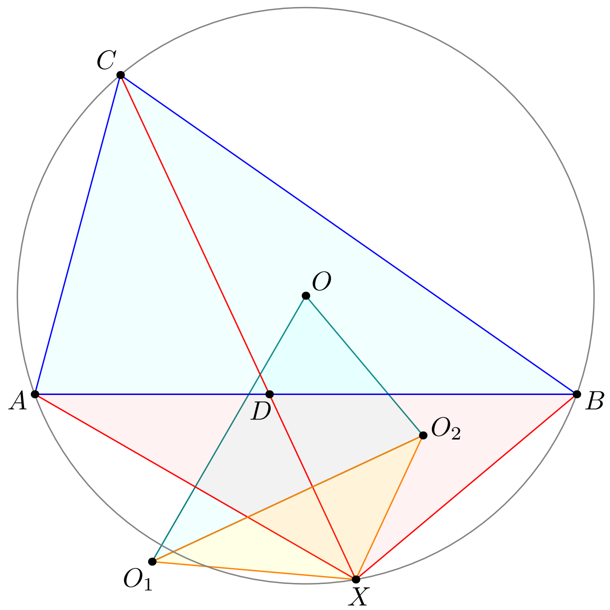
So, we find
$$[OO_1O_2] = \frac{1}{2} \cdot MN \cdot O_1O_2 \ge \frac{1}{8}h_a \cdot AB = \frac{1}{4}[ABC].$$
Note that equality occurs in both cases if and only if $\overline{CX} \perp \overline{AB}$. So the area is minimized exactly when this occurs.

Second approach (Evan's solution) We need two claims.

Claim — We have $\triangle OO_1O_2 \sim \triangle CBA$, with opposite orientation.

Proof. Notice that $\overline{OO_1} \perp \overline{AX}$ and $\overline{O_1O_2} \perp \overline{CX}$, so $\measuredangle OO_1O_2 = \measuredangle AXC = \measuredangle ABC$. Similarly $\measuredangle OO_2O_1 = \measuredangle BAC$. □

Therefore, the problem is equivalent to minimizing O_1O_2.



Claim (Salmon theorem) — We have $\triangle XO_1O_2 \sim \triangle XAB$.

Proof. It follows from the fact that $\triangle AO_1X \sim \triangle BO_2X$ (since $\measuredangle ADX = \measuredangle XDB \implies \measuredangle XO_1A = \measuredangle XO_2B$) and that spiral similarities come in pairs. □

Let $\theta = \angle ADX$. The ratio of similarity in the previous claim is equal to $\frac{XO_1}{XA} = \frac{1}{2\sin\theta}$. In other words,

$$O_1O_2 = \frac{AB}{2\sin\theta}.$$

This is minimized when $\theta = 90^\circ$, in which case $O_1O_2 = AB/2$ and $[OO_1O_2] = \frac{1}{4}[ABC]$. This completes the solution.

§2 USAMO 2020/2, proposed by Alex Zhai

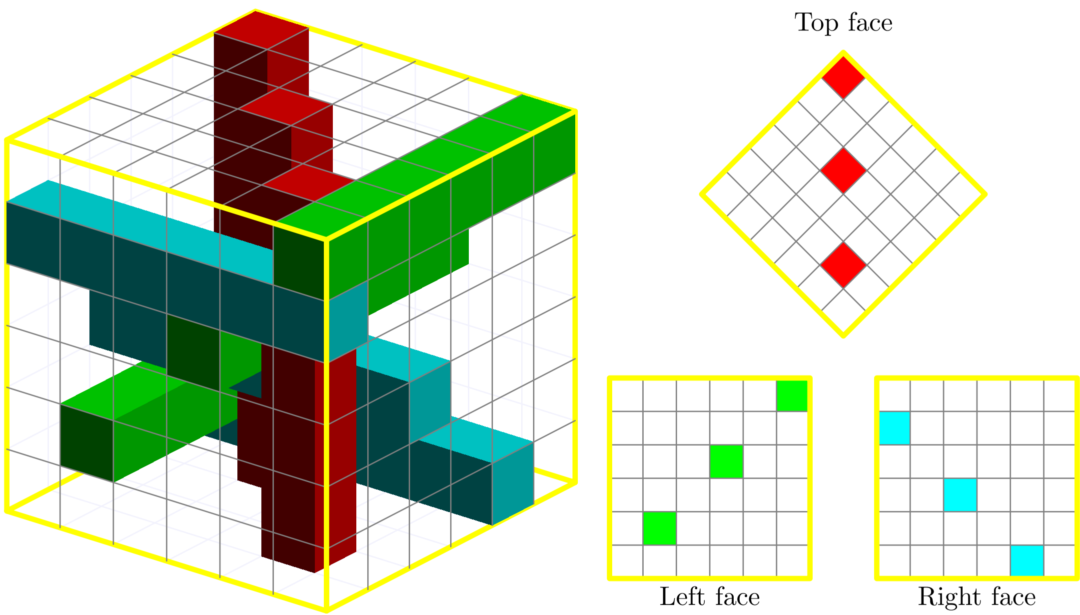
An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

Answer: 3030 beams.

Construction: We first give a construction with $3n/2$ beams for any $n \times n \times n$ box, where n is an even integer. Shown below is the construction for $n = 6$, which generalizes. (The left figure shows the cube in 3d; the right figure shows a direct view of the three visible faces.)



To be explicit, impose coordinate axes such that one corner of the cube is the origin. We specify a beam by two opposite corners. The $3n/2$ beams come in three directions, $n/2$ in each direction:

- $(0,0,0) \to (1,1,n)$, $(2,2,0) \to (3,3,n)$, $(4,4,0) \to (5,5,n)$, and so on;
- $(1,0,0) \to (2,n,1)$, $(3,0,2) \to (4,n,3)$, $(5,0,4) \to (6,n,5)$, and so on;
- $(0,1,1) \to (n,2,2)$, $(0,3,3) \to (n,4,4)$, $(0,5,5) \to (n,6,6)$, and so on.

This gives the figure we drew earlier and shows 3030 beams is possible.

Necessity: We now show at least $3n/2$ beams are necessary. Maintain coordinates, and call the beams x-beams, y-beams, z-beams according to which plane their long edges are perpendicular too. Let N_x, N_y, N_z be the number of these.

Claim — If $\min(N_x, N_y, N_z) = 0$, then at least n^2 beams are needed.

Proof. Assume WLOG that $N_z = 0$. Orient the cube so the z-plane touches the ground. Then each of the n layers of the cube (from top to bottom) must be completely filled, and so at least n^2 beams are necessary, $\square$

We henceforth assume $\min(N_x, N_y, N_z) > 0$.

Claim — If $N_z > 0$, then we have $N_x + N_y \geq n$.

Proof. Again orient the cube so the z-plane touches the ground. We see that for each of the n layers of the cube (from top to bottom), there is at least one x-beam or y-beam. (Pictorially, some of the x and y beams form a "staircase".) This completes the proof. $\square$

Proceeding in a similar fashion, we arrive at the three relations

$$\begin{aligned} N_x + N_y &\geq n \\ N_y + N_z &\geq n \\ N_z + N_x &\geq n. \end{aligned}$$

Summing gives $N_x + N_y + N_z \geq 3n/2$ too.

Remark. The problem condition has the following "physics" interpretation. Imagine the cube is a metal box which is sturdy enough that all beams must remain orthogonal to the faces of the box (i.e. the beams cannot spin). Then the condition of the problem is exactly what is needed so that, if the box is shaken or rotated, the beams will not move.

Remark. Walter Stromquist points out that the number of constructions with 3030 beams is actually enormous: not dividing out by isometries, the number is $(2 \cdot 1010!)^3$.

§3 USAMO 2020/3, proposed by Richard Stong and Toni Bluher

Let p be an odd prime. An integer x is called a *quadratic non-residue* if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

The answer is that $\prod_{a \in A} a \equiv 2 \pmod p$ regardless of the value of p. In the following solution, we work in $\mathbb{F}_p$ always and abbreviate "quadratic residue" and "non-quadratic residue" to "qr" and "non-qr", respectively.

We define

$$A = \{a \in \mathbb{F}_p \mid a, 4-a \text{ non-qr}\}$$
$$B = \{b \in \mathbb{F}_p \mid b, 4-b \text{ qr}, b \neq 0, b \neq 4\}.$$

Notice that

$$A \cup B = \left\{ n \in \mathbb{F}_p \mid \left(\frac{n}{p}\right) = \left(\frac{4-n}{p}\right), \; n \neq 0, 4 \right\}.$$

We now present two approaches both based on the set B.

First approach (based on Holden Mui's presentation in Mathematics Magazine) We prove two claims.

Claim — Let $n \in \mathbb{F}_p$. Then $n(4-n) \in B$ if and only if $n \in A \cup B \setminus \{2\}$.

Proof. Note that $4 - n(4-n) = (n-2)^2$ is always a qr for $n \neq 2$. Hence, $n(4-n) \in B$ if and only if

- $n(4-n) \neq 4$, which just means $n \neq 2$, and
- $n(4-n)$ is a nonzero qr, which is equivalent to n and $4-n$ either both being nonzero qr's or non-qr's.

The latter condition just says $n \in A \cup B$ so we're done. □

Claim — The map

$$A \cup B \setminus \{2\} \to B \qquad \text{by} \quad n \mapsto n(4-n)$$

is a two-to-one map, i.e. every $b \in B$ has exactly two pre-images.

Proof. Choose $b \in B$. The quadratic equation $n(4-n) = b$ in n rewrites as $n^2 - 4n + b = 0$, and has discriminant $4(4-b)$, which is a nonzero quadratic residue. Hence there are exactly two values of n, as desired. □

Therefore, it follows that

$$\prod_{n \in A \cup B \setminus \{2\}} n = \prod_{b \in B} b.$$

So, $\prod_{a \in A} a = 2$.

Second calculation approach (along the lines of official solution) We now do the following magical calculation in $\mathbb{F}_p$:

$$\begin{aligned}
\prod_{b\in B} b = \prod_{b \in B}(4-b) &= \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is qr}}} (4-y^2) \\
&= \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is qr}}} (2+y) \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is qr}}} (2-y) \\
&= \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is qr}}} (2+y) \prod_{\substack{(p+1)/2\le y\le p-1 \\ y\neq p-2 \\ 4-y^2 \text{ is qr}}} (2+y) \\
&= \prod_{\substack{1\le y\le p-1 \\ y\neq 2,p-2 \\ 4-y^2 \text{ is qr}}} (2+y) \\
&= \prod_{\substack{3\le z\le p+1 \\ z\neq 4,p \\ z(4-z) \text{ is qr}}} z \\
&= \prod_{\substack{0\le z\le p-1 \\ z\neq 0,4,2 \\ z(4-z) \text{ is qr}}} z.
\end{aligned}$$

Note $z(4-z)$ is a nonzero quadratic residue if and only if $z \in A \cup B$. So the right-hand side is almost the product over $z \in A\cup B$, except it is missing the $z = 2$ term. Adding it in gives

$$\prod_{b\in B} b = \frac{1}{2} \prod_{\substack{0\le z\le p-1 \\ z\neq 0,4 \\ z(4-z) \text{ is qr}}} z = \frac{1}{2}\prod_{a\in A} a \prod_{b\in B} b.$$

This gives $\prod_{a\in A} a = 2$ as desired.

§4 USAMO 2020/4, proposed by Ankan Bhattacharya

Suppose that (a_1, b_1), (a_2, b_2), ..., (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

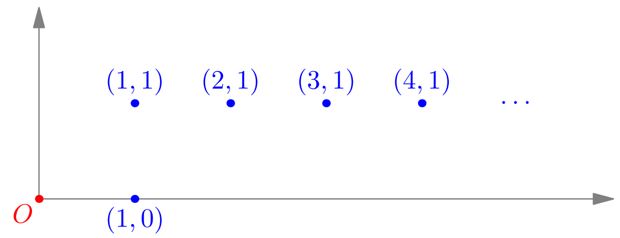
The answer is 197. In general, if 100 is replaced by $n \ge 2$ the answer is $2n - 3$.

The idea is that if we let $P_i = (a_i, b_i)$ be a point in the coordinate plane, and let $O = (0, 0)$ then we wish to maximize the number of triangles $\triangle OP_iP_j$ which have area $1/2$. Call such a triangle *good*.

Construction of 197 points: It suffices to use the points $(1, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, ..., $(99, 1)$ as shown. Notice that:

- There are 98 good triangles with vertices $(0, 0)$, $(k, 1)$ and $(k+1, 1)$ for $k = 1, \dots, 98$.
- There are 99 good triangles with vertices $(0, 0)$, $(1, 0)$ and $(k, 1)$ for $k = 1, \dots, 99$.

This is a total of $98 + 99 = 197$ triangles.

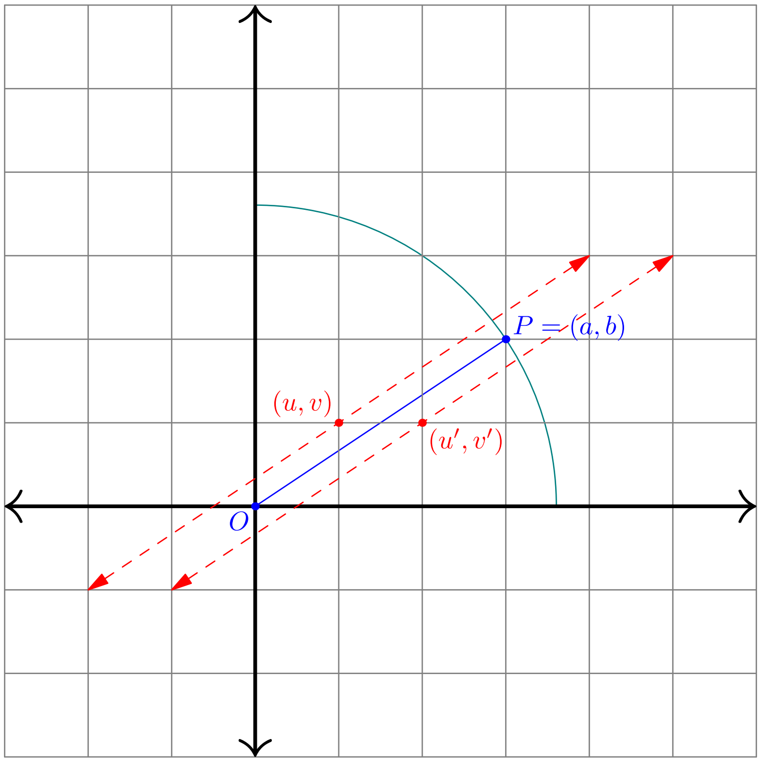


Proof that 197 points is optimal: We proceed by induction on n to show the bound of $2n - 3$. The base case $n = 2$ is evident.

For the inductive step, suppose (without loss of generality) that the point $P = P_n = (a, b)$ is the farthest away from the point O among all points.

Claim — This farthest point $P = P_n$ is part of at most two good triangles.

Proof. We must have $\gcd(a, b) = 1$ for P to be in any good triangles at all, since otherwise any divisor of $\gcd(a, b)$ also divides $2[OPQ]$. Now, we consider the locus of all points Q for which $[OPQ] = 1/2$. It consists of two parallel lines passing with slope OP, as shown.



Since $\gcd(a, b) = 1$, see that only two lattice points on this locus actually lie inside the quarter-circle centered at O with radius OP. Indeed if one of the points is (u, v) then the others on the line are $(u \pm a, v \pm b)$ where the signs match. This proves the claim. $\square$

This claim allows us to complete the induction by simply deleting P_n.

§5 USAMO 2020/5, proposed by Carl Schildkraut

A finite set S of points in the coordinate plane is called *overdetermined* if $|S| \ge 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S|-2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \ge 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is *not* overdetermined, but has k overdetermined subsets.

We claim the answer is $k = 2^{n-1} - n$. We denote the n points by A.

Throughout the solution we will repeatedly use the following fact:

Lemma

If S is a finite set of points in the plane there is at most one polynomial with real coefficients and of degree at most $|S|-1$ whose graph passes through all points of S.

Proof. If any two of the points have the same x-coordinate then obviously no such polynomial may exist at all.

Otherwise, suppose f and g are two such polynomials. Then $f-g$ has degree at most $|S|-1$, but it has $|S|$ roots, so is the zero polynomial. □

> **Remark.** Actually Lagrange interpolation implies that such a polynomial exists as long as all the x-coordinates are different!

Construction: Consider the set $A = \{(1, a), (2, b), (3, b), (4, b), \dots, (n, b)\}$, where a and b are two distinct nonzero real numbers. Any subset of the latter $n-1$ points with at least one element is overdetermined, and there are $2^{n-1} - n$ such sets.

Bound: Say a subset S of A is *flooded* if it is not overdetermined. For brevity, an *m-set* refers simply to a subset of A with m elements.

Claim — If S is an flooded m-set for $m \ge 3$, then at most one $(m-1)$-subset of S is not flooded.

Proof. Let $S = \{p_1, \dots, p_m\}$ be flooded. Assume for contradiction that $S - \{p_i\}$ and $S - \{p_j\}$ are both overdetermined. Then we can find polynomials f and g of degree at most $m-3$ passing through $S - \{p_i\}$ and $S - \{p_j\}$, respectively.

Since f and g agree on $S - \{p_i, p_j\}$, which has $m-2$ elements, by the lemma we have $f = g$. Thus this common polynomial (actually of degree at most $m-3$) witnesses that S is overdetermined, which is a contradiction. □

Claim — For all $m = 2, 3, \dots, n$ there are at least $\binom{n-1}{m-1}$ flooded m-sets of A.

Proof. The proof is by downwards induction on m. The base case $m = n$ is by assumption.

For the inductive step, suppose it's true for m. Each of the $\binom{n-1}{m-1}$ flooded m-sets has at least $m-1$ flooded $(m-1)$-subsets. Meanwhile, each $(m-1)$-set has exactly $n-(m-1)$ parent m-sets. We conclude the number of flooded sets of size $m-1$ is at least

$$\frac{m-1}{n-(m-1)} \binom{n-1}{m-1} = \binom{n-1}{m-2}$$

as desired. □

This final claim completes the proof, since it shows there are at most

$$\sum_{m=2}^{n} \left(\binom{n}{m} - \binom{n-1}{m-1} \right) = 2^{n-1} - n$$

overdetermined sets, as desired.

Remark (On repeated x-coordinates)**.** Suppose A has two points p and q with repeated x-coordinates. We can argue directly that A satisfies the bound. Indeed, any overdetermined set contains at most one of p and q. Moreover, given $S \subseteq A - \{p, q\}$, if $S \cup \{p\}$ is overdetermined then $S \cup \{q\}$ is not, and vice-versa. (Recall that overdetermined sets always have distinct x-coordinates.) This gives a bound $\left[2^{n-2} - (n-2) - 1\right] + \left[2^{n-2} - 1\right] = 2^{n-1} - n$ already.

Remark (Alex Zhai)**.** An alternative approach to the double-counting argument is to show that any overdetermined m-set has an flooded m-superset. Together with the first claim, this lets us pair overdetermined sets in a way that implies the bound.

§6 USAMO 2020/6, proposed by David Speyer and Kiran Kedlaya

Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be $2n$ real numbers such that

$$\begin{aligned} 0 &= x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n, \\ \text{and} \quad 1 &= x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2. \end{aligned}$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

We present two approaches. In both approaches, it's helpful motivation that for even n, equality occurs at

$$\begin{aligned} (x_i) &= \Big(\underbrace{\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\frac{1}{\sqrt{n}}, \dots, -\frac{1}{\sqrt{n}}}_{n/2} \Big) \\ (y_i) &= \Big(\frac{n-1}{\sqrt{n(n-1)}}, \underbrace{-\frac{1}{\sqrt{n(n-1)}}, \dots, -\frac{1}{\sqrt{n(n-1)}}}_{n-1} \Big) \end{aligned}$$

First approach (expected value) For a permutation σ on $\{1, 2, \dots, n\}$ we define

$$S_\sigma = \sum_{i=1}^{n} x_i y_{\sigma(i)}.$$

Claim — For random permutations σ, $\mathbb{E}[S_\sigma] = 0$ and $\mathbb{E}[S_\sigma^2] = \frac{1}{n-1}$.

Proof. The first one is clear.

Since $\sum_{i<j} 2x_i x_j = -1$, it follows that (for fixed i and j), $\mathbb{E}[y_{\sigma(i)} y_{\sigma(j)}] = -\frac{1}{n(n-1)}$, Thus

$$\begin{aligned} \sum_i x_i^2 \cdot \mathbb{E}\left[y_{\sigma(i)}^2\right] &= \frac{1}{n} \\ 2 \sum_{i<j} x_i x_j \cdot \mathbb{E}\left[y_{\sigma(i)} y_{\sigma(j)}\right] &= (-1) \cdot \frac{1}{n(n-1)} \end{aligned}$$

the conclusion follows. □

Claim (A random variable in $[0,1]$ has variance at most $1/4$) — If A is a random variable with mean μ taking values in the closed interval $[m, M]$ then

$$\mathbb{E}[(A-\mu)^2] \leq \frac{1}{4}(M-m)^2.$$

Proof. By shifting and scaling, we may assume $m = 0$ and $M = 1$, so $A \in [0,1]$ and hence $A^2 \le A$. Then

$$\mathbb{E}[(A-\mu)^2] = \mathbb{E}[A^2] - \mu^2 \le \mathbb{E}[A] - \mu^2 = \mu - \mu^2 \le \frac{1}{4}.$$

This concludes the proof. □

Thus the previous two claims together give

$$\max_\sigma S_\sigma - \min_\sigma S_\sigma \ge \sqrt{\frac{4}{n-1}} = \frac{2}{\sqrt{n-1}}.$$

But $\sum_i x_i y_i = \max_\sigma S_\sigma$ and $\sum_i x_i y_{n+1-i} = \min_\sigma S_\sigma$ by rearrangement inequality and therefore we are done.

Outline of second approach (by convexity, due to Alex Zhai) We will instead prove a converse result: given the hypotheses

- $x_1 \ge \dots \ge x_n$
- $y_1 \ge \dots \ge y_n$
- $\sum_i x_i = \sum_i y_i = 0$
- $\sum_i x_i y_i - \sum_i x_i y_{n+1-i} = \frac{2}{\sqrt{n-1}}$

we will prove that $\sum x_i^2 \sum y_i^2 \le 1$.

Fix the choice of y's. We see that we are trying to maximize a convex function in n variables $(x_1, \dots, x_n)$ over a convex domain (actually the intersection of two planes with several half planes). So a maximum can only happen at the boundaries: when at most two of the x's are different.

An analogous argument applies to y. In this way we find that it suffices to consider situations where $x_\bullet$ takes on at most two different values. The same argument applies to $y_\bullet$.

At this point the problem can be checked directly.

2021 USAMO Problems

Contents

Day 1

$\textbf{Note:}$ For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

Problem 1

$(*)$ Rectangles $BCC_1B_2,$ $CAA_1C_2,$ and ABB_1A_2 are erected outside an acute triangle $ABC.$ Suppose that

$$\angle BC_1C+\angle CA_1A+\angle AB_1B=180^{\circ}.$$

Prove that lines $B_1C_2,$ $C_1A_2,$ and A_1B_2 are concurrent.

Solution

Problem 2

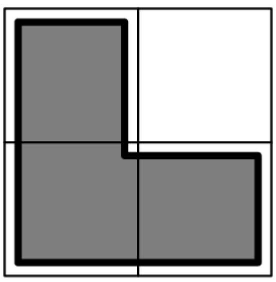
The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions.

A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Solution

Problem 3

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves: If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells. If all cells in a column have a stone, you may remove all stones from that column. If all cells in a row have a stone, you may remove all stones from that row.



For which n is it possible that, after some non-zero number of moves, the board has no stones?

Solution

Day 2

Problem 4

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

Solution

Problem 5

Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7 \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1
\end{aligned}$$

Solution

Problem 6

$(*)$ Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Solution

2021 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=2021))	
Preceded by **2020 USAMO**	Followed by **2022 USAMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAMO Problems and Solutions	

USAMO 2021 Solution Notes

Compiled by Evan Chen

27 January 2023

This is an compilation of solutions for the 2021 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

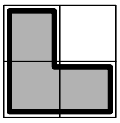
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

2. The Planar National Park is a undirected 3-regular planar graph (i.e. all vertices have degree 3). A visitor walks through the park as follows: she begins at a vertex and starts walking along an edge. When she reaches the other endpoint, she turns left. On the next vertex she turns right, and so on, alternating left and right turns at each vertex. She does this until she gets back to the vertex where she started. What is the largest possible number of times she could have entered any vertex during her walk, over all possible layouts of the park?

3. Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?

4. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

5. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}$$

6. Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

§1 USAMO 2021/1, proposed by Ankan Bhattacharya

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

The angle condition implies the circumcircles of the three rectangles concur at a single point P. Then $\measuredangle CPB_2 = \measuredangle CPA_1 = 90^\circ$, hence P lies on A_1B_2 etc., so we're done.

> **Remark.** As one might guess from the two-sentence solution, the entire difficulty of the problem is getting the characterization of the concurrence point.

§2 USAMO 2021/2, proposed by Zoran Sunic

The Planar National Park is a undirected 3-regular planar graph (i.e. all vertices have degree 3). A visitor walks through the park as follows: she begins at a vertex and starts walking along an edge. When she reaches the other endpoint, she turns left. On the next vertex she turns right, and so on, alternating left and right turns at each vertex. She does this until she gets back to the vertex where she started. What is the largest possible number of times she could have entered any vertex during her walk, over all possible layouts of the park?

The answer is 3.

We consider the trajectory of the visitor as an ordered sequence of *turns*. A turn is defined by specifying a vertex, the incoming edge, and the outgoing edge. Hence there are six possible turns for each vertex.

Claim — Given one turn in the sequence, one can reconstruct the entire sequence of turns.

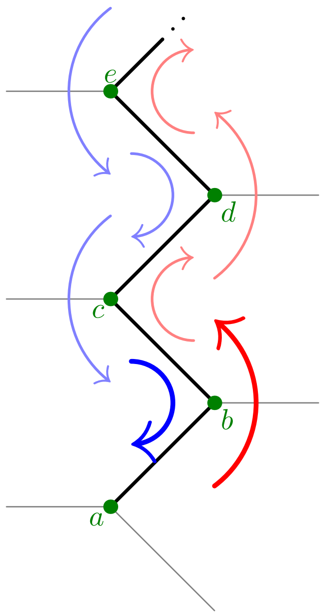
Proof. This is clear from the process's definition: given a turn t, one can compute the turn after it and the turn before it. □

This implies already that the trajectory of the visitor, when extended to an infinite sequence, is totally periodic (not just eventually periodic), because there are finitely many possible turns, so some turn must be repeated. So, any turn appears at most once in the period of the sequence, giving a naïve bound of 6 for the original problem.

However, the following claim improves the bound to 3.

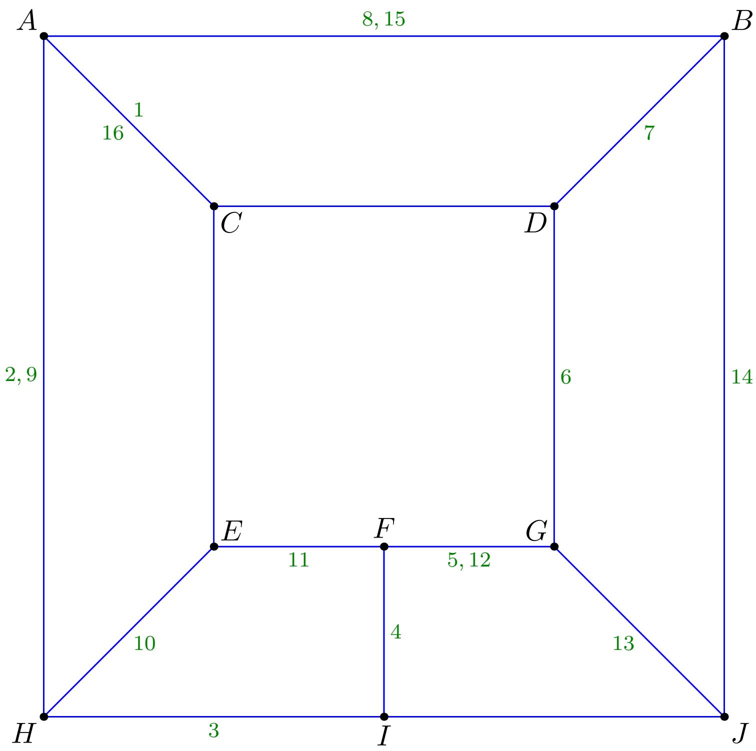
Claim — It is impossible for both of the turns $a \to b \to c$ and $c \to b \to a$ to occur in the same trajectory.

Proof. If so, then extending the path, we get $a \to b \to c \to d \to e \to \cdots$ and $\cdots \to e \to d \to c \to b \to a$, as illustrated below in red and blue respectively.



However, we assumed for contradiction the red and blue paths were part of the same trajectory, yet they clearly never meet. □

It remains to give a construction showing 3 can be achieved. There are many, many valid constructions. One construction due to Danielle Wang is given here, who provided the following motivation: "I was lying in bed and drew the first thing I could think of". The path is $CAHIFGDBAHEFGJBAC$ which visits A three times.

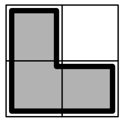


> **Remark.** As the above example shows it is possible to transverse an edge more than once even in the same direction, as in edge AH above.

§3 USAMO 2021/3, proposed by Alex Zhai, Shaunak Kishore

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.

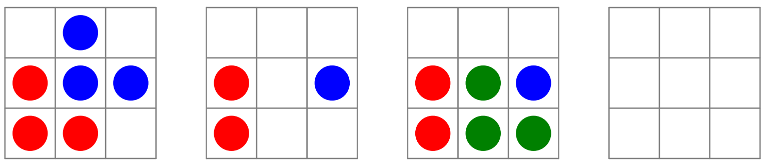


- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?

The answer is $3 \mid n$.

Construction: For $n = 3$, the construction is fairly straightforward, shown below.



This can be extended to any $3 \mid n$.

Polynomial-based proof of converse: Assume for contradiction $3 \nmid n$. We will show the task is impossible even if we allow stones to have real weights in our process. A valid elimination corresponds to a polynomial $P \in \mathbb{R}[x, y]$ such that

$$\begin{aligned} \deg_x P &\leq n-2 \\ \deg_y P &\leq n-2 \\ (1+x+y)P(x,y) &\in \left\langle 1+x+\dots+x^{n-1},\ 1+y+\dots+y^{n-1} \right\rangle. \end{aligned}$$

(Here $\langle \dots \rangle$ is an ideal of $\mathbb{R}[x,y]$.) In particular, if S is the set of nth roots of unity other than 1, we should have

$$(1+z_1+z_2)P(z_1,z_2) = 0$$

for any $z_1, z_2 \in S$. Since $3 \nmid n$, it follows that $1+z_1+z_2 \neq 0$ always.

So P vanishes on $S \times S$, a contradiction to the bounds on $\deg P$ (by, say, combinatorial nullstellensatz on any nonzero term).

Linear algebraic proof of converse (due to William Wang): Suppose there is a valid sequence of moves. Call r_j the number of operations clearing row j, indexing from bottom-to-top. The idea behind the solution is that we are going to calculate, for each cell, the number of times it is operated on entirely as a function of r_j. For example, a hypothetical illustration with $n = 6$ is partially drawn below, with the number in each cell denoting how many times it was the corner of an L.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 = r_3 & c_4 = r_5 - r_4 & c_5 = r_5 & 0 \\ \vdots & \vdots & 2r_4 + r_3 + r_2 - 2r_5 & r_5 - r_3 & r_4 & 0 \\ \vdots & \vdots & r_4 + r_3 + r_2 + r_1 - 2r_5 & r_5 - r_2 & r_3 & 0 \\ \vdots & \vdots & r_4 + r_2 + r_1 - 2r_5 & r_5 - r_1 & r_2 & 0 \\ \vdots & \vdots & r_4 + r_1 - r_5 & r_5 & r_1 & 0 \end{bmatrix}$$

Let $a_{i,j}$ be the expression in (i,j). It will also be helpful to define c_i in the obvious way as well.

Claim — We have $c_n = r_n = 0$, $a_{n-1,j} = r_j$ and $a_{i,n-1} = c_i$.

Proof. The first statement follows since (n,n) may never obtain a stone. The equation $a_{n-1,j} = r_j$ follows as both equal the number of times that cell (n,j) obtains a stone. The final equation is similar. □

Claim — For $1 \le i,j \le n-1$, the following recursion holds:

$$a_{i,j} + a_{i+1,j} + a_{i+1,j-1} = r_i + c_{j+1}.$$

Proof. Focus on cell $(i+1,j)$. The left-hand side counts the number of times that gains a stone while the right-hand side counts the number of times it loses a stone; they must be equal. □

We can coerce the table above into matrix form now as follows. Define

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}.$$

Then define a sequence of matrices M_i recursively by $M_{n-1} = \mathrm{id}$, and

$$M_i = \mathrm{id} + KM_{i+1} = \mathrm{id} + K + \dots + K^{n-1-i}.$$

The matrices are chosen so that, by construction,

$$\langle r_1, \dots, r_{n-1} \rangle \, M_i = \langle a_{i,1}, \dots, a_{i,n-1} \rangle$$

for $i = 1, 2, \dots, n-1$. On the other hand, we can extend the recursion one level deeper and obtain

$$\langle r_1, \dots, r_{n-1} \rangle \, M_0 = \langle 0, \dots, 0 \rangle .$$

However, the crux of the solution is the following.

Claim — The eigenvalues of K are exactly $-(1 + e^{\frac{2\pi i k}{n}})$ for $k = 1, 2, \dots, n-1$.

Proof. The matrix $-(K + \mathrm{id})$ is fairly known to have roots of unity as the coefficients. □

However, we are told that apparetnly

$$0 = \det M_0 = \det \left(\mathrm{id} + K + K^2 + \dots + K^{n-1} \right)$$

which means $\det(K^n - \mathrm{id}) = 0$. This can only happen if K^n has eigenvalue 1, meaning that

$$[-(1+\omega)]^n = 1$$

for ω some nth root of unity, not necessarily primitive. This can only happen if $|1+\omega| = 1$, ergo $3 \mid n$.

§4 USAMO 2021/4, proposed by Carl Schildkraut

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

The answer is that $|S|$ must be a power of 2 (including 1), or $|S| = 0$ (a trivial case we do not discuss further).

Construction: For any nonnegative integer k, a construction for $|S| = 2^k$ is given by

$$S = \{(p_1 \text{ or } q_1) \times (p_2 \text{ or } q_2) \times \cdots \times (p_k \text{ or } q_k)\}$$

for $2k$ distinct primes $p_1, \ldots, p_k, q_1, \ldots, q_k$.

Converse: the main claim is as follows.

Claim — In any valid set S, for any prime p and $x \in S$, $\nu_p(x) \le 1$.

Proof. Assume for contradiction $e = \nu_p(x) \ge 2$.

- On the one hand, by taking x in the statement, we see $\frac{e}{e+1}$ of the elements of S are divisible by p.
- On the other hand, consider a $y \in S$ such that $\nu_p(y) = 1$ which must exist (say if $\gcd(x,y) = p$). Taking y in the statement, we see $\frac{1}{2}$ of the elements of S are divisible by p.

So $e = 1$, contradiction. □

Now since $|S|$ equals the number of divisors of any element of S, we are done.

§5 USAMO 2021/5, proposed by Mohsen Jamaali

Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

The answer is that the only solution is $(1, 2, 1, 2, \dots, 1, 2)$ which works.
We will prove a_{2k} is a constant sequence, at which point the result is obvious.

First approach (Andrew Gu) Apparently, with indices modulo $2n$, we should have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

for every index k (this eliminates all a_{odd}'s). Define

$$m = \min_k a_{2k} \qquad \text{and} \qquad M = \max_k a_{2k}.$$

Look at the indices i and j achieving m and M to respectively get

$$\begin{aligned} m &= \frac{2}{m} + \frac{1}{a_{2i-2}} + \frac{1}{a_{2i+2}} \geq \frac{2}{m} + \frac{1}{M} + \frac{1}{M} = \frac{2}{m} + \frac{2}{M} \\ M &= \frac{2}{M} + \frac{1}{a_{2j-2}} + \frac{1}{a_{2j+2}} \leq \frac{2}{M} + \frac{1}{m} + \frac{1}{m} = \frac{2}{m} + \frac{2}{M}. \end{aligned}$$

Together this gives $m \geq M$, so $m = M$. That means a_{2i} is constant as i varies, solving the problem.

Second approach (author's solution) As before, we have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

The proof proceeds in three steps.

- Define
$$S = \sum_k a_{2k}, \quad \text{and} \quad T = \sum_k \frac{1}{a_{2k}}.$$
Summing gives $S = 4T$. On the other hand, Cauchy-Schwarz says $S \cdot T \geq n^2$, so $T \geq \frac{1}{2}n$.

- On the other hand,
$$1 = \frac{1}{a_{2k-2}a_{2k}} + \frac{2}{a_{2k}^2} + \frac{1}{a_{2k}a_{2k+2}}$$
Sum this modified statement to obtain
$$n = \sum_k \left(\frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 \overset{\text{QM-AM}}{\geq} \frac{1}{n} \left(\sum_k \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 = \frac{1}{n} (2T)^2$$
So $T \leq \frac{1}{2}n$.
- Since $T \leq \frac{1}{2}n$ and $T \geq \frac{1}{2}n$, we must have equality everywhere above. This means a_{2k} is a constant sequence.

Remark. The problem is likely intractable over $\mathbb{C}$, in the sense that one gets a high-degree polynomial which almost certainly has many complex roots. So it seems likely that most solutions must involve some sort of inequality, using the fact we are over $\mathbb{R}_{>0}$ instead.

§6 USAMO 2021/6, proposed by Ankan Bhattacharya

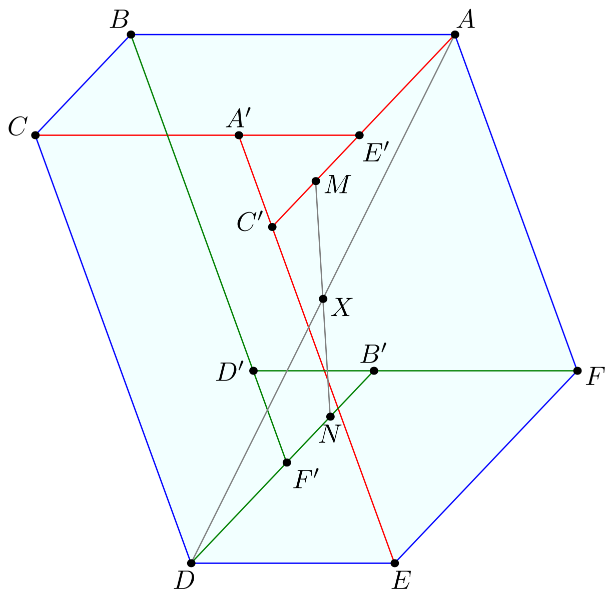
Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

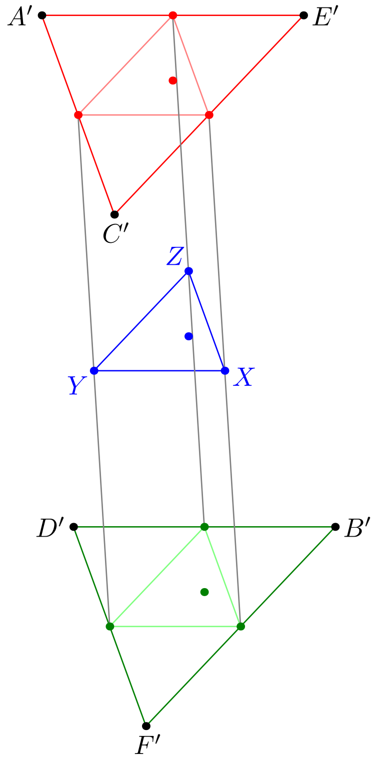
We present two solutions.

Parallelogram solution found by contestants Note that the following figure is intentionally *not* drawn to scale, to aid legibility. We construct parallelograms $ABCE'$, etc as shown. Note that this gives two congruent triangles $A'C'E'$ and $B'D'F'$. (Assuming that triangle XYZ is non-degenerate, the triangles $A'C'E'$ and $B'D'F'$ will also be non-degenerate.)



Claim — If $AB \cdot DE = BC \cdot EF = CD \cdot FA = k$, then the circumcenters of ACE and $A'C'E'$ coincide.

Proof. The power of A to $(A'C'E')$ is $AE' \cdot AC' = BC \cdot EF = k$; same for C and E. □



Claim — Triangle XYZ is the vector average of the (congruent) medial triangles of triangles $A'C'E'$ and $B'D'F'$.

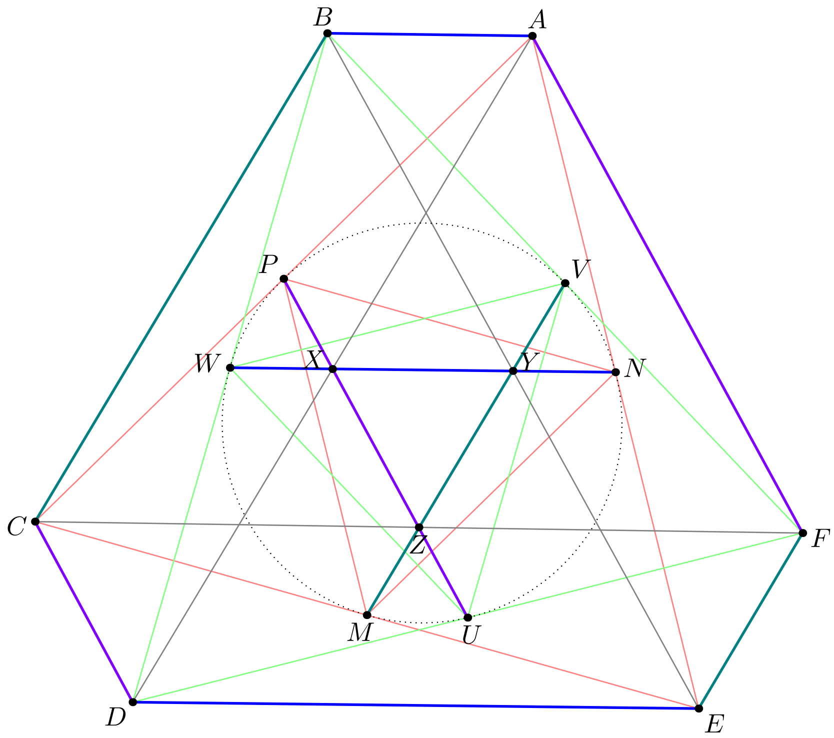
Proof. If M and N are the midpoints of $\overline{C'E'}$ and $\overline{B'F'}$, then X is the midpoint of $\overline{MN}$ by vector calculation:

$$\begin{aligned}\frac{\vec{M}+\vec{N}}{2} &= \frac{\frac{\vec{C}'+\vec{E}'}{2}+\frac{\vec{B}'+\vec{F}'}{2}}{2} \\ &= \frac{\vec{C}'+\vec{E}'+\vec{B}'+\vec{F}'}{4} \\ &= \frac{(\vec{A}+\vec{E}-\vec{F})+(\vec{C}+\vec{A}-\vec{B})+(\vec{D}+\vec{F}-\vec{E})+(\vec{B}+\vec{D}-\vec{C})}{4} \\ &= \frac{\vec{A}+\vec{D}}{2} = \vec{X}.\end{aligned}$$

□

Hence the orthocenter of XYZ is the midpoint of the orthocenters of the medial triangles of $A'C'E'$ and $B'D'F'$ — that is, their circumcenters.

Author's solution Call MNP and UVW the medial triangles of ACE and BDF.



Claim — In trapezoid $ABDE$, the perpendicular bisector of $\overline{XY}$ is the same as the perpendicular bisector of the midline $\overline{WN}$.

Proof. This is true for any trapezoid: because $WX = \frac{1}{2}AB = YN$. □

Claim — The points V, W, M, N are cyclic.

Proof. By power of a point from Y, since

$$WY \cdot YN = \frac{1}{2}DE \cdot \frac{1}{2}AB = \frac{1}{2}EF \cdot \frac{1}{2}BC = VY \cdot YM.$$ □

Applying all the cyclic variations of the above two claims, it follows that all six points U, V, W, M, N, P are concyclic, and the center of that circle coincides with the circumcenter of $\triangle XYZ$.

Remark. It is also possible to implement ideas from the first solution here, by showing all six midpoints have equal power to (XYZ).

Claim — The orthocenter of XYZ is the midpoint of the circumcenters of $\triangle ACE$ and $\triangle BDF$.

Proof. Apply complex numbers with the unit circle coinciding with the circumcircle of

$NVPWMU$. Then

$$\begin{aligned} \text{orthocenter}(XYZ) &= x+y+z = \frac{a+b+c+d+e+f}{2} \\ \text{circumcenter}(ACE) &= \text{orthocenter}(MNP) \\ &= m+n+p = \frac{c+e}{2} + \frac{e+a}{2} + \frac{a+c}{2} = a+c+e \\ \text{circumcenter}(BDF) &= \text{orthocenter}(UVW) \\ &= u+v+w = \frac{d+f}{2} + \frac{f+b}{2} + \frac{b+d}{2} = b+d+f. \qquad \square \end{aligned}$$

2022 USAMO Problems

Contents

Day 1

Problem 1

Let a and b be positive integers. The cells of an $(a+b+1)\times(a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

Solution

Problem 2

Let $b\geq 2$ and $w\geq 2$ be fixed integers, and $n=b+w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b=3$ and $w=2$ is shown below, as well as the resulting polygons B and W.

Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Solution

Problem 3

Let $\mathbb{R}_{>0}$ be the set of all positive real numbers. Find all functions $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ such that for all $x,y\in\mathbb{R}_{>0}$ we have

$$f(x)=f(f(f(x))+y)+f(xf(y))f(x+y).$$

Solution

Day 2

Problem 4

Find all pairs of primes (p,q) for which $p-q$ and $pq-q$ are both perfect squares.

Solution

Problem 5

A function $f:\mathbb{R}\to\mathbb{R}$ is *essentially increasing* if $f(s)\leq f(t)$ holds whenever $s\leq t$ are real numbers such that $f(s)\neq 0$ and $f(t)\neq 0$.

Find the smallest integer k such that for any 2022 real numbers $x_1,x_2,\ldots,x_{2022}$, there exist k essentially increasing functions $f_1,\ldots,f_k$ such that

$$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \ldots 2022.$$

Solution

Problem 6

There are 2022 users on a social network called Mathbook, and some of them are Mathbook-friends. (On Mathbook, friendship is always mutual and permanent.)

Starting now, Mathbook will only allow a new friendship to be formed between two users if they have *at least two* friends in common. What is the minimum number of friendships that must already exist so that every user could eventually become friends with every other user?

Solution

2022 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=2022))	
Preceded by **2021 USAMO**	Followed by **2023 USAMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAMO Problems and Solutions	

USAMO 2022 Solution Notes

Compiled by Evan Chen

27 January 2023

This is an compilation of solutions for the 2022 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

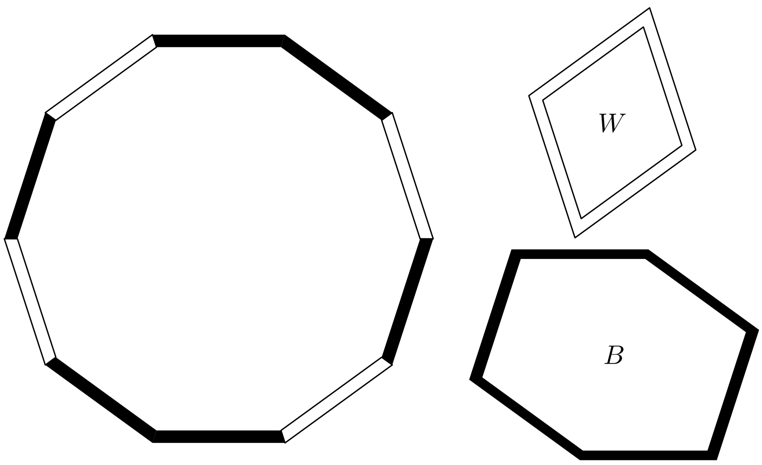
Contents

§0 Problems

1. Let a and b be positive integers. The cells of an $(a+b+1)\times(a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

2. Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

   We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



   Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

3. Solve over positive real numbers the functional equation

$$f(x) = f(f(f(x)) + y) + f(xf(y))f(x + y).$$

4. Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

5. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *essentially increasing* if $f(s) \leq f(t)$ holds whenever $s \leq t$ are real numbers such that $f(s) \neq 0$ and $f(t) \neq 0$.

   Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \dots, x_{2022}$, there exist k essentially increasing functions $f_1, \dots, f_k$ such that

$$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \dots, 2022.$$

6. There are 2022 users on a social network called Mathbook, and some of them are Mathbook-friends. (On Mathbook, friendship is always mutual and permanent.)

   Starting now, Mathbook will only allow a new friendship to be formed between two users if they have at least two friends in common. What is the minimum number of friendships that must already exist so that every user could eventually become friends with every other user?

§1 USAMO 2022/1, proposed by Ankan Bhattacharya

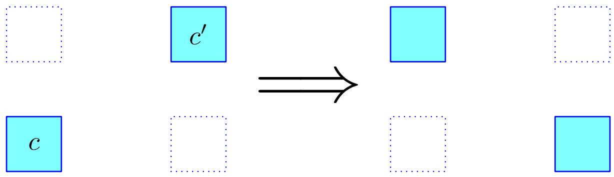
Let a and b be positive integers. The cells of an $(a+b+1) \times (a+b+1)$ grid are colored amber and bronze such that there are at least $a^2 + ab - b$ amber cells and at least $b^2 + ab - a$ bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a + b$ chosen cells lie in the same row or column.

Claim — There exists a transversal T_a with at least a amber cells. Analogously, there exists a transversal T_b with at least b bronze cells.

Proof. If one picks a random transversal, the expected value of the number of amber cells is at least

$$\frac{a^2 + ab - b^2}{a+b+1} = (a-1) + \frac{1}{a+b+1} > a - 1. \qquad \square$$

Now imagine we transform T_a to T_b in some number of steps, by repeatedly choosing cells c and c' and swapping them with the two other corners of the rectangle formed by their row/column, as shown in the figure.

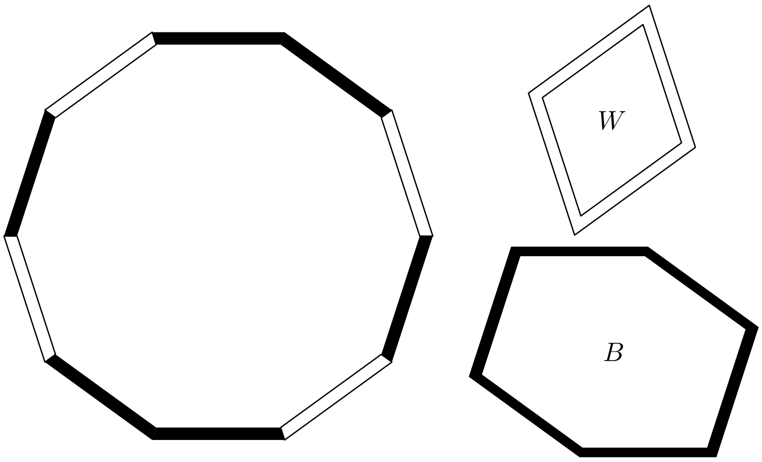


By "discrete intermediate value theorem", the number of amber cells will be either a or $a + 1$ at some point during this transformation. This completes the proof.

§2 USAMO 2022/2, proposed by Ankan Bhattacharya

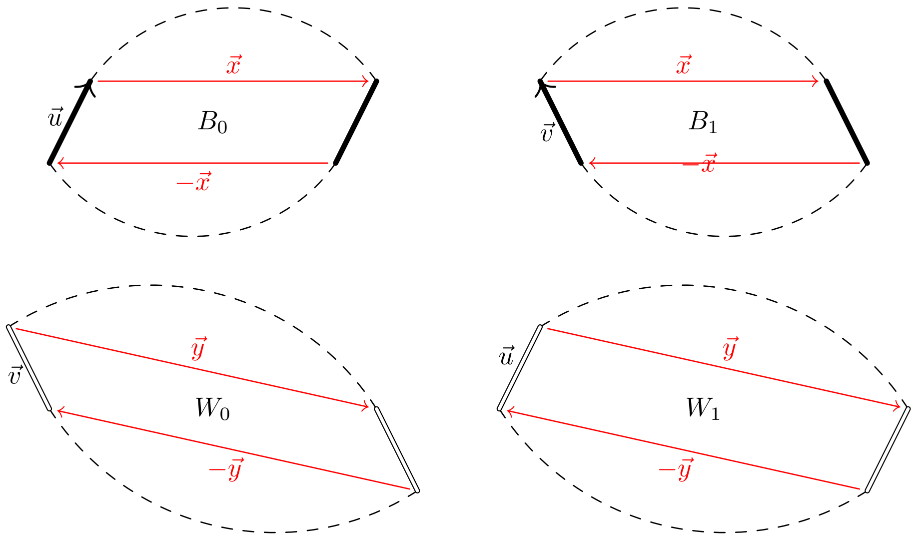
Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n$-gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

We are going to prove that one may swap a black rod with an adjacent white rod (as well as the rods parallel to them) without affecting the difference in the areas of $B - W$. Let $\vec{u}$ and $\vec{v}$ denote the originally black and white vectors that were adjacent on the $2n$-gon and are now going to be swapped. Let $\vec{x}$ denote the sum of all the other black vectors between $\vec{u}$ and $-\vec{u}$, and define $\vec{y}$ similarly. See the diagram below, where B_0 and W_0 are the polygons before the swap, and B_1 and W_1 are the resulting changed polygons.



Observe that the only change in B and W is in the parallelograms shown above in each diagram. Letting $\wedge$ denote the wedge product, we need to show that

$$\vec{u} \wedge \vec{x} - \vec{v} \wedge \vec{y} = \vec{v} \wedge \vec{x} - \vec{u} \wedge \vec{y}$$

which can be rewritten as

$$(\vec{u}-\vec{v}) \wedge (\vec{x}+\vec{y}) = 0.$$

In other words, it would suffice to show $\vec{u}-\vec{v}$ and $\vec{x}+\vec{y}$ are parallel. (Students not familiar with wedge products can replace every $\wedge$ with the cross product $\times$ instead.)

Claim — Both $\vec{u}-\vec{v}$ and $\vec{x}+\vec{y}$ are perpendicular to vector $\vec{u}+\vec{v}$.

Proof. We have $(\vec{u}-\vec{v}) \perp (\vec{u}+\vec{v})$ because $\vec{u}$ and $\vec{v}$ are the same length.

For the other perpendicularity, note that $\vec{u}+\vec{v}+\vec{x}+\vec{y}$ traces out a diameter of the circumcircle of the original $2n$-gon; call this diameter AB, so

$$A+\vec{u}+\vec{v}+\vec{x}+\vec{y} = B.$$

Now point $A+\vec{u}+\vec{v}$ is a point on this semicircle, which means (by the inscribed angle theorem) the angle between $\vec{u}+\vec{v}$ and $\vec{x}+\vec{y}$ is 90°. $\square$

§3 USAMO 2022/3, proposed by Hung-Hsun Hans Yu

Solve over positive real numbers the functional equation

$$f(x) = f(f(f(x)) + y) + f(xf(y))f(x + y).$$

The answer is $f(x) \equiv c/x$ for any $c > 0$. This works, so we'll prove this is the only solution. The following is based on the solution posted by pad on AoPS.

In what follows, f^n as usual denotes f iterated n times, and $P(x, y)$ is the given statement. Also, we introduce the notation Q for the statement

$$Q(a, b) : \qquad f(a) \geq f(b) \implies f(f(b)) \geq a.$$

To see why this statement Q is true, assume for contradiction that $a > f(f(b))$; then consider $P(b, a - f(f(b)))$ to get a contradiction.

The main idea of the problem is the following:

Claim — Any function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ obeying statement Q satisfies $f^2(x) = f^4(x)$.

Proof. From $Q(t, t)$ we get

$$f^2(t) \geq t \qquad \text{for all } t > 0.$$

So this already implies $f^4(x) \geq f^2(x)$ by choosing $t = f^2(x)$. It also gives $f(x) \leq f^3(x) \leq f^5(x)$ by choosing $t = f(x)$, $t = f^3(x)$.

Then $Q(f^4(x), x)$ is valid and gives $f^2(x) \geq f^4(x)$, as needed. □

Claim — The function f is injective.

Proof. Suppose $f(u) = f(v)$ for some $u > v$. From $Q(u, v)$ and $Q(v, u)$ we have $f^2(v) \geq u$ and $f^2(u) \geq v$. Note that for all $x > 0$ we have statements

$$P(f^2(x), u) \implies f^3(x) = f(x + u) + f(xf(u))f(x + u) = (1 + f(xf(u)))f(x + u)$$
$$P(f^2(x), v) \implies f^3(x) = f(x + v) + f(xf(v))f(x + v) = (1 + f(xf(v)))f(x + v).$$

It follows that $f(x + u) = f(x + v)$ for all $x > 0$.

This means that f is periodic with period $T = u - v > 0$. However, this is incompatible with Q, because we would have $Q(1 + nT, 1)$ for all positive integers n, which is obviously absurd. □

Since f is injective, we obtain that $f^2(x) = x$. Thus $P(x, y)$ now becomes the statement

$$P(x, y) : \qquad f(x) = f(x + y) \cdot \Big[1 + f(xf(y))\Big].$$

In particular

$$P(1, y) \implies f(1 + y) = \frac{f(1)}{1 + y}$$

so f is determined on inputs greater than 1. Finally, if $a, b > 1$ we get

$$P(a, b) \implies \frac{1}{a} = \frac{1}{a + b} \cdot \left[1 + f\left(\frac{a}{b} f(1)\right)\right]$$

which is enough to determine f on all inputs, by varying (a, b).

§4 USAMO 2022/4, proposed by Holden Mui

Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

The answer is $(3, 2)$ only.

Set

$$a^2 = p - q$$
$$b^2 = pq - q.$$

Note that $0 < a < p$, and $0 < b < p$ (because $q \le p$). Now subtracting gives

$$\underbrace{(b-a)}_{<p}\underbrace{(b+a)}_{<2p} = b^2 - a^2 = p(q-1)$$

The inequalities above now force $b + a = p$. Hence $q - 1 = b - a$.

This means p and $q - 1$ have the same parity, which can only occur if $q = 2$. Finally, taking mod 3 shows $p \equiv 0 \pmod 3$. So $(3, 2)$ is the only possibility (and it does work).

§5 USAMO 2022/5, proposed by Gabriel Carroll

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *essentially increasing* if $f(s) \le f(t)$ holds whenever $s \le t$ are real numbers such that $f(s) \neq 0$ and $f(t) \neq 0$.

Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \dots, x_{2022}$, there exist k essentially increasing functions $f_1, \dots, f_k$ such that

$$f_1(n) + f_2(n) + \dots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \dots, 2022.$$

The answer is 11 and, more generally, if 2022 is replaced by N then the answer is $\lfloor \log_2 N \rfloor + 1$.

Bound Suppose for contradiction that $2^k - 1 > N$ and choose $x_n = -n$ for each $n = 1, \dots, N$. Now for each index $1 \le n \le N$, define

$$S(n) = \{\text{indices } i \text{ for which } f_i(n) \neq 0\} \subseteq \{1, \dots, k\}.$$

As each $S(nt)$ is nonempty, by pigeonhole, two $S(n)$'s coincide, say $S(n) = S(n')$ for $n < n'$. But it's plainly impossible that $x_n > x_{n'}$ in that case due to the essentially increasing condition.

Construction It suffices to do $N = 2^k - 1$. Rather than drown the reader in notation, we'll just illustrate an example of the (inductive) construction for $k = 4$. Empty cells are zero.

	f_1	f_2	f_3	f_4
$x_1 = 3$	**3**			
$x_2 = 1$	10	**−9**		
$x_3 = 4$		**4**		
$x_4 = 1$	100	200	**−299**	
$x_5 = 5$		200	**−195**	
$x_6 = 9$	100		**−91**	
$x_7 = 2$			**2**	
$x_8 = 6$	1000	2000	4000	**−6994**
$x_9 = 5$		2000	4000	**−5995**
$x_{10} = 3$	1000		4000	**−4997**
$x_{11} = 5$			4000	**−3995**
$x_{12} = 8$	1000	2000		**−2992**
$x_{13} = 9$		2000		**−1991**
$x_{14} = 7$	1000			**−993**
$x_{15} = 9$				**9**

The general case is handled in the same way with powers of 10 replaced by powers of B, for a sufficiently large number B.

§6 USAMO 2022/6, proposed by Yannick Yao

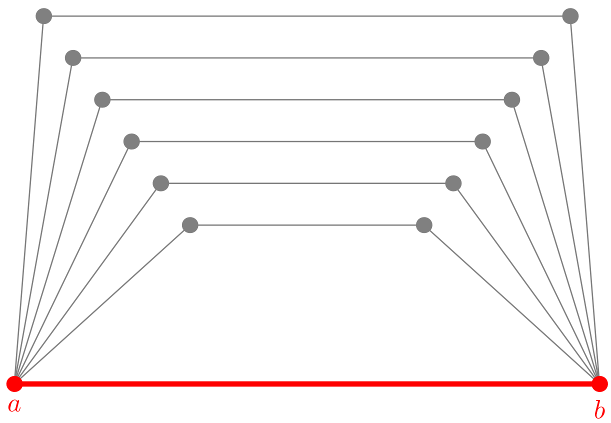
There are 2022 users on a social network called Mathbook, and some of them are Mathbook-friends. (On Mathbook, friendship is always mutual and permanent.)

Starting now, Mathbook will only allow a new friendship to be formed between two users if they have at least two friends in common. What is the minimum number of friendships that must already exist so that every user could eventually become friends with every other user?

With 2022 replaced by n, the answer is $\left\lceil \frac{3}{2}n \right\rceil - 2$.

Terminology Standard graph theory terms: starting from a graph G on n vertices, we're allowed to take any C_4 in the graph and complete it to a K_4. The problem asks the minimum number of edges needed so that this operation lets us transform G to K_n.

Construction For even n, start with an edge ab, and then create $n/2 - 1$ copies of C_4 that use ab as an edge, as shown below for $n = 14$ (six copies of C_4).



This can be completed into K_n by first completing the $n/2 - 1$ C_4's into K_4, then connecting red vertices to every grey vertex, and then finishing up.

The construction for odd n is the same except with one extra vertex c which is connected to both a and b.

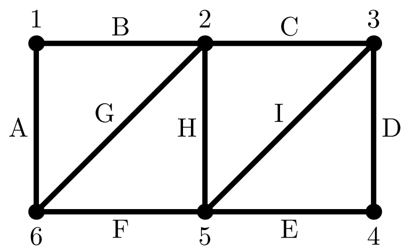
Bound Notice that additional operations or connections can never hurt. So we will describe a *specific* algorithm that performs operations on the graph until no more operations are possible. This means that if this algorithm terminates with anything other $G = K_n$, the graph was never completable to K_n to begin with.

The algorithm uses the following data: it keeps a list $\mathcal{C}$ of cliques of G, and a labeling $\mathcal{L}\colon E(G) \to \mathcal{C}$ which assigns to every edge one of the cliques that contains it.

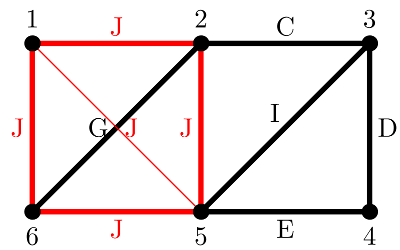
- Initially, $\mathcal{C}$ consists of one K_2 for every edge of G, and each edge is labeled in the obvious way.
- At each step, the algorithm arbitrarily takes any $C_4 = abcd$ whose four edges ab, bc, cd, da do not all have the same label. Consider these labels that appear (at least two, and up to four), and let V be the union of all vertices in any of these 2-4 cliques.
- Do the following graph operations: connect ac and bd, then connect every vertex in $V - \{a, b, c, d\}$ to each of $\{a, b, c, d\}$. Finally, complete this to a clique on V.

- Update $\mathcal{C}$ by merging these 2-4 cliques into a single clique K_V.
- Update $\mathcal{L}$ by replacing every edge that was labeled with one of these 2-4 cliques with the label K_V. Also, update every *newly* created edge to have label K_V. However, if there were existing edges not labeled with one of the 2-4 cliques, then we do *not* update these!
- Stop once every C_4 has only one label appearing among its edges. When this occurs, no operations are possible at all on the graph.

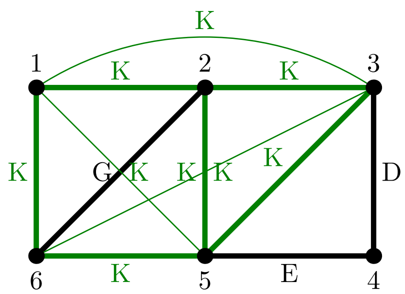
A few steps of the process are illustrated below for a graph on six vertices with nine initial edges. There are initially nine K_2's labeled A, B, ..., I. Original edges are always bolder than added edges. The relabeled edges in each step are highlighted in color. Notice how we need an entirely separate operation to get G to become L, even though no new edges are drawn in the graph.



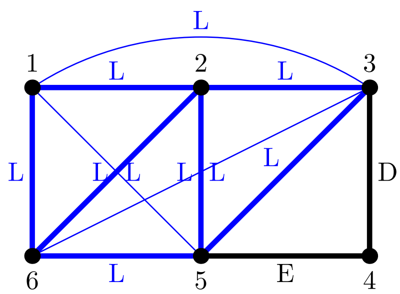
Initial setup



Step 1: Operate on 1256.
Merges ABFH into J.
$\theta(\text{J}) = 4$



Step 2: Operate on 1235.
Merges CIJ into K.
$\theta(\text{K}) = 6$



Step 3: Operate on 2356.
Merges GK into L.
$\theta(\text{L}) = 7$

As we remarked, if the graph is going to be completable to K_n at all, then this algorithm must terminate with $\mathcal{C} = \{K_n\}$. We will use this to prove our bound.

We proceed by induction in the following way. For a clique K, let $\theta(K)$ denote the number of edges of the *original* graph G which are labeled by K (this does *not* include new edges added by the algorithm); hence the problem amounts to estimating how small $\theta(K_n)$ can be. We are trying to prove:

Claim — At any point in the operation, if K is a clique in the cover $\mathcal{C}$, then

$$\theta(K) \geq \frac{3|K|}{2} - 2.$$

where $|K|$ is the number of vertices in K.

Proof. By induction on the time step of the algorithm. The base case is clear, because then K is just a single edge of G, so $\theta(K) = 1$ and $|K| = 2$.

The inductive step is annoying casework based on the how the merge occurred. Let $C_4 = abcd$ be the 4-cycle operated on. In general, the θ value of a newly created K is exactly the sum of the θ values of the merged cliques, by definition. Meanwhile, $|K|$ is the number of vertices in the union of the merged cliques; so it's the sum of the sizes of these cliques minus some error due to overcounting of vertices appearing more than once. To be explicit:

- Suppose we merged four cliques W, X, Y, Z. By definition,

$$\begin{aligned}\theta(K) &= \theta(W) + \theta(X) + \theta(Y) + \theta(Z) \\ &\geq \frac{3}{2}(|W| + |X| + |Y| + |Z|) - 8 = \frac{3}{2}(|W| + |X| + |Y| + |Z| - 4) - 2.\end{aligned}$$

  On the other hand $|K| \leq |W| + |X| + |Y| + |Z| - 4$; the -4 term comes from each of $\{a, b, c, d\}$ being in two (or more) of $\{W, X, Y, Z\}$. So this case is OK.

- Suppose we merged three cliques X, Y, Z. By definition,

$$\begin{aligned}\theta(K) &= \theta(X) + \theta(Y) + \theta(Z) \\ &\geq \frac{3}{2}(|X| + |Y| + |Z|) - 6 = \frac{3}{2}\left(|X| + |Y| + |Z| - \frac{8}{3}\right) - 2.\end{aligned}$$

  On the other hand, $|K| \leq |X| + |Y| + |Z| - 3$, since at least 3 of $\{a, b, c, d\}$ are repeated among X, Y, Z. Note in this case the desired inequality is actually strict.

- Suppose we merged two cliques Y, Z. By definition,

$$\begin{aligned}\theta(K) &= \theta(Y) + \theta(Z) \\ &\geq \frac{3}{2}(|Y| + |Z|) - 4 = \frac{3}{2}\left(|Y| + |Z| - \frac{4}{3}\right) - 2.\end{aligned}$$

  On the other hand, $|K| \leq |Y| + |Z| - 2$, since at least 2 of $\{a, b, c, d\}$ are repeated among Y, Z. Note in this case the desired inequality is actually strict. $\square$

Remark. Several subtle variations of this method do not seem to work.

- It does not seem possible to require the cliques in $\mathcal{C}$ to be disjoint, which is why it's necessary to introduce a label function $\mathcal{L}$ as well.
- It seems you do have to label the newly created edges, even though they do not count towards any θ value. Otherwise the termination of the algorithm doesn't tell you enough.
- Despite this, relabeling existing edges, like G in step 1 of the example, 1 seems to cause a lot of issues. The induction becomes convoluted if $\theta(K)$ is not exactly the sum of θ-values of the subparts, while the disappearance of an edge from a clique will also break induction.

2023 USAMO Problems

Contents

Day 1

Problem 1

In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Solution

Problem 2

Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2$$

Solution

Problem 3

Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum value of $k(C)$ as a function of n.

Solution

Day 2

Problem 4

A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Solution

Problem 5

Let $n \geq 3$ be an integer. We say that an arrangement of the numbers $1, 2, \dots, n^2$ in a $n \times n$ table is row-valid if the numbers in each row can be permuted to form an arithmetic progression, and column-valid if the numbers in each column can be permuted to form an arithmetic progression. For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

Solution

Problem 6

Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c opposite A, B, and C, respectively. Given an arbitrary point D on the circumcircle of $\triangle ABC$ that does not lie on any of the lines IIa, I_bI_c, or BC, suppose the circumcircles of $\triangle DIIa$ and $\triangle DI_bI_c$ intersect at two distinct points D and F. If E is the intersection of lines DF and BC, prove that $\angle BAD = \angle EAC$.

Solution

2023 USAMO (**Problems** • Resources (http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=27&year=2023))	
Preceded by **2022 USAMO**	Followed by **2024 USAMO**
1 • 2 • 3 • 4 • 5 • 6	
All USAMO Problems and Solutions	

USAMO 2023 Solution Notes

Evan Chen 《陳誼廷》

2 June 2023

This is a compilation of solutions for the 2023 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

2. Solve over the positive real numbers the functional equation
$$f(xy + f(x)) = xf(y) + 2.$$

3. Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes.

   Find all possible values of $k(C)$ as a function of n.

4. Positive integers a and N are fixed, and N positive integers are written on a blackboard. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

   After analyzing the N integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these N integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

5. Let $n \geq 3$ be an integer. We say that an arrangement of the numbers $1, 2, \dots, n^2$ in an $n \times n$ table is *row-valid* if the numbers in each row can be permuted to form an arithmetic progression, and *column-valid* if the numbers in each column can be permuted to form an arithmetic progression.

   For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

6. Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c opposite A, B, and C, respectively. Given an arbitrary point D on the circumcircle of $\triangle ABC$ that does not lie on any of the lines II_a, I_bI_c, or BC, suppose the circumcircles of $\triangle DII_a$ and $\triangle DI_bI_c$ intersect at two distinct points D and F. If E is the intersection of lines DF and BC, prove that $\angle BAD = \angle EAC$.

§1 Solutions to Day 1

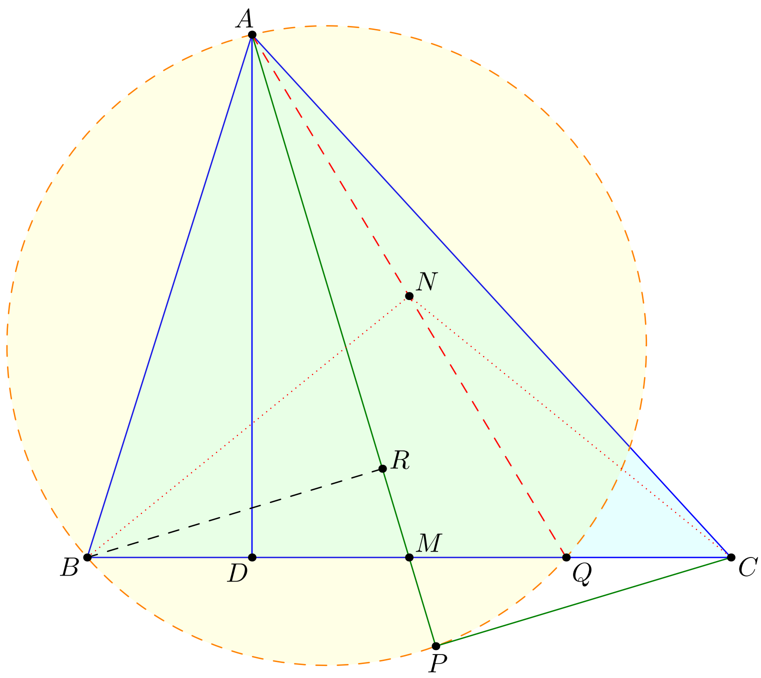
§1.1 USAMO 2023/1, proposed by Holden Mui

Available online at https://aops.com/community/p27349297.

Problem statement

In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

We show several different approaches. In all solutions, let D denote the foot of the altitude from A.



¶ **Most common synthetic approach** The solution hinges on the following claim:

Claim — Q coincides with the reflection of D across M.

Proof. Note that $\measuredangle ADC = \measuredangle APC = 90°$, so $ADPC$ is cyclic. Then by power of a point (with the lengths directed),

$$MB \cdot MQ = MA \cdot MP = MC \cdot MD.$$

Since $MB = MC$, the claim follows. □

It follows that $\overline{MN} \parallel \overline{AD}$, as M and N are respectively the midpoints of $\overline{AQ}$ and $\overline{DQ}$. Thus $\overline{MN} \perp \overline{BC}$, and so N lies on the perpendicular bisector of $\overline{BC}$, as needed.

> **Remark** (David Lin)**.** One can prove the main claim without power of a point as well, as follows: Let R be the foot from B to $\overline{AM}$, so $BRCP$ is a parallelogram. Note that $ABDR$ is cyclic, and hence
>
> $$\measuredangle DRM = \measuredangle DBA = QBA = \measuredangle QPA = \measuredangle QPM.$$
>
> Thus, $\overline{DR} \parallel \overline{PQ}$, so DRQ is also a parallelogram.

¶ Synthetic approach with no additional points at all

> **Claim —** $\triangle BPC \sim \triangle ANM$ (oppositely oriented).

Proof. We have $\triangle BMP \sim \triangle AMQ$ from the given concyclicity of $ABPQ$. Then

$$\frac{BM}{BP} = \frac{AM}{AQ} \implies \frac{2BM}{BP} = \frac{AM}{AQ/2} \implies \frac{BC}{BP} = \frac{AM}{AN}$$

implying the similarity (since $\measuredangle MAQ = \measuredangle BPM$). □

This similarity gives us the equality of directed angles

$$\measuredangle(BC, MN) = -\measuredangle(PC, AM) = 90^\circ$$

as desired.

¶ Synthetic approach using only the point R Again let R be the foot from B to $\overline{AM}$, so $BRCP$ is a parallelogram.

> **Claim —** $ARQC$ is cyclic; equivalently, $\triangle MAQ \sim \triangle MCR$.

Proof. $MR \cdot MA = MP \cdot MA = MB \cdot MQ = MC \cdot MQ$. □

Note that in $\triangle MCR$, the M-median is parallel to $\overline{CP}$ and hence perpendicular to $\overline{RM}$. The same should be true in $\triangle MAQ$ by the similarity, so $\overline{MN} \perp \overline{MQ}$ as needed.

¶ Cartesian coordinates approach with power of a point Suppose we set $B = (-1, 0)$, $M = (0, 0)$, $C = (1, 0)$, and $A = (a, b)$. One may compute:

$$\begin{aligned} \overleftrightarrow{AM} &: 0 = bx - ay \iff y = \frac{b}{a}x \\ \overleftrightarrow{CP} &: 0 = a(x-1) + by \iff y = -\frac{a}{b}(x-1) = -\frac{a}{b}x + \frac{a}{b}. \\ P &= \left(\frac{a^2}{a^2+b^2}, \frac{ab}{a^2+b^2} \right) \end{aligned}$$

Now note that

$$AM = \sqrt{a^2+b^2}, \qquad PM = \frac{a}{\sqrt{a^2+b^2}}$$

together with power of a point

$$AM \cdot PM = BM \cdot QM$$

to immediately deduce that $Q = (a, 0)$. Hence $N = (0, b/2)$ and we're done.

¶ Cartesian coordinates approach without power of a point (outline) After computing A and P as above, one could also directly calculate

$$\begin{aligned}
\text{Perpendicular bisector of } \overline{AB} &: y = -\frac{a+1}{b}x + \frac{a^2+b^2-1}{2b} \\
\text{Perpendicular bisector of } \overline{PB} &: y = -\left(\frac{2a}{b}+\frac{b}{a}\right)x - \frac{b}{2a} \\
\text{Perpendicular bisector of } \overline{PA} &: y = -\frac{a}{b}x + \frac{a+a^2+b^2}{2b}. \\
\text{Circumcenter of } \triangle PAB &= \left(-\frac{a+1}{2}, \frac{2a^2+2a+b^2}{2b}\right).
\end{aligned}$$

This is enough to extract the coordinates of $Q = (\bullet, 0)$, because $B = (-1, 0)$ is given, and the x-coordinate of the circumcenter should be the average of the x-coordinates of B and Q. In other words, $Q = (-a, 0)$. Hence, $N = \left(0, \frac{b}{2}\right)$, as needed.

¶ Ill-advised barycentric approach (outline) Use reference triangle ABC. The A-median is parametrized by $(t : 1 : 1)$ for $t \in \mathbb{R}$. So because of $\overline{CP} \perp \overline{AM}$, we are looking for t such that

$$\left(\frac{t\vec{A}+\vec{B}+\vec{C}}{t+2} - \vec{C}\right) \perp \left(A - \frac{\vec{B}+\vec{C}}{2}\right).$$

This is equivalent to

$$\left(t\vec{A}+\vec{B}-(t+1)\vec{C}\right) \perp \left(2\vec{A}-\vec{B}-\vec{C}\right).$$

By the perpendicularity formula for barycentric coordinates (EGMO 7.16), this is equivalent to

$$\begin{aligned}
0 &= a^2t - b^2 \cdot (3t+2) + c^2 \cdot (2-t) \\
&= \left(a^2 - 3b^2 - c^2\right)t - 2(b^2-c^2) \\
\implies t &= \frac{2(b^2-c^2)}{a^2-3b^2-c^2}.
\end{aligned}$$

In other words,

$$P = \left(2(b^2-c^2) : a^2-3b^2-c^2 : a^2-3b^2-c^2\right).$$

A long calculation gives $a^2 y_P z_P + b^2 z_P x_P + c^2 x_P y_P = (a^2-3b^2-c^2)(a^2-b^2+c^2)(a^2-2b^2-2c^2)$. Together with $x_P + y_P + z_P = 2a^2 - 4b^2 - 4c^2$, this makes the equation of (ABP) as

$$0 = -a^2yz - b^2zx - c^2xy + \frac{a^2-b^2+c^2}{2}z(x+y+z).$$

To solve for Q, set $x = 0$ to get to get

$$a^2yz = \frac{a^2-b^2+c^2}{2}z(y+z) \implies \frac{y}{z} = \frac{a^2-b^2+c^2}{a^2+b^2-c^2}.$$

In other words,

$$Q = \left(0 : a^2-b^2+c^2 : a^2+b^2-c^2\right).$$

Taking the average with $A = (1, 0, 0)$ then gives

$$N = \left(2a^2 : a^2-b^2+c^2 : a^2+b^2-c^2\right).$$

The equation for the perpendicular bisector of $\overline{BC}$ is given by (see EGMO 7.19)

$$0 = a^2(z-y) + x(c^2-b^2)$$

which contains N, as needed.

¶ Extremely ill-advised complex numbers approaches (outline) Suppose we pick a, b, c as the unit circle, and let $m = (b+c)/2$. Using the fully general "foot" formula, one can get

$$p = \frac{(a-m)\overline{c} + (\overline{a}-\overline{m})c + \overline{a}m - a\overline{m}}{2(\overline{a}-\overline{m})} = \frac{a^2b - a^2c - ab^2 - 2abc - ac^2 + b^2c + 3bc^2}{4bc - 2a(b+c)}$$

Meanwhile, an extremely ugly calculation will eventually yield

$$q = \frac{\frac{bc}{a} + b + c - a}{2}$$

so

$$n = \frac{a+q}{2} = \frac{a+b+c+\frac{bc}{a}}{4} = \frac{(a+b)(a+c)}{2a}.$$

There are a few ways to then verify $NB = NC$. The simplest seems to be to verify that

$$\frac{n - \frac{b+c}{2}}{b-c} = \frac{a-b-c+\frac{bc}{a}}{4(b-c)} = \frac{(a-b)(a-c)}{2a(b-c)}$$

is pure imaginary, which is clear.

§1.2 USAMO 2023/2, proposed by Carl Schildkraut

Available online at https://aops.com/community/p27349314.

Problem statement

Solve over the positive real numbers the functional equation

$$f(xy + f(x)) = xf(y) + 2.$$

The answer is $f(x) \equiv x + 1$, which is easily verified to be the only linear solution.

We show conversely that f is linear. Let $P(x, y)$ be the assertion.

Claim — f is weakly increasing.

Proof. Assume for contradiction $a > b$ but $f(a) < f(b)$. Choose y such that $ay + f(a) = by + f(b)$, that is $y = \frac{f(b)-f(a)}{a-b}$. Then $P(a, y)$ and $P(b, y)$ gives $af(y) + 2 = bf(y) + 2$, which is impossible. □

Claim (Up to an error of 2, f is linear) — We have

$$|f(x) - (Kx + C)| \le 2$$

where $K := \frac{2}{f(1)}$ and $C := f(f(1)) - 2$ are constants.

Proof. Note $P(1, y)$ gives $\boxed{f(y + f(1)) = f(y) + 2}$. Hence, $f(nf(1)) = 2(n-1)+f(f(1))$ for $n \ge 1$. Combined with weakly increasing, this gives

$$2\left\lfloor \frac{x}{f(1)} \right\rfloor + C \le f(x) \le 2\left\lceil \frac{x}{f(1)} \right\rceil + C$$

which implies the result. □

Rewrite the previous claim to the simpler $f(x) = Kx + O(1)$. Then for any x and y, the above claim gives

$$K\left(xy + Kx + O(1)\right) + O(1) = xf(y) + 2$$

which means that

$$x \cdot \left(Ky + K^2 - f(y)\right) = O(1).$$

If we fix y and consider large x, we see this can only happen if $Ky + K^2 - f(y) = 0$, i.e. f is linear.

§1.3 USAMO 2023/3, proposed by Holden Mui

Available online at https://aops.com/community/p27349464.

Problem statement

Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes.

Find all possible values of $k(C)$ as a function of n.

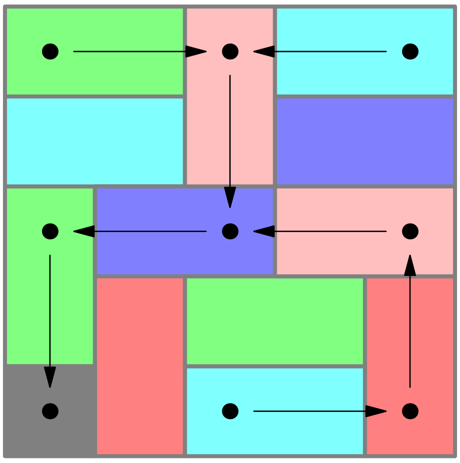
The answer is that

$$k(C) \in \left\{1, 2, \dots, \left(\frac{n-1}{2}\right)^2\right\} \cup \left\{\left(\frac{n+1}{2}\right)^2\right\}.$$

Index the squares by coordinates $(x, y) \in \{1, 2, \dots, n\}^2$. We say a square is *special* if it is empty or it has the same parity in both coordinates as the empty square.

We now proceed in two cases:

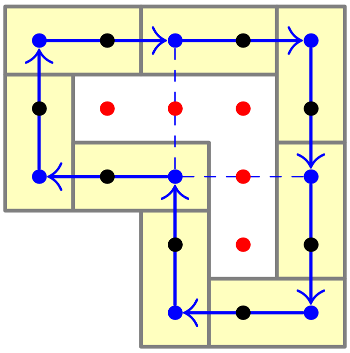
¶ The special squares have both odd coordinates We construct a directed graph $G = G(C)$ whose vertices are special squares as follows: for each domino on a special square s, we draw a directed edge from s to the special square that domino points to. Thus all special squares have an outgoing edge except the empty cell.



Claim — Any undirected connected component of G is acyclic unless the cycle contains the empty square inside it.

Proof. Consider a cycle of G; we are going to prove that the number of chessboard cells enclosed is always odd.

This can be proven directly by induction, but for theatrical effect, we use Pick's theorem. Mark the center of every chessboard cell on or inside the cycle to get a lattice. The dominoes of the cycle then enclose a polyominoe which actually consists of 2×2 squares, meaning its area is a multiple of 4.



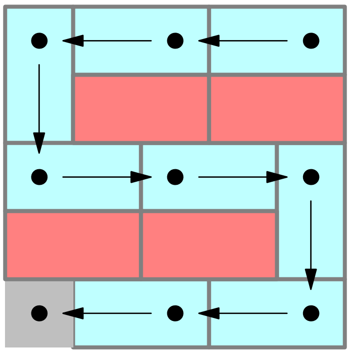
Hence $B/2 + I - 1$ is a multiple of 4, in the notation of Pick's theorem. As B is twice the number of dominoes, and a parity argument on the special squares shows that number is even, it follows that B is also a multiple of 4 (these correspond to blue and black in the figure above). This means I is odd (the red dots in the figure above), as desired. □

Let T be the connected component containing the empty cell. By the claim, T is acyclic, so it's a tree. Now, notice that all the arrows point along T towards the empty cell, and moving a domino corresponds to flipping an arrow. Therefore:

Claim — $k(C)$ is exactly the number of vertices of T.

Proof. Starting with the underlying tree, the set of possible graphs is described by picking one vertex to be the sink (the empty cell) and then directing all arrows towards it. □

This implies that $k(C) \le \left(\frac{n+1}{2}\right)^2$ in this case. Equality is achieved if T is a spanning tree of G. One example of a way to achieve this is using the snake configuration below.

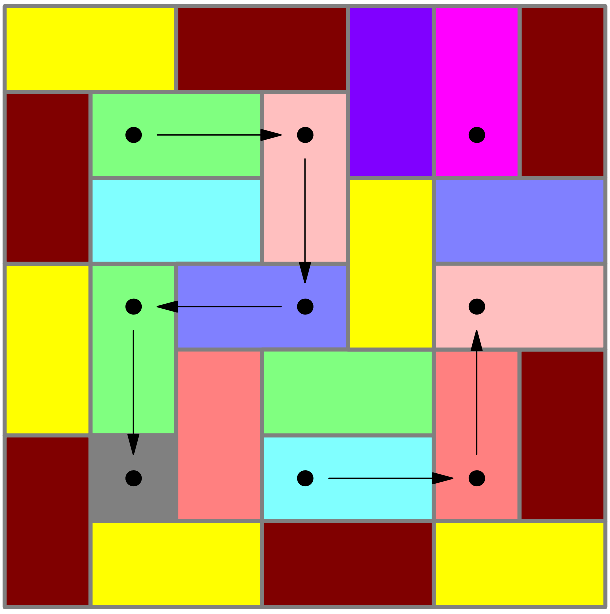


Remark. In Russia 1997/11.8 it's shown that as long as the missing square is a corner, we have $G = T$. The proof is given implicitly from our work here: when the empty cell is in a corner, it cannot be surrounded, ergo the resulting graph has no cycles at all. And since the overall graph has one fewer edge than vertex, it's a tree.

Conversely, suppose T was *not* a spanning tree, i.e. $T \neq G$. Since in this odd-odd case, G has one fewer edge than vertex, if G is not a tree, then it must contain at least one cycle. That cycle encloses every special square of T. In particular, this means that T

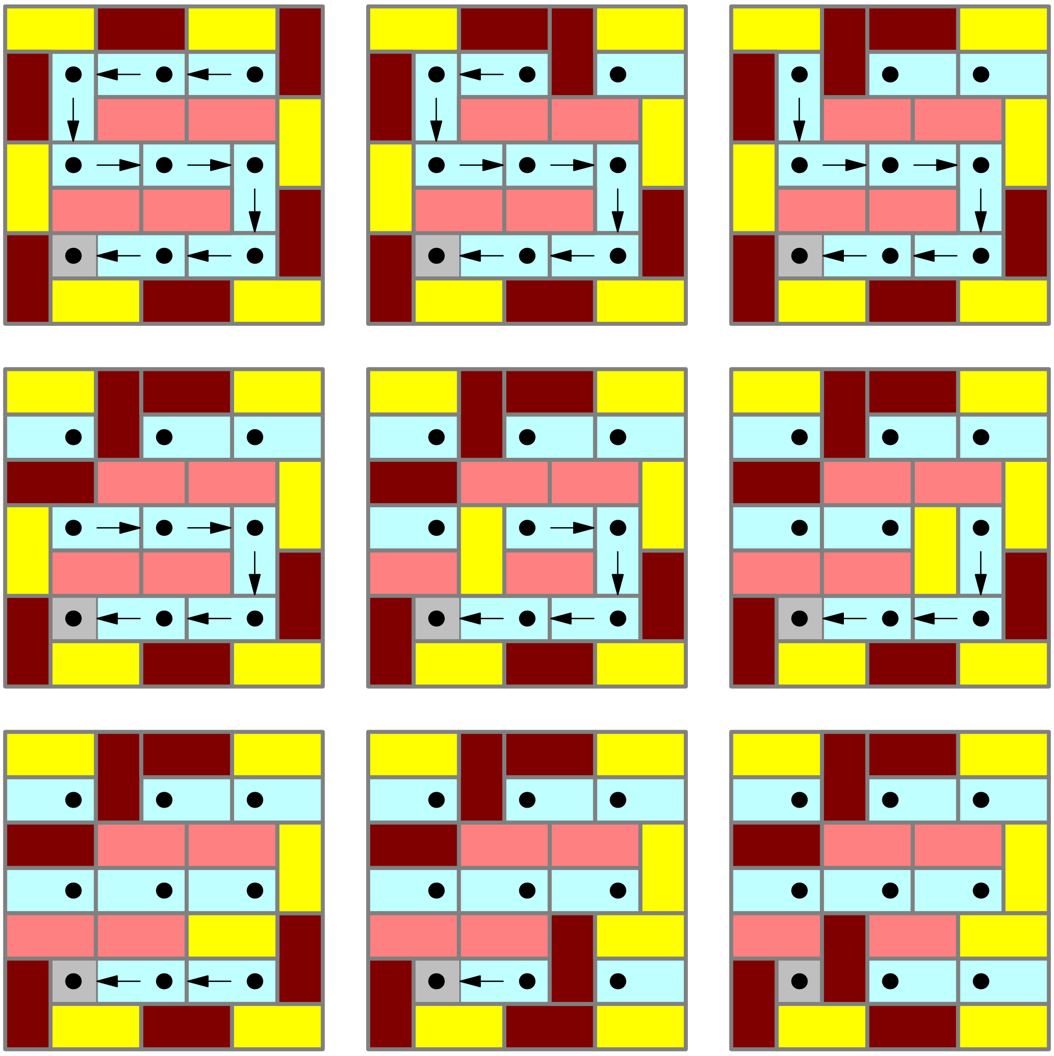
can't contain any special squares from the outermost row or column of the $n \times n$ grid. In this situation, we therefore have $k(C) \le \left(\frac{n-3}{2}\right)^2$.

¶ The special squares have both even coordinates We construct the analogous graph G on the same special squares. However, in this case, some of the points may not have outgoing edges, because their domino may "point" outside the grid.



As before, the connected component T containing the empty square is a tree, and $k(C)$ is exactly the number of vertices of T. Thus to finish the problem we need to give, for each $k \in \{1, 2, \dots, \left(\frac{n-1}{2}\right)^2\}$, an example of a configuration where G has exactly k vertices.

The construction starts with a "snake" picture for $k = \left(\frac{n-1}{2}\right)^2$, then decreases k by one by perturbing a suitable set of dominoes. Rather than write out the procedure in words, we show the sequence of nine pictures for $n = 7$ (where $k = 9, 8, \dots, 1$); the generalization to larger n is straightforward.



§2 Solutions to Day 2

§2.1 USAMO 2023/4, proposed by Carl Schildkraut

Available online at https://aops.com/community/p27349336.

Problem statement

Positive integers a and N are fixed, and N positive integers are written on a blackboard. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the N integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these N integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

For $N = 1$, there is nothing to prove. We address $N \geq 2$ only henceforth. Let S denote the numbers on the board.

Claim — When $N \geq 2$, if $\nu_2(x) < \nu_2(a)$ for all $x \in S$, the game must terminate no matter what either player does.

Proof. The ν_2 of a number is unchanged by Alice's move and decreases by one on Bob's move. The game ends when every ν_2 is zero.

Hence, in fact the game will always terminate in exactly $\sum_{x \in S} \nu_2(x)$ moves in this case, regardless of what either player does. □

Claim — When $N \geq 2$, if there exists a number x on the board such that $\nu_2(x) \geq \nu_2(a)$, then Alice can cause the game to go on forever.

Proof. Denote by x the first entry of the board (its value changes over time). Then Alice's strategy is to:

- Operate on the first entry if $\nu_2(x) = \nu_2(a)$ (the new entry thus has $\nu_2(x+a) > \nu_2(a)$);
- Operate on any other entry besides the first one, otherwise.

A double induction then shows that

- Just before each of Bob's turns, $\nu_2(x) > \nu_2(a)$ always holds; and
- After each of Bob's turns, $\nu_2(x) \geq \nu_2(a)$ always holds.

In particular Bob will never run out of legal moves, since halving x is always legal. □

§2.2 USAMO 2023/5, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p27349487.

Problem statement

Let $n \geq 3$ be an integer. We say that an arrangement of the numbers 1, 2, ..., n^2 in an $n \times n$ table is *row-valid* if the numbers in each row can be permuted to form an arithmetic progression, and *column-valid* if the numbers in each column can be permuted to form an arithmetic progression.

For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

Answer: n prime only.

¶ Proof for n prime Suppose $n = p$. In an arithmetic progression with p terms, it's easy to see that either every term has a different residue modulo p (if the common difference is not a multiple of p), or all of the residues coincide (when the common difference is a multiple of p).

So, look at the multiples of p in a row-valid table; there is either 1 or p per row. As there are p such numbers total, there are two cases:

- If all the multiples of p are in the same row, then the common difference in each row is a multiple of p. In fact, it must be exactly p for size reasons. In other words, up to permutation the rows are just the $k \pmod p$ numbers in some order, and this is obviously column-valid because we can now permute such that the kth column contains exactly $\{(k-1)p+1, (k-1)p+2, \dots, kp\}$.
- If all the multiples of p are in different rows, then it follows each row contains every residue modulo p exactly once. So we can permute to a column-valid arrangement by ensuring the kth column contains all the $k \pmod p$ numbers.

¶ Counterexample for n composite (due to Anton Trygub) Let p be any prime divisor of n. Construct the table as follows:

- Row 1 contains 1 through n.
- Rows 2 through $p+1$ contain the numbers from $p+1$ to $np+p$ partitioned into arithmetic progressions with common difference p.
- The rest of the rows contain the remaining numbers in reading order.

For example, when $p = 2$ and $n = 10$, we get the following table:

1	2	3	4	5	6	7	8	9	10
11	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**
12	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

We claim this works fine. Assume for contradiction the rows may be permuted to obtain a column-valid arrangement. Then the n columns should be arithmetic progressions whose smallest element is in $[1, n]$ and whose largest element is in $[n^2 - n + 1, n^2]$. These two elements must be congruent modulo $n - 1$, so in particular the column containing 2 must end with $n^2 - n + 2$.

Hence in that column, the common difference must in fact be exactly n. And yet $n+2$ and $2n + 2$ are in the same row, contradiction.

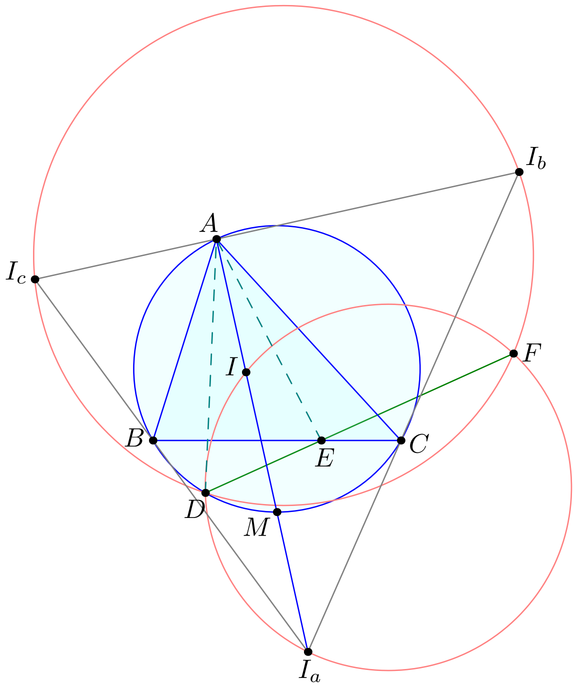
§2.3 USAMO 2023/6, proposed by Zack Chroman

Available online at https://aops.com/community/p27349354.

Problem statement

Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c opposite A, B, and C, respectively. Given an arbitrary point D on the circumcircle of $\triangle ABC$ that does not lie on any of the lines II_a, I_bI_c, or BC, suppose the circumcircles of $\triangle DII_a$ and $\triangle DI_bI_c$ intersect at two distinct points D and F. If E is the intersection of lines DF and BC, prove that $\angle BAD = \angle EAC$.

Here are a two approaches.



¶ Barycentric coordinates (Carl Schildkraut) With reference triangle $\triangle ABC$, set $D = (r : s : t)$.

Claim — The equations of (DII_a) and (DI_bI_c) are, respectively,

$$0 = -a^2yz - b^2zx - c^2xy + (x+y+z) \cdot \left(bcx - \frac{bcr}{cs - bt}(cy - bz) \right)$$
$$0 = -a^2yz - b^2zx - c^2xy + (x+y+z) \cdot \left(-bcx + \frac{bcr}{cs + bt}(cy + bz) \right).$$

Proof. Since $D \in (ABC)$, we have $a^2st + b^2tr + c^2rs = 0$. Now each equation can be verified by direct substitution of three points. □

By EGMO Lemma 7.24, the radical axis is then given by

$$\overline{DF} : bcx - \frac{bcr}{cs - bt}(cy - bz) = -bcx + \frac{bcr}{cs + bt}(cy + bz).$$

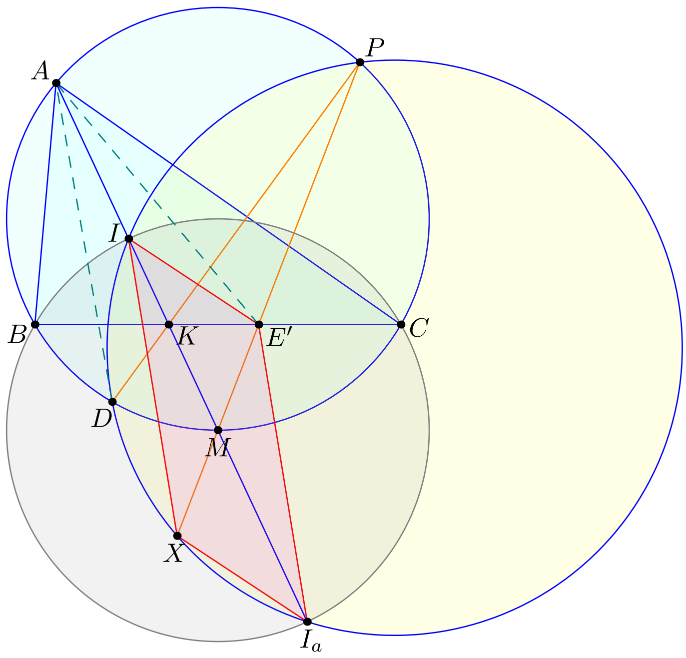
Now the point

$$\left(0 : \frac{b^2}{s} : \frac{c^2}{t}\right) = (0 : b^2t : c^2s)$$

lies on line DF by inspection, and is obviously on line BC, hence it coincides with E. This lies on the isogonal of $\overline{AD}$ (by EGMO Lemma 7.6), as needed.

¶ Synthetic approach (Anant Mudgal) Focus on just (DII_a). Let P be the second intersection of (DII_a) with (ABC), and let M be the midpoint of minor arc $\widehat{BC}$. Then by radical axis, lines AM, DP, and BC are concurrent at a point K.

Let $E' = \overline{PM} \cap \overline{BC}$.



Claim — We have $\measuredangle BAD = \measuredangle E'AC$.

Proof. By shooting lemma, $AKE'P$ is cyclic, so

$$\measuredangle KAE' = \measuredangle KPE' = \measuredangle DPM = \measuredangle DAM. \qquad \square$$

Claim — The power of point E' with respect to (DII_a) is $2E'B \cdot E'C$.

Proof. Construct parallelogram $IE'I_aX$. Since $MI^2 = ME' \cdot MP$, we can get

$$\measuredangle XI_aI = \measuredangle I_aIE' = \measuredangle MIE' = \measuredangle MPI = \measuredangle XPI.$$

Hence X lies on (DII_a), and $E'X \cdot E'P = 2E'M \cdot E'P = 2E'B \cdot E'C$. $\square$

Repeat the argument on (DI_bI_c); the same point E' (because of the first claim) then has power $2E'B \cdot E'C$ with respect to (DI_bI_c). Hence E' lies on the radical axis of (DII_a) and (DI_bI_c), ergo $E' = E$. The first claim then solves the problem.